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WEAK AND STRONG CONVERGENCE OF SUBGRADIENT EXTRAGRADIENT METHODS FOR PSEUDOMONOTONE EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we introduce three subgradient extragradient algorithms for solving pseudomonotone equilibrium problems. The paper originates from the subgradient extragradient algorithm for variational inequalities and the extragradient method for pseudomonotone equilibrium problems in which we have to solve two optimization programs onto feasible set. The main idea of the proposed algorithms is that at every iterative step, we have replaced the second optimization program by that one on a specific half-space which can be performed more easily. The weakly and strongly convergent theorems are established under widely used assumptions for bifunctions.

1. Introduction

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $f : H \times H \to \mathbb{R}$ be a bifunction with $f(x, x) = 0$ for all $x \in C$. The equilibrium problem (EP) is stated as follows:

$$\text{Find } x^* \in C \text{ such that } f(x^*, y) \geq 0,\ \forall y \in C. \tag{1}$$

The solution set of EP (1) is denoted by $EP(f)$. The EP includes, as special cases, many mathematical models such as: variational inequality problems, fixed point problems, optimization problems, Nash equilirium problems, complementarity problems, etc., see [8, 21] and the references therein. In recent years, many algorithms have been proposed for solving EPs [1, 2, 3, 4, 5, 6, 13, 16, 17, 18, 20, 25, 28]. In the case, the bifunction f is monotone, solution approximations of EPs are based on a regularization equilibrium problem, i.e., at the step n, known x_n, the next approximation x_{n+1} is the solution of the following problem:

$$\text{Find } x \in C \text{ such that: } f(x, y) + \frac{1}{r_n} \langle y - x, x - x_n \rangle \geq 0,\ \forall y \in C, \tag{2}$$

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where r_n is a suitable parameter. Note that problem (2) is strongly monotone, hence its solution exists and is unique. However, if the bifunction f is more general monotone, for instance, pseudomonotone then problem (2) in general is not strongly monotone. So, the unique solvability of problem (2) is not guaranteed, even its solution set can be not convex. In this case, the authors in [1, 25] replaced problem (2) by the following two strongly convex programs

$$y_n = \underset{y \in C}{\operatorname{argmin}}\{\lambda f(x_n, y) + \frac{1}{2}||x_n - y||^2\}, \tag{3}$$

$$z_n = \underset{y \in C}{\operatorname{argmin}}\{\lambda f(y_n, y) + \frac{1}{2}||x_n - y||^2\}, \tag{4}$$

where λ is a suitable parameter. If $f(x, y) = \langle A(x), y - x\rangle$, where $A : H \to H$ is a nonlinear operator, then EP (1) becomes the following variational inequality problem (VIP):

$$\text{Find } x^* \in C \text{ such that } \langle A(x^*), y - x^*\rangle \geq 0, \ \forall y \in C. \tag{5}$$

The solution set of VIP (5) is denoted by $VI(A, C)$. In this case, two strongly convex programs (3)-(4) are reduced to the extragradient method (or double projection method) which was first introduced by Korpelevich [19] in Euclidean spaces for the saddle point problem and then it was extended to Hilbert spaces by Nadezhkina and Takahashi [22, 23] for variational inequalities for Lipschitz continuous and monotone operators as follows

$$y_n = P_C(x_n - \lambda A(x_n)), \tag{6}$$

$$z_n = P_C(x_n - \lambda A(y_n)). \tag{7}$$

If C has a simple structure as balls or half-spaces then problems (3)-(4) and (6)-(7) can be easily solved. In general, if C is any closed convex set, we have to solve two strongly convex optimization problems (or two projections) on C per each iteration. This can be costly and affect the efficiency of used methods. In 2011, the authors in [10, 11] proposed the subgradient extragradient method for variational inequalities in which they replaced projection (7) on C by that one on a specific half-space which can be computed explicitly.

In this paper, motivated and inspired by the results in [10, 11], we propose three subgradient extragradient algorithms for solving EPs for pseudomonotone bifunctions. In these algorithms, we replaced strongly convex optimization program (4) on feasible set C by that one on a specific half-space whose the bounding hyperplane supported on the feasible set C. It seems to be more easily performed than on the feasible set.

The paper is organized as follows: In Section 2, we collect some definitions and results for further use. Section 3 deals with the proposed algorithms and analyzing the convergence of iteration sequences generated by the algorithms.

2. Preliminaries

In this section, we recall some definitions and preliminary results. Let C be a nonempty closed convex subset of a real Hilbert space H. A mapping $S : C \to H$ is called Lipschitz continuous on C if there exists a positive constant L such that $||S(x) - S(y)|| \leq L||x - y||$ for all $x, y \in C$. If $L = 1$, then S is said to be nonexpansive on C. The fixed point set of S is denoted by $F(S)$. We begin with the following properties of a nonexpansive mapping.

