J. Korean Math. Soc. **53** (2016), No. 6, pp. 1225–1236
http://dx.doi.org/10.4134/JKMS.j150072
pISSN: 0304-9914 / eISSN: 2234-3008

ON n-ABSORBING IDEALS AND THE n-KRULL DIMENSION OF A COMMUTATIVE RING

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Abstract. Let R be a commutative ring with $1 \neq 0$ and n a positive integer. In this article, we introduce the n-Krull dimension of R, denoted $\dim_n R$, which is the supremum of the lengths of chains of n-absorbing ideals of R. We study the n-Krull dimension in several classes of commutative rings. For example, the n-Krull dimension of an Artinian ring is finite for every positive integer n. In particular, if R is an Artinian ring with k maximal ideals and $l(R)$ is the length of a composition series for R, then $\dim_n R = l(R) - k$ for some positive integer n. It is proved that a Noetherian domain R is a Dedekind domain if and only if $\dim_n R = n$ for every positive integer n if and only if $\dim_2 R = 2$. It is shown that Krull's (Generalized) Principal Ideal Theorem does not hold in general when prime ideals are replaced by n-absorbing ideals for some $n > 1$.

1. Introduction

We assume throughout this paper that all rings are commutative with $1 \neq 0$. The concept of 2-absorbing ideal, which is a generalization of prime ideal, was introduced by A. Badawi in [4] and studied in [5], [9]. Various generalizations of prime ideals are also studied in [1, 2, 10]. In recent years, 2-absorbing ideals have been generalized and studied in several directions (see, for example, [5, 6, 7, 8, 9]). As in [3], for a positive integer n, a proper ideal I of a commutative ring R is called an n-absorbing ideal if whenever $x_1 \cdots x_{n+1} \in I$ for $x_1, \ldots, x_{n+1} \in R$, then there are n of the x_i's whose product is in I. It is evident that 1-absorbing ideals are just prime ideals. This was our motivation for the following generalization of the Krull dimension of a ring.

Definition. Let R be a ring and n a positive integer. Then

$$I_0 \subset I_1 \subset \cdots \subset I_m,$$

Received January 28, 2015; Revised July 14, 2015.
2010 *Mathematics Subject Classification.* Primary 13A15; Secondary 13C15, 13E10, 13F05.
Key words and phrases. n-absorbing ideal, n-Krull dimension, n-height, Artinian ring, Dedekind domain.

where $I_0, I_1, \ldots, I_m$ are distinct n-absorbing ideals of R, is called a chain of n-absorbing ideals of length m. The *n-Krull dimension* of R, denoted by $\dim_n R$, is defined to be the supremum of the lengths of these chains. Thus $\dim_1 R$ is just the usual Krull dimension, $\dim R$, of R.

As a first example, let $R = \mathbb{Z}_{p^n}$ for a positive integer n and a prime integer p. By [3, Lemma 2.8, Theorem 2.1(c)], the set of k-absorbing ideals of R consists of all ideals of the form $p^i\mathbb{Z}_{p^n}$, where $1 \leq i \leq k$. Thus for each $1 \leq k \leq n$, $\dim_k(R) = k - 1$ and $\dim_k(R) = n - 1$ for all $k \geq n$.

By [3, Theorem 2.1(c)], every n-absorbing ideal of R is an m-absorbing ideal for all $m \geq n$. It follows immediately that

$$(*) \qquad \dim R = \dim_1 R \leq \dim_2 R \leq \dim_3 R \leq \cdots .$$

We will give several examples for which some of the inequalities in $(*)$ may be strict. There exists a Noetherian ring R such that $\dim_1 R = \infty$ ([11, Exercise 9.6]). Thus by $(*)$, the n-Krull dimension of a Noetherian ring may be infinity for each positive integer n. However, we shall see that for each positive integer n, $\dim_n(R)$ is finite in the case that R is an Artinian ring (Theorem 2.8) or a Dedekind domain (Theorem 2.19). We also show that if R is a Noetherian local domain with $\dim_1(R) = 1$, then $\dim_2 R$ is finite (Theorem 2.9). Furthermore, if [3, Conjecture 2] holds, then $\dim_n R$ is finite for all $n \geq 3$ (Theorem 2.10).

In the rest of paper, we assume that $l(R)$ denotes the length of a composition series for a ring R which is of finite length. It is shown that if $(R, \mathfrak{m})$ is an Artiniain local ring and n is the smallest positive integer such that $\mathfrak{m}^n = 0$, then $\dim_k R = l(R) - 1$ for each $k \geq n$ (Theorem 2.12). In particular, if $\mathfrak{m}$ is principal it is shown that $\dim_k(R) = n - 1$ if $k \geq n$ and $\dim_k(R) = k - 1$ if $1 \leq k \leq n$ (Corollary 2.14). It is shown that if $R = R_1 \times \cdots \times R_k$, where each R_i is a ring, then $\sum_{i=1}^{k} \dim_{n_i} R_i \leq \dim_n R$, for all positive integers $n_1, \ldots, n_k$ with $n = \sum_{i=1}^{k} n_i$. Moreover, if some of the R_i's are fields and $\dim_n R$ is finite for some positive integer n, then $\dim_n R \leq \sum_{i=t+1}^{k} \dim_n R_i + t$, where t is the number of fields in this product (Theorem 2.11). Using this fact and the structure theorem for Artinian rings, we prove that if R is an Artinian ring with k maximal ideals, then $\dim_n R = l(R) - k$ for some positive integer n (Theorem 2.13). As in [3], if I is an n-absorbing ideal of R for some positive integer n, define $\omega(I) = \min\{n : I \text{ is an } n\text{-absorbing ideal of } R\}$, otherwise, set $\omega(I) = \infty$. It is shown that if $I \subseteq J$ are ideals of a Dedekind domain R, then $I = J$ (respectively $I \subset J$) if and only if $\omega(J) = \omega(I)$ (respectively $\omega(J) < \omega(I)$) (Lemmas 2.17 and 2.18). It is shown that a Noetherian domain R is a Dedekind domain if and only if $\dim_n R = n$ for every positive integer n if and only if $\dim_2 R = 2$ (Theorem 2.19).

