

## FRACTIONAL POLYNOMIAL METHOD FOR SOLVING FRACTIONAL ORDER POPULATION GROWTH MODEL

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**ABSTRACT.** This paper presents an efficient fractional shifted Legendre polynomial method to solve the fractional Volterra's model for population growth model. The fractional derivatives are described based on the Caputo sense by using Riemann-Liouville fractional integral operator. The theoretical analysis, such as convergence analysis and error bound for the proposed technique has been demonstrated. In applications, the reliability of the technique is demonstrated by the error function based on the accuracy of the approximate solution. The numerical applications have provided the efficiency of the method with different coefficients of the population growth model. Finally, the obtained results reveal that the proposed technique is very convenient and quite accurate to such considered problems.

### 1. Introduction

Fractional calculus has been applied in physics in recent years, although it has a long history of mathematics. One possible explanation of such unpopularity could be the existence of multiple nonequivalent definitions of fractional derivatives [2]. Also, another difficulty of these definitions is that the fractional derivatives have no obvious geometrical interpretation in consequence of their nonlocal character [3]. However, during the last 10 years, fractional calculus starts to attract much more attention of physicists and mathematicians. It was found that various, especially interdisciplinary applications can be elegantly modeled with the help of the fractional derivatives. For example, the fractional derivatives could have modeled the nonlinear oscillation of earthquake [6]. The fluid-dynamic models with fractional derivatives [10], [11] can eliminate the deficiency arising from the assumption of continuum traffic flow and differential equations with fractional order have recently proved to be valuable tools for the modeling of many physical phenomena [2], [9]. Mainardi [9] discussed

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the applications of fractional calculus in statistical mechanics and continuum. Nowadays fractional differential equations have proved to be valuable tools in the modeling of physical phenomena [1], [2], [4], [5], [13], [15], [17]. Many researchers [2], [16] have proved the existence and uniqueness of solutions to different types of the fractional differential equations.

The population growth [20] is characterized by the nonlinear fractional Volterra integro-differential equation

$$(1) \quad \begin{aligned} \frac{d^\alpha y}{dt^\alpha} &= ay - by^2 - cy \int_0^t y(s)ds, \\ y(0) &= \beta, \quad 0 < \alpha \leq 1, \end{aligned}$$

where  $y(t)$  is the scaled population of identical individuals,  $t$  denotes the time,  $\alpha$  is a constant describing the order of time-fractional derivative,  $a > 0$  is the birth rate coefficient,  $b > 0$  is the crowding coefficient, and  $c > 0$  denotes the essential behavior of the population evolution before its level falls to zero in the long run is the toxicity coefficient as defined in [23],  $c/(ab)$  is a non-dimensional parameter. The last segment of Eq. (1) is a function integral representing the “total metabolism” or total amount of toxins accumulated from time zero. The individual death rate is proportional to this integral, and so the population death rate due to toxicity must include a factor  $\alpha$ . Since the system is closed, the presence of toxic always causes the population level to fall to zero in the long run. For more details about these investigations, we refer [7], [8], [20], [23].

Several analytical and numerical methods have been proposed to solve the population growth model Eq. (1). The fractional Volterra’s population model has been solved by the Adomian decomposition method, Pade approximation and the homotopy analysis method [12]. Moreover, various familiar methods such as the Sinc and rational Legendre collocation method [14], the rational Chebyshev and Hermite functions collocation approach [16], the second derivative multistep methods [15], the hybrid function approximation [10], the homotopy-Pade method [11], the spectral functions methods [18], and the derivative multistep methods [15] are used to obtain the numerical solutions of classical Volterra’s population model. In this paper, we intend to extend the application of fractional shifted Legendre polynomial method (FSLPM) to find the approximate solution of the population growth model.

This paper is organized as follows. In Section 2, we present some necessary definitions and mathematics preliminaries of the fractional calculus theory. In Section 3, fractional order Legendre functions and its properties are discussed. In Section 4, we discuss the convergence analysis and error analysis for the proposed function approximation. In Section 5, we demonstrate the accuracy of the proposed scheme by considering numerical examples. The conclusion is given in Section 6.

## 2. Preliminaries and notations

In this section, we give the definition of fractional-order integration and fractional-order differentiation [2], [3].

**Definition.** A real function  $f(x)$ ,  $x > 0$ , is said to be in the space  $C_\mu$ ,  $\mu \in \mathbb{R}$  if there exists a real number  $p > \mu$ , such that  $f(x) = x^p f_1(x)$ , where  $f_1(x) \in C[0, \infty]$ . Clearly  $C_\mu < C_\beta$  if  $\beta < \mu$ .

**Definition.** The fractional derivative of  $f(x)$  in the Caputo sense is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t) dt$$

for  $m-1 < \alpha \leq m$ ,  $m \in \mathbb{N}$ ,  $x > 0$ ,  $f \in C_{-1}^m$ .