Lemma 2.1 ([15])**.** *Assume that $S : C \to C$ is a nonexpansive mapping. If S has a fixed point, then*

(i) *$F(S)$ is a closed convex subset of C.*
(ii) *$I - S$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in C weakly converging to some point $x \in C$ and the sequence $\{(I - S)x_n\}$ strongly converges to some point y, it follows that $(I - S)x = y$.*

For solving EP (1), we assume that the bifunction f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$ and f is pseudomonotone, i.e., for all $x, y \in C$,
$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0;$$
(A2) f is Lipschitz-type continuous, i.e., there exist two positive constants c_1, c_2 such that
$$f(x, y) + f(y, z) \geq f(x, z) - c_1||x - y||^2 - c_2||y - z||^2, \quad \forall x, y, z \in H;$$
(A3) f is jointly weakly upper semicontinuous on the product $C \times C$, in the sense that if $x,\ y \in C$ and $\{x_n\},\ \{y_n\} \subset H$ converge weakly to x and y, respectively, then $\limsup_{n\to\infty} f(x_n, y_n) \leq f(x, y)$;
(A4) $f(x, \cdot)$ is convex and subdifferentiable on H for every fixed $x \in H$.

If $A : H \to H$ is a L-Lipschitz continuous mapping on H, then the bifunction $f(x, y) = \langle A(x), y - x\rangle$ satisfies hypothesis (A2) with $c_1 = c_2 = L/2$. The class of other bifunctions, which is generalized from the Cournot-Nash equilibrium model [14, 25, 26] as

$$f(x, y) = \langle F(x) + Qy + q, y - x\rangle\,,\ x, y \in \mathbb{R}^n, \tag{8}$$

where $F : \mathbb{R}^n \to \mathbb{R}$, $Q \in \mathbb{R}^{n\times n}$ is a symmetric positive semidefinite matrix and $q \in \mathbb{R}^n$ also satisfies condition (A2) under some suitable assumptions imposed on F [25]. It is easy to show that if f satisfies conditions (A1)-(A4), then the solution set $EP(f)$ is closed and convex, see for instance [25]. In this paper, we assume that $EP(f)$ is nonempty.

The metric projection $P_C : H \to C$ is defined by
$$P_C x = \arg\min\{\|y - x\| : y \in C\}\,.$$
Since C is nonempty closed and convex, $P_C x$ exists and is unique. It is also known that P_C has the following characteristic properties.

Lemma 2.2. *Let $P_C : H \to C$ be the metric projection from H onto C. Then*

(i) *P_C is firmly nonexpansive, i.e.,*

$$\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \ \forall x, y \in H.$$

(ii) *For all $x \in C, y \in H$,*

$$\|x - P_C y\|^2 + \|P_C y - y\|^2 \leq \|x - y\|^2. \tag{9}$$

(iii) *$z = P_C x$ if and only if*

$$\langle x - z, z - y\rangle \geq 0, \quad \forall y \in C. \tag{10}$$

Note that any closed convex subset C of H can be represented as the sublevel set of an appropriate convex function $c : H \to \mathbb{R}$, i.e., $C = \{v \in H : c(v) \leq 0\}$. The subdifferential of c at x is defined by

$$\partial c(x) = \{w \in H : c(y) - c(x) \geq \langle w, y - x\rangle, \ \forall y \in H\}.$$

For each $z \in H$ and $w \in \partial c(z)$, we denote

$$T(z) = \{v \in H : c(z) + \langle w, v - z\rangle \leq 0\}.$$

If $z \notin \mathrm{int}C$, then $T(z)$ is a half-space whose bounding hyperplane separates the set C from the point z. Otherwise, $T(z)$ is the entire space H. We recall that the normal cone of C at $x \in C$ is defined by

$$N_C(x) = \{w \in H : \langle w, y - x\rangle \leq 0, \ \forall y \in C\}.$$

Lemma 2.3 ([12])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H and $g : C \to \mathbb{R}$ be a convex, lower semicontinuous and subdifferentiable function on C. Then, x^* is a solution to the convex optimization problem $\min\{g(x) : x \in C\}$ if and only if $0 \in \partial g(x^*) + N_C(x^*)$, where $\partial g(\cdot)$ denotes the subdifferential of g and $N_C(x^*)$ is the normal cone of C at x^*.*

Lemma 2.4 ([29])**.** *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $\{x_n\} \subset H$ be a Fejér-monotone sequence with respect to C, i.e., for every $u \in C$,*

$$||x_{n+1} - u|| \leq ||x_n - u||, \ \forall n \geq 0.$$

Then the sequence $\{P_C(x_n)\}$ converges strongly to some $z \in C$.

3. Main results

The following algorithm can be considered as an extension of the results in [10, 11] to equilibrium problems.