Finally, among several examples of the n-Krull dimension of a ring, some examples are given to show that Krull's (Generalized) Principal Ideal Theorem can not be generalized when n-absorbing ideals for $n > 1$ are considered rather than prime ideals.

2. The n-Krull dimension of a ring

An n-absorbing ideal I of R is called a minimal n-absorbing ideal of the ideal J if $J \subseteq I$ and there is no n-absorbing ideal I' such that $J \subseteq I' \subset I$. By a minimal n-absorbing ideal of R, we mean a minimal n-absorbing ideal of (0). Although every prime ideal of R is an n-absorbing ideal for each $n \geq 2$, there exists a minimal prime ideal which is not a minimal n-absorbing ideal for each $n \geq 2$. For example, if $R = K[X]$ is the polynomial ring in one variable X over a field K, the minimal prime ideal $P = RX$ of (0) is not a minimal 2-absorbing ideal of (0), since by [3, Lemma 2.8], RX^2 is a 2-absorbing ideal of R.

Theorem 2.1. *Let R be a ring. Then for each positive integer n, there is an n-absorbing ideal of R which is minimal among all n-absorbing ideals of R.*

Proof. Let Σ be the set of all n-absorbing ideals of R. Since every maximal ideal of R is an n-absorbing ideal for each $n > 1$, Σ is not empty. It is clear that $(\Sigma, \leq)$ is a partially ordered set in which $I \leq I'$ if and only if $I \supseteq I'$ for all $I, I' \in \Sigma$. Let $C = \{I_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary non-empty chain of elements of Σ and set $J = \bigcap_{\lambda \in \Lambda} I_\lambda$. We show that J is an n-absorbing ideal of R. Since C is non-empty, $J \neq R$. Let $a_1 \cdots a_{n+1} \in J$ for some $a_1, \ldots, a_{n+1} \in R$. Let $\widehat{a_i} = \prod_{j \neq i} a_j$, the product of all a_j's except a_i. Assume that $\widehat{a_i} \notin J$ for each $1 \leq i \leq n$. Then, for each $1 \leq i \leq n$, there exists $I_{\lambda_i} \in C$ such that $\widehat{a_i} \notin I_{\lambda_i}$. We may assume that $I_{\lambda_1} \subseteq \cdots \subseteq I_{\lambda_n}$. For $\mu \in \Lambda$, we have the following cases:

(1) If $I_\mu \subseteq I_{\lambda_1} \subseteq \cdots \subseteq I_{\lambda_n}$, then $\widehat{a_i} \notin I_\mu$ for each $1 \leq i \leq n$. Now since $a_1 \cdots a_{n+1} \in I_\mu$ and I_μ is an n-absorbing ideal of R, we have $\widehat{a_{n+1}} \in I_\mu$.

(2) If there exists $1 < j \leq n$ such that

$$I_{\lambda_1} \subseteq \cdots \subseteq I_{\lambda_{j-1}} \subseteq I_\mu \subseteq I_{\lambda_j} \subseteq \cdots \subseteq I_{\lambda_n},$$

then $\widehat{a_i} \notin I_{\lambda_1}$ for each $1 \leq i \leq n$. Now since $a_1 \cdots a_{n+1} \in I_{\lambda_1}$ and I_{λ_1} is an n-absorbing ideal of R, we have $\widehat{a_{n+1}} \in I_{\lambda_1} \subseteq I_\mu$.

Thus $\widehat{a_{n+1}} \in I_\mu$ for each $\mu \in \Lambda$, and therefore $\widehat{a_{n+1}} \in J$. Hence by Zorn's Lemma, $(\Sigma, \leq)$ has a maximal element, i.e., there is a minimal n-absorbing ideal of R. □

Corollary 2.2. *Let R be a ring and I a proper ideal of R. Then for each positive integer n, there is an n-absorbing ideal of R which is minimal among all n-absorbing ideals of R containing I.*

Proof. Use Theorem 2.1 and [2, Corollary 4.3(b)]. □

Remark 2.3. Every n-absorbing ideal of R is contained in a maximal ideal of R (and, of course, maximal ideals are n-absorbing ideals). Also, every n-absorbing ideal of R contains a minimal n-absorbing ideal of (0) by Theorem 2.1. It follows that $\dim_n R$ is equal to the supremum of lengths of chains

$$I_0 \subset I_1 \subset \cdots \subset I_m$$

of n-absorbing ideals of R in which I_m is a maximal ideal of R and I_0 is a minimal n-absorbing ideal of (0).

Definition. Let R be a ring and I an ideal of R.

(1) If I is an n-absorbing ideal of R, the *n-height* of I, denoted by $\mathrm{ht}_n(I)$, is defined to be the supremum of lengths of chains

$$I_0 \subset I_1 \subset \cdots \subset I_m$$

of n-absorbing ideals of R for which $I_m = I$ if this supremum exists, and ∞ otherwise.

(2) If I is a proper ideal of R (not necessarily n-absorbing ideal) and n a positive integer, the *n-height of I*, denoted by $\mathrm{ht}_n(I)$, is defined to be

$$\min\{\mathrm{ht}_n(J) : J \text{ is an } n\text{-absorbing ideal and } J \supseteq I\}.$$

Lemma 2.4. *Let $I \subseteq J$ be n-absorbing ideals of R. Then $\mathrm{ht}_n(I) \leq \mathrm{ht}_n(J)$. In particular, if $\mathrm{ht}_n(J) < \infty$, then $I = J$ if and only if $\mathrm{ht}_n(I) = \mathrm{ht}_n(J)$.*

Proof. If $\mathrm{ht}_n(J) = \infty$, there is noting to prove. So let $\mathrm{ht}_n(J) < \infty$. We may assume that $I \subset J$. First note that $\mathrm{ht}_n(I)$ is finite, since for each chain $I_0 \subset I_1 \subset \cdots \subset I_m = I$ of n-absorbing ideals of R, we have the chain $I_0 \subset I_1 \subset \cdots \subset I_m \subset J$ of n-absorbing ideals of R. Let $\mathrm{ht}_n(I) = m$, and $I_0 \subset I_1 \subset \cdots \subset I_m$ be a chain of n-absorbing ideals of R with $I_m = I$. Then, the chain $I_0 \subset I_1 \subset \cdots \subset I_m \subset J$ of n-absorbing ideals of R shows that $\mathrm{ht}_n(J) \geq m + 1$. The "in particular" statement follows immediately. □

Corollary 2.5. *Let R be a ring and I an ideal of R. Then for any positive integer n,*

$$\mathrm{ht}_n(I) = \min\{\mathrm{ht}_n(J) : J \textit{ is a minimal } n\textit{-absorbing ideal of } I\}.$$

Proof. Clearly,

$$\mathrm{ht}_n(I) \leq \min\{\mathrm{ht}_n(J) : J \text{ is a minimal } n\text{-absorbing ideal of } I\}.$$

Thus, if $\mathrm{ht}_n(I) = \infty$, then there is noting to prove. So let $\mathrm{ht}_n(I) = m < \infty$. Then there exists an n-absorbing ideal $J \supseteq I$ of R, such that $\mathrm{ht}_n(J) = \mathrm{ht}_n(I) = m$. By Corollary 2.2, there exists a minimal n-absorbing ideal J' of I such that $I \subseteq J' \subset J$. It follows from Lemma 2.4 that $\mathrm{ht}_n(I) \leq \mathrm{ht}_n(J') \leq \mathrm{ht}_n(J)$. Thus

$$\min\{\mathrm{ht}_n(J) : J \text{ is a minimal } n\text{-absorbing ideal of } I\} \leq \mathrm{ht}_n(I).$$

This completes the proof. □

Theorem 2.6. *Let $I \subset J$ be ideals of R, and J be an n-absorbing ideal of R such that $\mathrm{ht}_n(J)$ is finite. If $\mathrm{ht}_n(I) = \mathrm{ht}_n(J)$, then J is a minimal n-absorbing ideal of I.*

Proof. Suppose that J is not a minimal n-absorbing ideal of I. Then by Corollary 2.2, there exists a minimal n-absorbing ideal J' of I such that $I \subseteq J' \subset J$. In view of Lemma 2.4 and Corollary 2.5, we have $\mathrm{ht}_n(I) \leq \mathrm{ht}_n(J') < \mathrm{ht}_n(J)$, contrary to hypothesis. □

Theorem 2.7. *Let R be a ring and n a positive integer. If $\dim_n R$ is finite, then*

$$\begin{aligned}\dim_n R &= \sup\{\mathrm{ht}_n(I) : I \text{ is an } n\text{-absorbing ideal of } R\}\\ &= \sup\{\mathrm{ht}_n(\mathfrak{m}) : \mathfrak{m} \text{ is a maximal ideal of } R\}.\end{aligned}$$

Proof. To show the first equality, if I is an n-absorbing ideal of R, then it is clear that $\mathrm{ht}_n(I) \leq \dim_n R$. Thus, we have the "$\geq$" for the required equalities. In order to show the "$\leq$", let $\dim_n R = t$. Then there exists a chain $I_0 \subset I_1 \subset \cdots \subset I_t$ of n-absorbing ideals of R. Set $I_t = I$, then $\mathrm{ht}_n(I) = t$, and therefore we have "$\leq$" for the required equalities. The second equality immediately follows from Remark 2.3 and the first equality. □

Theorem 2.8. *If R is an Artinian ring, then $\dim_n R$ is finite for each positive integer n.*

Proof. Since R is Artinian, $\mathrm{Max}(R)$ is a finite set. Let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_k\}$. Since R has finite length, the lengths of all strict chains of ideals of R have an upper bound, say t. Thus $\mathrm{ht}_n(\mathfrak{m}_i) \leq t$ for each $1 \leq i \leq k$. Now the result follows from Remark 2.3. □