Algorithm 3.1. (Subgradient Extragradient Method)
Initialization. *Choose $x_0 \in H$ and the control parameter sequences $\{\lambda_k\}$, $\{\gamma_k\}$ satisfying $0 < \alpha \leq \lambda_k \leq \beta < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$ and $\gamma_k \in [\epsilon, \frac{1}{2}]$ for some*

$\epsilon \in (0, \frac{1}{2}]$. *Set* $n := 0$.
Step 1. *Solve a strongly convex optimization program*

$$y_n = \underset{y\in C}{\text{argmin}}\{\lambda_n f(x_n, y) + \frac{1}{2}||x_n - y||^2\}.$$

If $y_n = x_n$, *then stop.*
Step 2. *Solve a strongly convex optimization program*

$$x_{n+1} = \underset{y\in T_n}{\text{argmin}}\{\lambda_n f(y_n, y) + \frac{1}{2}||x_n - y||^2\},$$

where T_n *is the half-space whose bounding hyperplane supported on* C *at* y_n, *i.e.,* $T_n = \{v \in H : \langle (x_n - \lambda_n w_n) - y_n, v - y_n\rangle \le 0\}$ *and* $w_n \in \partial_2 f(x_n, y_n)$. *Set* $n := n + 1$ *and go back* ***Step 1****.*

Remark 3.2. Before analyzing the convergence, we give some advantages of Algorithm 3.1 in comparing with the extragradient method [25].

(a) If $f(x, y) = \langle A(x), y - x\rangle$, where $A : H \to H$ is a nonlinear operator, then the optimization problem in Step 2 of Algorithm 3.1 is reduced to

$$x_{n+1} = P_{T_n}(x_n - \lambda_n A(y_n)). \tag{11}$$

It is clear that projection (11) is performed on half-space T_n and it is explicit while the second optimization problem in the extragradient method [25] is a projection onto the feasible set C. Even in $\Re^m$ when C has a simple structure as a polyhedral convex set $C = \langle x \in \Re^m : Ex \le e\rangle$, where $E \in \Re^{k\times m}$ and $e \in \Re^k$ then the projection on C is often computed by cyclic (parallel or block) iterative methods. This can be costly if the number of linear inequalities k is large.

(b) We consider the bifunction $f : C \times C \to \Re$ which is generalized from the Cournot-Nash model [14, 25] as

$$f(x, y) = \langle Px + Qy + q, y - x\rangle\,, \tag{12}$$

where $q \in \Re^m$ and P, Q are two $m \times m$ matrices. Since T_n is a half-space, the optimization problem in Step 2 of Algorithm 3.1 is always a convex quadratic problem (only with one linear inequality constraint). This problem can be solved very effectively by the available methods of convex quadratic programming [9, Chapter 8] while both two solved optimization problems over C in the extragradient method may be costly, specially when C has a complex structure.

The following lemma gives us a stopping criterion of Algorithm 3.1.

Lemma 3.3. *If* $y_n = x_n$, *then* $x_n \in EP(f)$, *i.e.,* x_n *is a solution of EP* (1).

Proof. If $y_n = x_n$, then from the definition of y_n we obtain

$$x_n = \underset{y\in C}{\text{argmin}}\{\lambda_n f(x_n, y) + \frac{1}{2}||x_n - y||^2\}.$$

By [20, Proposition 2.1], one has $x_n \in EP(f)$. The proof of Lemma 3.3 is complete. $\square$

Lemma 3.4. *Assume that $x^* \in EP(f)$. Let $\{y_n\}, \{x_n\}$ be the sequences determined as in Algorithm 3.1. Then, there holds the relation*

$$||x_{n+1}-x^*||^2 \leq ||x_n-x^*||^2-(1-2\lambda_n c_1)\,||y_n-x_n||^2-(1-2\lambda_n c_2)\,||x_{n+1}-y_n||^2.$$

Proof. Since $x_{n+1} \in T_n$, we have

$$\langle (x_n - \lambda_n w_n) - y_n, x_{n+1} - y_n \rangle \leq 0.$$

Thus

$$\langle x_n - y_n, x_{n+1} - y_n \rangle \leq \lambda_n \langle w_n, x_{n+1} - y_n \rangle . \tag{13}$$

From $w_n \in \partial_2 f(x_n, y_n)$ and the definition of subdifferential, we obtain

$$f(x_n, y) - f(x_n, y_n) \geq \langle w_n, y - y_n \rangle , \ \forall y \in H.$$

The last inequality with $y = x_{n+1}$ and (13) imply that

$$\lambda_n \{f(x_n, x_{n+1}) - f(x_n, y_n)\} \geq \langle x_n - y_n, x_{n+1} - y_n \rangle . \tag{14}$$

By Lemma 2.3 and

$$x_{n+1} = \underset{y \in T_n}{\mathrm{argmin}}\{\lambda_n f(y_n, y) + \frac{1}{2}||x_n - y||^2\},$$

one has

$$0 \in \partial_2 \left\{ \lambda_n f(y_n, y) + \frac{1}{2}||x_n - y||^2 \right\} (x_{n+1}) + N_{T_n}(x_{n+1}).$$

Thus, there exist $w \in \partial_2 f(y_n, x_{n+1})$ and $\bar{w} \in N_{T_n}(x_{n+1})$ such that

$$\lambda_n w + x_{n+1} - x_n + \bar{w} = 0. \tag{15}$$

From the definition of the normal cone and $\bar{w} \in N_{T_n}(x_{n+1})$, we get

$$\langle \bar{w}, y - x_{n+1} \rangle \leq 0$$

for all $y \in T_n$. This together with (15) implies that

$$\lambda_n \langle w, y - x_{n+1} \rangle \geq \langle x_n - x_{n+1}, y - x_{n+1} \rangle$$

for all $y \in T_n$. Since $x^* \in T_n$,

$$\lambda_n \langle w, x^* - x_{n+1} \rangle \geq \langle x_n - x_{n+1}, x^* - x_{n+1} \rangle . \tag{16}$$