Theorem 2.9. *Let $(R, \mathfrak{m})$ be a Noetherian local domain with $\dim_1(R) = 1$. Then $\dim_2(R)$ is finite.*

Proof. Let $(0) = I_0 \subset I_1 \subset \cdots \subset I_t = \mathfrak{m}$ be a chain of 2-absorbing ideals of R. Since $\dim_1(R) = 1$, $\mathrm{Rad}(I_1) = \mathfrak{m}$, and so $\mathfrak{m}^2 \subseteq I_1$ by [4, Theorem 2.4]. Now, since by [14, Exercise 15.17], $l(R/\mathfrak{m}^2) < \infty$, we conclude that $t \leq l(R/\mathfrak{m}^2) < \infty$. Thus $\dim_2(R) < \infty$ by Remark 2.3. □

Let R be a ring. It is clear that $\mathrm{Rad}(\mathfrak{p}) = \mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R. If I is a 2-absorbing ideal of R, then $\mathrm{Rad}(I)^2 \subseteq I$ by [4, Theorem 2.4] and [3, Theorem 6.1]. In [3, Conjecture 2], it has been conjectured that if $n \geq 3$ and I is an n-absorbing ideal of R, then $\mathrm{Rad}(I)^n \subseteq I$. This is true, for example, when R is a Prüfer domain [3, Corollary 6.9]. The following result extends Theorem 2.9 if the conjecture holds.

Theorem 2.10. *Let $(R, \mathfrak{m})$ be a Noetherian local domain with $\dim_1(R) = 1$. Then $\dim_n(R)$ is finite for every positive integer n.*

Proof. The proof is essentially the same as the proof of Corollary 2.9, but by replacing [4, Theorem 2.4] by [3, Conjecture 2] and using n instead of 2. □

The following theorem will be used to show that for an Artinian ring R with k maximal ideals, $\dim_n R = l(R) - k$ for some positive integer n.

Theorem 2.11. *Let $R = R_1 \times \cdots \times R_k$, where each R_i $(1 \leq i \leq k)$ is a ring.*

(1) *If $n_1, \dots, n_k$ are positive integers and $n = \sum_{i=1}^{k} n_i$, then*

$$\sum_{i=1}^{k} \dim_{n_i} R_i \leq \dim_n R.$$

(2) *If R_i is not a field for each $1 \leq i \leq k$, and $\dim_n R$ is finite for some positive integer n, then*

$$\dim_n R \leq \sum_{i=1}^{k} \dim_n R_i.$$

(3) *If $R_1, \dots, R_t$ are fields and $R_{t+1}, \dots, R_k$ are not fields for some $1 \leq t \leq k$, and $\dim_n R$ is finite for some positive integer n, then*

$$\dim_n R \leq \sum_{i=t+1}^{k} \dim_n R_i + t.$$

Proof. (1) Assume that $I_{1i} \subset I_{2i} \subset \cdots \subset I_{ik_i}$ is a chain of n_i-absorbing ideals of R_i. Thus $I'_{1i} \subset I'_{2i} \subset \cdots \subset I'_{ik_i}$ is a chain of n_i-absorbing ideals of R, where for each $1 \leq j \leq k_i$

$$I'_{ji} = R_1 \times \cdots \times R_{i-1} \times I_{ji} \times R_{i+1} \times \cdots \times R_k$$

is an n_i-absorbing ideal of R. Thus $\dim_{n_i} R \geq \dim_{n_i} R_i$, and so $\dim_n R \geq \dim_{n_i} R_i$. Hence, if $\dim_{n_i} R_i = \infty$ for some $1 \leq i \leq k$, then $\dim_n R = \infty$. Now, we assume that for every $1 \leq i \leq k$, $\dim_{n_i} R_i = t_i < \infty$. Thus, for each $i = 1, \dots, k$, there exists a chain $I_{i0} \subset I_{i1} \subset I_{i2} \subset \cdots \subset I_{it_i}$ of n_i-absorbing ideals of R_i. By [3, Theorem 4.7], we have the following chain of n-absorbing ideals which is of the length $t_1 + \cdots + t_k$.

$$\begin{array}{l}
I_{10} \times I_{20} \times I_{30} \times \cdots \times I_{k0} \subset I_{11} \times I_{20} \times I_{30} \times \cdots \times I_{k0} \subset \cdots \subset I_{1t_1} \times I_{20} \times I_{30} \times \cdots \times I_{k0} \subset \\
I_{1t_1} \times I_{21} \times I_{30} \times \cdots \times I_{k0} \subset I_{1t_1} \times I_{22} \times I_{30} \times \cdots \times I_{k0} \subset \cdots \subset I_{1t_1} \times I_{2t_2} \times I_{30} \times \cdots \times I_{k0} \subset \\
\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
I_{1t_1} \times \cdots \times I_{k-1t_{k-1}} \times I_{k0} \subset I_{1t_1} \times \cdots \times I_{k-1t_{k-1}} \times I_{k1} \subset \cdots \subset I_{1t_1} \times \cdots \times I_{k-1t_{k-1}} \times I_{kt_k}
\end{array}$$

Thus $\dim_n R \geq \sum_{i=1}^{k} \dim_{n_i} R_i$.