By $w \in \partial_2 f(y_n, x_{n+1})$, we also obtain

$$f(y_n, y) - f(y_n, x_{n+1}) \geq \langle w, y - x_{n+1} \rangle , \ \forall y \in H.$$

This together with (16) implies that

$$\lambda_n \{f(y_n, x^*) - f(y_n, x_{n+1})\} \geq \langle x_n - x_{n+1}, x^* - x_{n+1} \rangle . \tag{17}$$

Note that $x^* \in EP(f)$, so $f(x^*, y_n) \geq 0$. The pseudomonotonicity of f implies that $f(y_n, x^*) \leq 0$. From (17), we get

$$\langle x_n - x_{n+1}, x_{n+1} - x^* \rangle \geq \lambda_n f(y_n, x_{n+1}). \tag{18}$$

The Lipschitz-type continuity of f leads to

$$f(y_n, x_{n+1}) \geq f(x_n, x_{n+1}) - f(x_n, y_n) - c_1||x_n - y_n||^2 - c_2||x_{n+1} - y_n||^2. \tag{19}$$

Combining the relations (18) and (19), we obtain

$$\begin{aligned} \langle x_n - x_{n+1}, x_{n+1} - x^* \rangle \geq\ & \lambda_n \{f(x_n, x_{n+1}) - f(x_n, y_n)\} \\ & - \lambda_n \left\{c_1||x_n - y_n||^2 + c_2||x_{n+1} - y_n||^2\right\}. \end{aligned} \tag{20}$$

By (14), (20), we obtain

$$\begin{aligned} \langle x_n - x_{n+1}, x_{n+1} - x^* \rangle \geq\ & \langle x_n - y_n, x_{n+1} - y_n \rangle \\ & - \lambda_n c_1||x_n - y_n||^2 - \lambda_n c_2||x_{n+1} - y_n||^2. \end{aligned} \tag{21}$$

We have the following facts

$$2\langle x_n - x_{n+1}, x_{n+1} - x^* \rangle = ||x_n - x^*||^2 - ||x_{n+1} - x_n||^2 - ||x_{n+1} - x^*||^2.$$

$$2\langle x_n - y_n, x_{n+1} - y_n \rangle = ||x_n - y_n||^2 + ||x_{n+1} - y_n||^2 - ||x_n - x_{n+1}||^2.$$

The last two relations and (21) lead to the desired conclusion of Lemma 3.4. □

Theorem 3.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the bifunction f satisfies all conditions* (A1)-(A4). *In addition the solution set $EP(f)$ is nonempty. Then, the sequences $\{x_n\}, \{y_n\}$ generated by Algorithm 3.1 converge weakly to some solution $u^* \in EP(f)$. Moreover, $u^* = \lim_{n\to\infty} P_{EP(f)}(x_n)$.*

Proof. From Lemma 3.3, we obtain

$$(1 - 2\lambda_n c_1)||y_n - x_n||^2 \leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2.$$

Therefore

$$\sum_{n=1}^{\infty} (1 - 2\lambda_n c_1)||y_n - x_n||^2 \leq ||x_0 - x^*||^2. \tag{22}$$

From the hypothesis of λ_n, we get $0 < 1 - 2\beta c_1 \leq 1 - 2\lambda_n c_1$. Thus, from (22), one has

$$\sum_{n=1}^{\infty} ||y_n - x_n||^2 \leq \frac{||x_0 - x^*||^2}{1 - 2\beta c_1} < \infty.$$

Hence,

$$\lim_{n\to\infty} ||y_n - x_n|| = 0. \tag{23}$$

By Lemma 3.3, the sequence $\{x_n\}$ is bounded. So, there exists a subsequence of $\{x_n\}$, denoted by $\{x_m\}$, converging weakly to u^*, i.e., $x_m \rightharpoonup u^*$. From (23), we also have $y_m \rightharpoonup u^*$. From the definition of y_m and Lemma 2.3, we obtain

$$0 \in \partial_2 \left\{\lambda_n f(x_m, y) + \frac{1}{2}||x_m - y||^2\right\}(y_m) + N_C(y_m).$$

Therefore, there exist $w \in \partial_2 f(x_m, y_m)$ and $\bar{w} \in N_C(y_m)$ such that

$$\lambda_n w + y_m - x_m + \bar{w} = 0. \tag{24}$$

Since $\bar{w} \in N_C(y_m)$, $\langle \bar{w}, y - y_m \rangle \leq 0$ for all $y \in C$. This together with (24) implies that

$$\lambda_n \langle w, y - y_m \rangle \geq \langle x_m - y_m, y - y_m \rangle \tag{25}$$

for all $y \in C$. Since $w \in \partial_2 f(x_n, y_m)$,

$$f(x_m, y) - f(x_m, y_m) \geq \langle w, y - y_m \rangle, \forall y \in C. \tag{26}$$

From (25) and (26), we get

$$\lambda_n \left(f(x_m, y) - f(x_m, y_m) \right) \geq \langle x_m - y_m, y - y_m \rangle, \forall y \in C. \tag{27}$$

Letting $m \to \infty$ in (27) and using the assumption of λ_n, (A1) and (A3), we can conclude that $f(u^*, y) \geq 0$ for all $y \in C$. Thus, $u^* \in EP(f)$. Next, we show that the whole sequence $\{x_n\}$ converges weakly to u^*. Indeed, from Lemma 3.3, the sequence $\{||x_n - x^*||\}$ is decreasing for each $x^* \in EP(f)$. Therefore, there exists the limit $c(x^*) = \lim_{n\to\infty} ||x_n - x^*||$ for each $x^* \in EP(f)$. Assume that $\bar{u}^*$ is any weak cluster point of $\{x_n\}$ such that $\bar{u}^* \neq u^*$ and $x_k \rightharpoonup \bar{u}^*$ where $\{x_k\}$ is some subsequence of $\{x_n\}$. From the Opial condition [24], we have

$$\begin{aligned} c(u^*) &= \lim_{m\to\infty} ||x_m - u^*|| = \lim_{m\to\infty} \inf ||x_m - u^*|| \\ &< \lim_{m\to\infty} \inf ||x_m - \bar{u}^*|| = \lim_{m\to\infty} ||x_m - \bar{u}^*|| = c(\bar{u}^*) \\ &= \lim_{k\to\infty} ||x_k - \bar{u}^*|| = \lim_{k\to\infty} \inf ||x_k - \bar{u}^*|| \\ &< \lim_{k\to\infty} \inf ||x_k - u^*|| = \lim_{k\to\infty} ||x_k - u^*|| = c(u^*). \end{aligned}$$

This is a contradiction, and so $u^* = \bar{u}^*$. Hence, $x_n \rightharpoonup u^*$ as $n \to \infty$. Clearly, $y_n \rightharpoonup u^*$ as $n \to \infty$ because $||x_n - y_n|| \to 0$. Let $u_n = P_{EP(f)} x_n$, by Lemma 2.4, we obtain $u_n \to z \in EP(f)$. Moreover, from Lemma 2.2 and $u^* \in EP(f)$, we also get

$$\langle u_n - u^*, x_n - u_n \rangle \geq 0.$$

Passing to the limit in the last inequality as $n \to \infty$, one has $\langle z - u^*, u^* - z \rangle \geq 0$. Thus $z = u^*$. Theorem 3.5 is proved. □

In order to obtain the strong convergence of the iterative sequences. We propose the following algorithm, so-called the hybrid subgradient extragradient method, which combines the subgradient extragradient method and the cutting-hyperplane. It can be considered as an extension of the algorithm in [11] to equilibrium problems.

Algorithm 3.6. (Hybrid Subgradient Extragradient Method)
Initialization. *Choose $x_0 \in H$ and the control parameter sequences $\{\lambda_k\}$, $\{\gamma_k\}$ satisfying $0 < \alpha \leq \lambda_k \leq \beta < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right)$ and $\gamma_k \in [\epsilon, \frac{1}{2}]$ for some $\epsilon \in (0, \frac{1}{2}]$. Set $n := 0$.*
Step 1. *Solve two strongly convex optimization programs*

$$y_n = \underset{y \in C}{\operatorname{argmin}} \{\lambda_n f(x_n, y) + \frac{1}{2}||x_n - y||^2\},$$

$$z_n = \underset{y \in T_n}{\operatorname{argmin}}\{\lambda_n f(y_n, y) + \frac{1}{2}||x_n - y||^2\},$$

where T_n is defined as in Algorithm 3.1.
Step 2. *Compute $x_{n+1} = P_{H_n \cap W_n}(x_0)$, where*

$$H_n = \{z \in H : \langle x_n - z_n, z - x_n - \gamma_n(z_n - x_n)\rangle \leq 0\}.$$
$$W_n = \{z \in H : \langle x_0 - x_n, x_n - z\rangle \geq 0\}.$$

Step 3. *If $x_{n+1} = x_n$, then stop. Otherwise, $n := n + 1$ and go back* ***Step 1.***

According to Algorithm 3.6, the sets H_n, W_n are either half-spaces or H. Therefore, the projection $x_{n+1} = P_{H_n \cap W_n}(x_0)$ can be found explicitly. We need the following results for proving the convergence of the sequences generated by Algorithm 3.6.