(2) Let $\dim_n R = s$. By induction on k, it suffices to show that the assertion holds for $k = 2$. In this case, there exists a chain

$$I_0 \times I'_0 \subset I_1 \times I'_1 \subset \cdots \subset I_s \times I'_s$$

of n-absorbing ideals of $R = R_1 \times R_2$. We may assume that there is a chain $I_0 \subset \cdots \subset I_t \subset I_{t+1} = R_1$ of n-absorbing ideals of R_1 for some $0 \leq t \leq s$. Then $\dim_n R_1 \geq t$, and we must have the chain $I'_{t+1} \subset \cdots \subset I'_s$ of n-absorbing ideals of R_2. Therefore $\dim_n R_2 \geq s - t$, and so $\dim_n R_1 + \dim_n R_2 \geq t + (s - t) = s$.

(3) By induction and part (2), we only need to show that $\dim_n(R_1 \times F_1) \leq \dim_n(R_1) + 1$, where F_1 is a field. Let $\dim_n(R_1 \times F_1) = s$. Then there exists a chain

$$I_{10} \times I_{20} \subset I_{11} \times I_{21} \subset \cdots \subset I_{1s} \times I_{2s}$$

of n-absorbing ideals of R. Let $I_{2j} = F_1$ for some $0 \leq j \leq s$. Then we have the chain $I_{1j} \subset I_{1j+1} \subset \cdots \subset I_{1s}$ of n-absorbing ideals of R_1. Now the length of this chain is s if $j = 0$ or $j > 0$ and $I_{1j-1} \neq I_{1j}$, and is $s-1$ if $j > 0$ and $I_{1j-1} = I_{1j}$. Thus $\dim_n R_1 \geq s-1$, i.e., $\dim_n R \leq \dim_n R_1 + 1$ as required. □

Theorem 2.12. *Let $(R, \mathfrak{m})$ be an Artiniain local ring and n be the smallest positive integer such that $\mathfrak{m}^n = (0)$. Then $\dim_k R = l(R) - 1$ for each $k \geq n$.*

Proof. By assumption $\mathfrak{m}^n = (0)$ and $\mathfrak{m}^{n-1} \neq (0)$. Let $k \geq n$ and $\dim_k R = t$. Then there exists a chain $(0) = I_0 \subset I_1 \subset \cdots \subset I_t = \mathfrak{m}$ of k-absorbing ideals of R. Since $\mathfrak{m}^k = (0)$, by [3, Theorem 3.1], every ideal of R is a k-absorbing ideal. It follows that the chain $(0) = I_0 \subset I_1 \subset \cdots \subset I_t = \mathfrak{m} \subset R$ is a composition series for R. Hence, $\dim_k R = l(R) - 1$. □

Theorem 2.13. *Let R be an Artinian ring with k maximal ideals. Then there exists a positive integer n such that $\dim_n R = l(R) - k$.*

Proof. Since R is Artinian, by [11, Corolary 2.16], there exist Artinian local rings R_i $(1 \leq i \leq k)$, such that $R = R_1 \times \cdots \times R_k$. For each $1 \leq i \leq k$, let $\mathfrak{m}_i$ be the unique maximal ideal of R_i and n_i a positive integer such that $\mathfrak{m}_i^{n_i} = (0)$ and $\mathfrak{m}_i^{n_i - 1} \neq (0)$. Thus by Theorem 2.12, $\dim_m R_i = l(R_i) - 1$ for all $i = 1, \ldots, k$ and $m \geq n_i$. Let $n = n_1 + \cdots + n_k$. Then by Theorem 2.11(1), we have

$$\dim_{n_1} R_1 + \cdots + \dim_{n_k} R_k \leq \dim_n R.$$

It follows that $l(R) - k \leq \dim_n R$. Now we have the following two cases:

(1) If R_i is not a field for all $1 \leq i \leq k$, then by Theorem 2.11(2), we have

$$\dim_n R \leq \sum_{i=1}^{k} \dim_n R_i = l(R) - k.$$

(2) Suppose that some of the R_i's are fields. We may assume that R is of the form $R = F_1 \times \cdots \times F_t \times R_{t+1} \times \cdots \times R_t$ for fields $F_1, \ldots, F_t$ $(1 \leq t \leq k)$. Thus by Theorem 2.11(3) and the proof of Theorem 2.12, we have

$$\begin{aligned} \dim_n R \leq \sum_{i=t+1}^{k} \dim_n R_i + t &= \sum_{i=t+1}^{k} (l(R_i) - 1) + t \\ &= (\sum_{i=t+1}^{k} l(R_i) + t) - k = l(R) - k. \end{aligned}$$

Therefore $\dim_n R = l(R) - k$. □

Corollary 2.14. *Let $(R, \mathfrak{m})$ be an Artinian local ring such that $\mathfrak{m} \neq (0)$ is principal. Let n be the smallest positive integer such that $\mathfrak{m}^n = (0)$. Then for each $k \geq n$, $\dim_k(R) = n - 1$, and for each $1 \leq k \leq n$, $\dim_k(R) = k - 1$.*

Proof. Since $\mathfrak{m}^n = (0)$ and $\mathfrak{m}^{n-1} \neq (0)$, by [14, Lemma 15.41], the chain

$$(0) = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset R$$

is a composition series for R. Thus by Theorem 2.12, $\dim_k(R) = n - 1$ for each $k \geq n$. Now let $1 \leq k < n$. Then by [14, Lemma 15.41] and [3, Theorem 3.1], the only chain of k-absorbing ideals of R is $\mathfrak{m}^k \subset \mathfrak{m}^{k-1} \subset \cdots \subset \mathfrak{m}$. Thus $\dim_k(R) = k - 1$. □

Example 2.15. Let K be a field.

(1) Let $R = K[X]/(X^n)$, where $n \geq 2$ is an integer. Then R is an Artinian local ring with maximal ideal $\mathfrak{m} = (X)/(X^n)$. Clearly $\mathfrak{m}^n = (0)$ and $\mathfrak{m}^{n-1} \neq (0)$. Thus, by Corollary 2.14, we have $\dim_k(R) = n - 1$ for each $k \geq n$, and $\dim_k(R) = k - 1$ for each $1 \leq k \leq n$.

(2) If $\mathfrak{m}$ is not principal, then Corollary 2.14 is not necessarily true. For instance, let $R = K[X, Y]/(X^2, Y^2)$. Then R is an Artinian local ring with unique maximal ideal $\mathfrak{m} = (X, Y)/(X^2, Y^2)$. Clearly $\mathfrak{m}^3 = (0)$ and $\mathfrak{m}^2 \neq (0)$. One can easily see that the chain

$$(0) \subset \mathfrak{m}^2 \subset (X^2, Y)/(X^2, Y^2) \subset \mathfrak{m} \subset R$$

is a composition series for R. Thus, by Theorem 2.12, $\dim_3(R) = 3$. Furthermore, $\mathfrak{m}^2 \subset (X^2, Y)/(X^2, Y^2) \subset \mathfrak{m}$ is a chain of 2-absorbing ideals of R, so $\dim_2 R \geq 2$. Note that (0) is a 3-absorbing ideal which is not 2-absorbing since $X(X + Y)Y \in (X^2, Y^2)$, and $X(X + Y), (X + Y)Y, XY \notin (X^2, Y^2)$. Thus $\dim_2 R = 2$. However, Corollary 2.14 is true for $k = 1$, i.e., $\dim_1 R = \dim R = 0$.

In the rest of this section, we determine the n-absorbing dimension of some special rings.

Theorem 2.16. ([3, Theorem 5.1]) *Let R be a Noetherian integral domain. Then the following statements are equivalent*:

(1) *R is a Dedekind domain*;

(2) *If I is an n-absorbing ideal of R, then $I = M_1 \cdots M_m$ for maximal ideals $M_1, \ldots, M_m$ of R with $1 \leq m \leq n$.*

Moreover, if $I = M_1 \cdots M_n$ for maximal ideals $M_1, \ldots, M_n$ of a Dedekind domain R which is not field, then $\omega(I) = n$.

Lemma 2.17. *Let R be a Dedekind domain. Assume that $I \subseteq J$ are ideals of R. Then $I = J$ if and only if $\omega(I) = \omega(J)$.*

Proof. Necessity is clear. For sufficiency, since R is a Dedekind domain and $I \subseteq J$, we have $I = P_1^{k_1} \cdots P_s^{k_s}$ and $J = P_1^{l_1} \cdots P_s^{l_s}$ for maximal ideals $P_1, \ldots, P_s$ of R and positive integers $k_1, \ldots, k_s$ and $l_1, \ldots, l_s$ with $l_i \leq k_i$ for all $1 \leq i \leq s$. Thus, by Theorem 2.16, $\omega(I) = k_1 + \cdots + k_s$ and $\omega(J) = l_1 + \cdots + l_s$. Since $\omega(I) = \omega(J)$ and $l_i \leq k_i$ for all $1 \leq i \leq s$, we conclude that $k_i = l_i$ for all i, and therefore $I = J$. □

Lemma 2.18. *Let R be a Dedekind domain. Assume that $I \subset J$ are ideals of R. Then $\omega(J) < \omega(I)$.*

Proof. In a similar way as the proof of Lemma 2.17, we have $I = P_1^{k_1} \cdots P_s^{k_s}$ and $J = P_1^{l_1} \cdots P_s^{l_s}$ for maximal ideals $P_1, \ldots, P_s$ of R and positive integers $k_1, \ldots, k_s$ and $l_1, \ldots, l_s$ such that $l_i \leq k_i$ for all $1 \leq i \leq s$. Furthermore, $\omega(I) = k_1 + \cdots + k_s$ and $\omega(J) = l_1 + \cdots + l_s$. Since $I \subset J$, we must have $l_i < k_i$ for some $1 \leq i \leq s$. Thus $\omega(J) < \omega(I)$. $\square$

Theorem 2.19. *Let R be a Noetherian integral domain which is not a field. Then the following statements are equivalent*:

(1) *R is a Dedekind domain*;
(2) $\dim_n R = n$*, for every positive integer n*;
(3) $\dim_2(R) = 2$.