Lemma 3.7. *Assume that $x^* \in EP(f)$. Let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by Algorithm 3.6. Then,*

(i) *There holds the relation*

$$||z_n - x^*||^2 \leq ||x_n - x^*||^2 - (1 - 2\lambda_n c_1)\,||y_n - x_n||^2 - (1 - 2\lambda_n c_2)\,||z_n - y_n||^2.$$

(ii) *$EP(f) \subset H_n \cap W_n$ for all $n \geq 0$.*

Proof. The proof of claim (i) is same to that one of Lemma 3.3. Next, we show conclusion (ii). From the definitions of H_n, W_n, we see that these sets are closed and convex. Now, we show that $EP(f) \subset H_n \cap W_n$ for all $n \geq 0$. Putting

$$C_n = \left\{z \in H : \left\langle x_n - z_n, z - x_n - \frac{1}{2}(z_n - x_n)\right\rangle \leq 0\right\}.$$

Since $\gamma_n \in [\epsilon, \frac{1}{2}]$, $C_n \subset H_n$. From (i) and the hypothesis of λ_n, we obtain $||z_n - x^*|| \leq ||x_n - x^*||$ for all $x^* \in EP(f)$. This implies that $EP(f) \subset C_n$. Next, we show that $EP(f) \subset C_n \cap W_n$ for all $n \geq 0$ by the induction. Indeed, we have $EP(f) \subset C_0 \cap W_0$. Assume that $EP(f) \subset C_n \cap W_n$ for some $n \geq 0$. From the definition of W_n, we see that $x_{n+1} = P_{H_n \cap W_n}(x_0)$. By (10), we obtain

$$\langle x_0 - x_{n+1}, x_{n+1} - z\rangle \geq 0,\ \forall y \in H_n \cap W_n.$$

By $EP(f) \subset C_n \cap W_n \subset H_n \cap W_n$,

$$\langle x_0 - x_{n+1}, x_{n+1} - z\rangle \geq 0,\ \forall y \in EP(f).$$

Thus $EP(f) \subset W_{n+1}$ because of the definition of W_{n+1}, and so $EP(f) \subset C_n \cap W_n \subset H_n \cap W_n$ for all $n \geq 0$. Since $EP(f)$ is nonempty. Therefore, x_{n+1} is well-defined. Lemma 3.7 is proved. □

Lemma 3.8. *If Algorithm 3.6 finishes at some iteration $n < \infty$, then $x_n \in EP(f)$.*

Proof. Assume that $x_{n+1} = x_n$. Since $x_{n+1} = P_{H_n \cap W_n}(x_0)$, $x_n = x_{n+1} \in H_n$. This together with the definition of H_n implies that $\gamma_n ||x_n - z_n|| \le 0$. Thus, $x_n = z_n$. From Lemma 3.7, we obtain $y_n = x_n$. Thus

$$x_n = \arg\min \left\{ \lambda_n f(x_n, y) + \frac{1}{2}||x_n - y||^2 : y \in C \right\}.$$

Thus, from [20, Proposition 2.1], one has $x_n \in EP(f)$. The proof of Lemma 3.8 is complete. □

Lemma 3.9. *Let $\{x_n\}, \{y_n\}, \{z_n\}$ be the sequences generated by Algorithm 3.6. Then, there hold the relations*

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||y_n - x_n|| = \lim_{n\to\infty} ||z_n - x_n|| = 0.$$

Proof. Note that $x_n = P_{W_n}(x_0)$. For each $u \in EP(f) \in W_n$, from (9), one has

$$||x_n - x_0|| \le ||u - x_0||. \tag{28}$$

Thus, the sequence $\{||x_n - x_0||\}$ is bounded, and so $\{x_n\}$ is also bounded. By $x_{n+1} = P_{H_n \cap W_n}(x_0) \in W_n$ and relation (9), we get

$$||x_n - x_0|| \le ||x_{n+1} - x_0||.$$

Thus, the sequence $\{||x_n - x_0||\}$ is non-decreasing, and so there exists the limit of the sequence $\{||x_n - x_0||\}$. By $x_{n+1} \in W_n$, $x_n = P_{W_n}(x_0)$ and relation (9), we also have

$$||x_{n+1} - x_n||^2 \le ||x_{n+1} - x_0||^2 - ||x_n - x_0||^2.$$

Letting $n \to \infty$ in the last inequality, one gets

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{29}$$

Since $x_{n+1} = P_{H_n \cap W_n}(x_0) \in H_n$,

$$\gamma_n ||z_n - x_n||^2 \le \langle x_n - z_n, x_n - x_{n+1} \rangle \le ||x_n - z_n|| ||x_n - x_{n+1}||.$$

Thus, $\gamma_n ||z_n - x_n|| \le ||x_n - x_{n+1}||$. From $\gamma_n \ge \epsilon > 0$ and (29), one has

$$\lim_{n\to\infty} ||z_n - x_n|| = 0. \tag{30}$$

Lemma 3.7 and the triangle inequality lead to

$$\begin{aligned}
(1 - 2\lambda_n c_1) ||y_n - x_n||^2 &\le ||x_n - x^*||^2 - ||z_n - x^*||^2 \\
&\le (||x_n - x^*|| + ||z_n - x^*||)(||x_n - x^*|| - ||z_n - x^*||) \\
&\le (||x_n - x^*|| + ||z_n - x^*||)||x_n - z_n||.
\end{aligned}$$