Proof. (1) $\Rightarrow$ (2) Let n be a positive integer. By Theorem 2.7 and the fact that $\dim_n R$ is equal to the supremum of lengths of chains $I_0 \subset I_1 \subset \cdots \subset I_m$ of n-absorbing ideals of R in which I_m is a maximal ideal of R, it suffices to show that $\mathrm{ht}_n(\mathfrak{m}) = n$ for each maximal ideal $\mathfrak{m}$ of R. Suppose $\mathfrak{m}$ is a maximal ideal of R. Hence, we have the following chain of n-absorbing ideals of R

$$(0) \subset \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m}.$$

Thus $\mathrm{ht}_n(\mathfrak{m}) \geq n$. Assume to the contrary that $\mathrm{ht}_n(\mathfrak{m}) > n$. Then there exists a chain $0 \subset I_1 \subset \cdots \subset I_{t-1} \subset I_t = \mathfrak{m}$ of n-absorbing ideals of R with $t > n$. Hence by Lemma 2.18, we have

$$\omega(I_t) < \omega(I_{t-1}) < \cdots < \omega(I_1),$$

and therefore $t - 1 < \omega(I_1) \leq n$, which is a contradiction.

(2) $\Rightarrow$ (3) Trivial.

(3) $\Rightarrow$ (1) Let $\mathfrak{m}$ be a maximal ideal of R. Since R is a domain which is not a field, $\mathfrak{m}^2 \neq (0)$. Thus by [3, Lemma 2.8], $(0) \subset \mathfrak{m}^2 \subset \mathfrak{m}$ is a chain of 2-absorbing ideals of R. Every ideal between $\mathfrak{m}^2$ and $\mathfrak{m}$ is an $\mathfrak{m}$-primary ideal of R, and hence a 2-absorbing ideal of R by [3, Theorem 3.1]. Now the hypothesis $\dim_2 R = 2$ implies that there are no ideals of R properly between $\mathfrak{m}^2$ and $\mathfrak{m}$. Thus R is a Dedekind domain by [12, Theorem 6.20]. $\square$

Example 2.20. If R is a principal ideal domain, then by Theorem 2.19, $\dim_n R = n$ for every positive integer n. In particular, $\dim_n \mathbb{Z} = \dim_n \mathbb{Z}[i] = \dim_n K[X] = \dim_n K[[X]] = n$, where $K[X]$ and $K[[X]]$ are the ring of polynomials and the ring of formal power series over a field K, respectively, and $\mathbb{Z}[i]$ is the ring of Gaussian integers. Moreover, let $\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}$. It is well-known that $\mathbb{Z}[\sqrt{-5}]$ is a Dedekind domain that is not a principal ideal domain, and so $\dim_n \mathbb{Z}[\sqrt{-5}] = n$ by Theorem 2.19.

If $(R, \mathfrak{m})$ is a discrete valuation ring, it is well known that every non-zero ideal of R is uniquely of the form $\mathfrak{m}^n$, where n is a positive integer. Furthermore, by [3, Lemma 2.8], the ideal $\mathfrak{m}^n$ is an n-absorbing ideal with $\omega(\mathfrak{m}^n) = n$. Thus every ideal of R is an n-absorbing ideal for some positive integer n. In particular, $0, \mathfrak{m}, \ldots, \mathfrak{m}^n$ are the only n-absorbing ideals of R. This leads us to the following result.

For a finite dimensional vector space V over a field F, we shall denote the dimension of V by $\mathrm{vdim}_F V$.

Theorem 2.21. *Let $(R, \mathfrak{m})$ be a discrete valuation ring and I an ideal of R. Then*

(1) *I is an n-absorbing ideal for some positive integer n and $\omega(I)=l_R(R/I)$.*
(2) *For every positive integer n, $\dim_n(R) = l_R(R/\mathfrak{m}^n) = n$.*

Proof. (1) Since $(R, \mathfrak{m})$ is a discrete valuation ring, $I = \mathfrak{m}^n$ for a unique positive integer n. Further, by [3, Lemma 2.8], $\omega(I) = n$. Consider the following saturated chain of ideals of R

$$I = \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m} \subset R.$$

Then $l_R(R/I) = \sum_{i=0}^{n-1} l_R(\mathfrak{m}^i/\mathfrak{m}^{i+1})$. Now

$$l_R(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = \mathrm{vdim}_{R/\mathfrak{m}}(\mathfrak{m}^i/\mathfrak{m}^{i+1}) = 1$$

is the dimension of the vector space $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ over $R/\mathfrak{m}$. Hence $l_R(R/I) = n = \omega(I)$.

(2) This follows immediately from part (1) and Theorem 2.19. □

If R is a one-dimensional valuation domain with principal maximal ideal $\mathfrak{m}$, then by [13, Theorem 11.2], R is a principal ideal domain and so $\dim_n R = n$. In the following example, we show that this is not necessarily true for every valuation domain R, even if $\dim_1 R = 1$.

Example 2.22. (See [3, Example 5.6])

(1) Let R be a one-dimensional valuation domain with non-principal maximal ideal $\mathfrak{m}$. For every positive integer n, the only n-absorbing ideals of R are (0) and $\mathfrak{m}$. Therefore $\dim_n R = 1$ for every positive integer n.
(2) Let R be a two-dimensional valuation domain with prime ideals $0 \subset \mathfrak{p} \subset \mathfrak{m}$ and value group G. Let n be a positive integer. If $G = \mathbb{Z} \oplus \mathbb{Z}$ (all direct sums have the lexicographic order), then $\mathfrak{p}^{i+1} \neq \mathfrak{p}^i$ and $\mathfrak{m}^{i+1} \neq \mathfrak{m}^i$ for all i; so $(0), \mathfrak{p}^k$, and $\mathfrak{m}^k$ with $1 \leq k \leq n$ are the only n-absorbing ideals of R. Thus, the longest chain of n-absorbing ideals of R is the chain $(0) \subset \mathfrak{p}^n \subset \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \cdots \subset \mathfrak{m}$, and therefore $\dim_n R = n+1$. If $G = \mathbb{Q} \oplus \mathbb{Q}$, then $\mathfrak{p}^2 = \mathfrak{p}$ and $\mathfrak{m}^2 = \mathfrak{m}$; so $(0), \mathfrak{p}$, and $\mathfrak{m}$ are the only n-absorbing ideals of R. Thus $\dim_n R = 2$. If $G = \mathbb{Z} \oplus \mathbb{Q}$, then $\mathfrak{m}^2 = \mathfrak{m}$ and $\mathfrak{p}^{i+1} \neq \mathfrak{p}^i$ for all i; so $(0), \mathfrak{p}^k$ with $1 \leq k \leq n$, and $\mathfrak{m}$ are the only n-absorbing ideals of R. Thus $(0) \subset \mathfrak{p}^n \subset \mathfrak{p}^{n-1} \subset \cdots \subset \mathfrak{p} \subset \mathfrak{m}$ is the longest chain of n-absorbing ideals of R. Hence $\dim_n R = n+1$. If $G = \mathbb{Q} \oplus \mathbb{Z}$, then $\mathfrak{p}^2 = \mathfrak{p}$ and $\mathfrak{m}^{i+1} \neq \mathfrak{m}^i$ for all i; so $(0), \mathfrak{p}$, and $\mathfrak{m}^k$

with $1 \leq k \leq n$ are the only n-absorbing ideals of R. Thus, the longest chain of n-absorbing ideals of R is $(0) \subset \mathfrak{p} \subset \mathfrak{m}^n \subset \mathfrak{m}^{n-1} \subset \cdots \subset \mathfrak{m}$. Hence $\dim_n R = n + 1$.

In the following examples, we show that a number of results concerning the 1-Krull dimension for a ring R can not be generalized to n-Krull dimension in case $n > 1$.

Example 2.23. Let R be a Noetherian ring and I a nilpotent ideal of R. It is easily seen that $\dim_n R = \dim_n R/I$ if $n = 1$. But if $n > 1$, this is not necessarily true. For instance, let $R = K[X]$, where K is a field. Let $I = (X^2)$ and $S = R/I$. Then $\dim_2 S = 1$ by Example 2.15. For the ideal $J = (X)/(X^2)$ of S, it is clear that J is a nilpotent ideal of S and $S/J \cong K$. Thus $\dim_2(S/J) = 0$, and therefore $\dim_2 S \neq \dim_2 S/J$.

Example 2.24. Let R be a commutative Noetherian ring and I a prime (1-absorbing) ideal of R which is generated by m elements. Then $\mathrm{ht}_n(I) \leq m$ for $n = 1$ (Krull's Generalized Principal Ideal Theorem [14, Theorem 15.4]). In the following examples, we show that this is not necessarily true if $n > 1$, whether I is a prime ideal or not.

(1) Let K be a field, $R = K[X]$, and $I = RX$. Since $(0) \subset RX^2 \subset RX$ is a chain of 2-absorbing ideals of R, we have $\mathrm{ht}_2(RX) = 2$.
(2) Let K be a field and $R = K[X]/(X^3)$. Then R is an Artinian local ring with maximal ideal $\mathfrak{m} = (X)/(X^3)$. Clearly $\mathfrak{m}^3 = 0$ and $\mathfrak{m}^2 \neq 0$. By [3, Theorem 3.1], $\mathfrak{m}^2$ is a 2-absorbing ideal, but not a prime ideal of R. Now by [14, Lemma 15.41], $(0) \subset \mathfrak{m}^3 \subset \mathfrak{m}^2 \subset \mathfrak{m}$ is the only chain of 3-absorbing ideals of R. Hence $\mathrm{ht}_3(\mathfrak{m}^2) = 2$.
(3) Let K be a field and $R = K[X,Y]/(X^3,Y^2)$. Then R is an Artinian local ring with maximal ideal $\mathfrak{m} = (X,Y)/(X^3,Y^2)$. Clearly $\mathfrak{m}^4 = 0$ and $\mathfrak{m}^3 \neq 0$. Now if $I = (X^2,Y)/(X^3,Y^2)$, then by [3, Theorem 3.1], I is a 4-absorbing ideal of R. Since
$$(0) \subset \mathfrak{m}^3 \subset (X^2,Y^2)/(X^3,Y^2) \subset \mathfrak{m}^2 \subset I \subset \mathfrak{m} \subset R$$
is a composition series for R, we have $\mathrm{ht}_4(I) = 4$.
(4) Let R be a Dedekind domain and $\mathfrak{m}$ a maximal ideal of R. Then by [13, Exercise 11.5], $\mathfrak{m}$ is generated by at most two elements . However, by the proof of Theorem 2.19, $\mathrm{ht}_n(\mathfrak{m}) = n$ for every positive integer n.

Acknowledgments. We would like to thank the referee for a careful reading of our article and many insightful comments.

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