The last inequality together with (30), the hypothesis of λ_n and the boundedness of $\{x_n\}, \{z_n\}$ implies that

$$\lim_{n\to\infty} ||y_n - x_n|| = 0.$$

Lemma 3.8 is proved. □

Theorem 3.10. *Let C be a nonempty closed convex subset of a real Hilbert space H. Assume that the bifunction f satisfies all conditions* (A1)-(A4). *In addition the solution set $EP(f)$ is nonempty. Then, the sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ generated by Algorithm 3.6 converge strongly to $P_{EP(f)}(x_0)$.*

Proof. Lemmas 3.7 and 3.9 ensure that the sets H_n, W_n are closed and convex for all $n \geq 0$ and the sequence $\{x_n\}$ is bounded. Assume that p is any weak limit point of the sequence $\{x_n\}$. Then, there exists a subsequence of $\{x_n\}$ converging weakly to p. For the sake of simplicity, we denote again this subsequence by $\{x_n\}$ and $x_n \rightharpoonup p$ as $n \to \infty$. Next, we show that $p \in EP(f)$. Indeed, from the definition of y_n and Lemma 2.3, one gets

$$0 \in \partial_2 \left\{ \lambda_n f(x_n, y) + \frac{1}{2} ||x_n - y||^2 \right\} (y_n) + N_C(y_n).$$

Thus, there exist $\bar{w} \in N_C(y_n)$ and $w \in \partial_2 f(x_n, y_n)$ such that

$$\lambda_n w + y_n - x_n + \bar{w} = 0. \tag{31}$$

From the definition of the normal cone $N_C(y_n)$, we have $\langle \bar{w}, y - y_n \rangle \leq 0$ for all $y \in C$. Taking into account (31), we obtain

$$\lambda_n \langle w, y - y_n \rangle \geq \langle x_n - y_n, y - y_n \rangle \tag{32}$$

for all $y \in C$. Since $w \in \partial_2 f(x_n, y_n)$,

$$f(x_n, y) - f(x_n, y_n) \geq \langle w, y - y_n \rangle, \forall y \in C. \tag{33}$$

Combining (32) and (33), one has

$$\lambda_n \left(f(x_n, y) - f(x_n, y_n) \right) \geq \langle x_n - y_n, y - y_n \rangle, \forall y \in C. \tag{34}$$

From Lemma 3.9 and $x_n \rightharpoonup p$, we also have $y_n \rightharpoonup p$. Passing to the limit in the inequality (34) and employing the assumption of λ_n, (A1) and (A3), we can conclude that $f(p, y) \geq 0$ for all $y \in C$. Hence, $p \in EP(f)$. Finally, we show that $x_n \to p$. Let $x^\dagger = P_{EP(f)}(x_0)$. Using inequality (28) with $u = x^\dagger$, we get

$$||x_n - x_0|| \leq ||x^\dagger - x_0||.$$

By the weakly lower semicontinuity of the norm $|| \cdot ||$ and $x_n \rightharpoonup p$, we have

$$||p - x_0|| \leq \lim \inf_{n \to \infty} ||x_n - x_0|| \leq \lim \sup_{n \to \infty} ||x_n - x_0|| \leq ||x^\dagger - x_0||.$$

By the definition of $x^\dagger$, $p = x^\dagger$ and $\lim_{n \to \infty} ||x_n - x_0|| = ||x^\dagger - x_0||$. Since $x_n \rightharpoonup x^\dagger$, $x_n - x_0 \rightharpoonup x^\dagger - x_0$. By the Kadec-Klee property of H, we have $x_n - x_0 \to x^\dagger - x_0$ as $n \to \infty$. Thus $x_n \to x^\dagger = P_{EP(f)} x_0$. From Lemma 3.9, we also see that $\{y_n\}, \{z_n\}$ converge strongly to $P_{EP(f)} x_0$. Theorem 3.10 is proved. □

Next, we propose an algorithm, so-called the modified hybrid subgradient extragradient - Mann algorithm, which combines the subgradient extragadient method and Mann's iteration method for finding a common element of the solution set of an equilibrium problem for a bifunction f and the fixed point set of a mapping $S : H \to H$. The algorithm is designed as follows:

Algorithm 3.11. (Modified Hybrid Subgradient Extragradient Method)
Initialization *Choose* $x_0 \in H$ *and set* $n := 0$. *The control parameter sequences* $\{\lambda_k\}$, $\{\gamma_k\}$, $\{\alpha_k\}$ *satisfy the following conditions*:

(a) $0 < \alpha \leq \lambda_k \leq \beta < \min\left(\frac{1}{2c_1}, \frac{1}{2c_2}\right), \gamma_k \in [\epsilon, \frac{1}{2}]$ *for some* $\epsilon \in (0, \frac{1}{2}]$.

(b) $\{\alpha_k\} \subset (0, 1)$ *such that* $\lim_{k\to\infty} \sup \alpha_k < 1$.

Step 1. *Solve two strongly convex optimization programs*

$$y_n = \underset{y\in C}{\operatorname{argmin}}\{\lambda_n f(x_n, y) + \frac{1}{2}||x_n - y||^2\},$$

$$z_n = \underset{y\in T_n}{\operatorname{argmin}}\{\lambda_n f(y_n, y) + \frac{1}{2}||x_n - y||^2\},$$

where T_n *is defined as in Algorithm 3.6.*
Step 2. *Calculate* $u_n = \alpha_n x_n + (1 - \alpha_n)Sz_n$.
Step 3. *Compute* $x_{n+1} = P_{H_n\cap W_n}(x_0)$, *where*

$$H_n = \{z \in H : \langle x_n - u_n, z - x_n - \gamma_n(u_n - x_n)\rangle \leq 0\}\,;$$
$$W_n = \{z \in H : \langle x_0 - x_n, x_n - z\rangle \geq 0\}\,.$$

Step 4. *If* $x_{n+1} = x_n = z_n$, *then stop. Otherwise,* $n := n + 1$ *and go back* ***Step 1.***

Remark 3.12. If Algorithm 3.11 finishes at the iteration step $n < \infty$, then $x_n \in EP(f) \cap F(S)$.

Theorem 3.13. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Assume that the bifunction* f *satisfies all conditions* (A1)-(A4) *and* $S : H \to H$ *is a nonexpansive mapping. In addition the solution set* $F := EP(f) \cap F(S)$ *is nonempty. Then, the sequences* $\{x_n\}, \{y_n\}, \{z_n\}, \{u_n\}$ *generated by Algorithm 3.11 converge strongly to* $P_F(x_0)$.

Proof. From Lemma 2.1, the set F is closed and convex. By arguing similarly to the proof of Lemma 3.7, we also obtain $F \subset H_n \cap W_n$. Next, we show that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||y_n - x_n|| = \lim_{n\to\infty} ||z_n - x_n|| = 0.$$
$$\lim_{n\to\infty} ||u_n - x_n|| = \lim_{n\to\infty} ||S(x_n) - x_n|| = 0.$$

Indeed, repeating the proofs of (29), (30) we obtain

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} ||u_n - x_n|| = 0. \tag{35}$$

By the triangle inequality, we have

$$\left|||x_n - x^*||^2 - ||u_n - x^*||^2\right| \leq ||x_n - u_n||(||x_n - x^*|| + ||u_n - x^*||)$$

for each $x^* \in F$. The last inequality together with (35) and the boundedness of $\{x_n\}, \{u_n\}$ one has

$$\lim_{n\to\infty} \left(||x_n - x^*||^2 - ||u_n - x^*||^2\right) = 0. \tag{36}$$

For each $x^* \in F$, from the convexity of $||\cdot||^2$ and Lemma 3.7 we get

$$\begin{aligned}
&||u_n - x^*||^2\\
&= ||\alpha_n(x_n - x^*) + (1-\alpha_n)(Sz_n - x^*)||^2\\
&\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)||Sz_n - x^*||^2\\
&\le \alpha_n||x_n - x^*||^2 + (1-\alpha_n)||z_n - x^*||^2\\
&= ||x_n - x^*||^2 + (1-\alpha_n)\left\{z_n - x^*||^2 - ||x_n - x^*||^2\right\}\\
&\le ||x_n - x^*||^2 - (1-\alpha_n)\left\{(1-2\lambda_n^i c_1)||x_n - y_n||^2 + (1-2\lambda_n c_2)||z_n - y_n||^2\right\}.
\end{aligned}$$

Therefore

$$(1-2\lambda_n c_1)||x_n - y_n||^2 + (1-2\lambda_n c_2)||z_n - y_n||^2 \le \frac{||x_n - x^*||^2 - ||u_n - x^*||^2}{1-\alpha_n}.$$

Combining this inequality with relation (36) and hypothesises (a), (b), we obtain

$$\lim_{n\to\infty} ||x_n - y_n|| = \lim_{n\to\infty} ||z_n - y_n|| = 0. \tag{37}$$

Thus, from $||x_n - z_n|| \le ||x_n - y_n|| + ||y_n - z_n||$ and (37), we obtain

$$\lim_{n\to\infty} ||x_n - z_n|| = 0.$$

Moreover, from $u_n = \alpha_n x_n + (1-\alpha_n)Sz_n$, we obtain

$$||u_n - x_n|| = (1-\alpha_n)||x_n - Sz_n||. \tag{38}$$

From (35), (38) and hypothesis (b), we can conclude that

$$\lim_{n\to\infty} ||x_n - Sz_n|| = 0.$$

This together with the inequality $||x_n - Sx_n|| \le ||x_n - Sz_n|| + ||Sz_n - Sx_n|| \le ||x_n - Sz_n|| + ||z_n - x_n||$ implies that

$$\lim_{n\to\infty} ||x_n - Sx_n|| = 0. \tag{39}$$

Next, since $\{x_n\}$ is bounded, there exists a subsequence of $\{x_n\}$ converging weakly to p. For the sake of simplicity, we denote again this subsequence by $\{x_n\}$, and so, $x_n \rightharpoonup p$ as $n \to \infty$. Lemma 2.1 and relation (39) ensure that $p \in F(S)$. Repeating the proof of Theorem 3.10, we can conclude that $p \in F$ and $x_n \to p$ as $n \to \infty$. The proof of Theorem 3.13 is complete. □

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