Caputo fractional derivative first computes an ordinary derivative followed by a fractional integral to achieve the desired order of fractional derivative. Some properties of the operator  $D^\alpha$ , which are needed here, are as follows:

$$(2) \quad \begin{aligned} D^\alpha D^\beta f(x) &= D^{\alpha+\beta} f(x), \\ D^\alpha C &= 0, \quad (C \text{ is a constant}) \\ D^\alpha x^\beta &= \begin{cases} 0 & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta < \lceil \alpha \rceil, \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha} & \text{for } \beta \in \mathbb{N}_0 \text{ and } \beta \geq \lceil \alpha \rceil \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > \lfloor \alpha \rfloor. \end{cases} \end{aligned}$$

We use the ceiling function  $\lceil \alpha \rceil$  to denote the smallest integer greater than or equal to  $\alpha$ , and the floor function  $\lfloor \alpha \rfloor$  to denote the largest integer less than or equal to  $\alpha$ . Also  $\mathbb{N} = \{1, 2, \dots\}$  and  $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ . Similar to the integer-order derivative, the Caputo fractional derivative is a linear operation:

$$(3) \quad D^\alpha \left( \sum_{i=1}^n c_i f_i(t) \right) = \sum_{i=1}^n c_i D^\alpha f_i(t),$$

where  $\{c_i\}_{i=1}^n$  are constants.

## 3. Fractional shifted Legendre polynomials

In this section, we study another application of fractional shifted Legendre polynomials, introduced by Kazem et al. [6].

### 3.1. Shifted Legendre polynomial

The Legendre polynomials are orthogonal functions defined over  $(-1, 1)$ . The shifted Legendre polynomials  $P_n(2t-1)$  is obtained by changing the variable  $z = 2t-1$  in the Legendre polynomial  $P_n(z)$  [22], where  $t \in (0, 1)$  is generally denoted by  $L_n(t)$ . The shifted Legendre polynomials are orthogonal with respect to the weight function  $w_s(t) = 1$  in the interval  $(0, 1)$  with the orthogonality property  $\int_0^1 L_n(t) L_m(t) dt = \frac{1}{2n+1} \delta_{nm}$ .

Also it satisfies

$$(4) \quad \begin{aligned} L_{i+1}(t) &= \frac{(2i+1)(2t-1)}{i+1} L_i(t) - \frac{i}{i+1} L_{i-1}(t), \quad i = 1, 2, \dots, \\ L_0(t) &= 1 \quad \text{and} \quad L_1(t) = 2t - 1. \end{aligned}$$

The analytic form of the shifted Legendre polynomial  $L_n(t)$  of degree  $n$  given by  $L_n(t) = \sum_{i=0}^n (-1)^{n+i} \frac{(n+i)!}{(n-i)!} \frac{t^i}{(i!)^2}$ . Note that  $L_n(0) = (-1)^n$  and  $L_n(1) = 1$ .

### 3.2. Fractional-order Legendre definition for FDE

For solving FDEs of order  $\alpha$ , Rida and Yousef [19] have introduced a fractional Legendre polynomials based method to solve FDEs. The other common and efficient methods to solve the FDEs of order  $\alpha$  is based on the series expansion of the form  $\sum_{i=0}^N c_i x^{i\alpha}$ . In this way, Kazem et al. [6] has introduced a fractional shifted Legendre polynomial method, replacing  $t$  by  $x^\alpha$  and it is denoted by  $FL_i^\alpha(x)$ . The so called  $FL_i^\alpha(x)$  are the particular solution of the normalized eigen functions of the singular Sturm-Liouville problem

$$(5) \quad \left( (x - x^{1+\alpha}) FL_i'^\alpha(x) \right)' + \alpha^2 i(i+1) x^{\alpha-1} FL_i^\alpha(x) = 0, \quad x \in (0, 1).$$

Also it satisfies

$$\begin{aligned} FL_i^\alpha(x) &= \frac{(2i+1)(2x^\alpha-1)}{i+1} FL_i^\alpha(x) - \frac{i}{i+1} FL_{i-1}^\alpha(x), \quad i = 1, 2, \dots, \\ FL_0^\alpha(x) &= 1 \quad \text{and} \quad FL_1^\alpha(x) = 2x^\alpha - 1. \end{aligned}$$

The analytic form of  $FL_i^\alpha(x)$  of degree  $i\alpha$  given by

$$(6) \quad FL_i^\alpha(x) = \sum_{s=0}^i b_{s,i} x^{s\alpha},$$

where  $b_{s,i} = \frac{(-1)^{i+s} (i+s)!}{(i-s)! (s!)^2}$ . And  $FL_i^\alpha(0) = (-1)^i$  and  $FL_i^\alpha(1) = 1$ .

The FLFs are orthogonal with respect to the weight function  $w(x) = x^{\alpha-1}$  in the interval  $(0, 1]$  with the orthogonality property

$$(7) \quad \int_0^1 FL_n^\alpha(x) FL_m^\alpha(x) w(x) dx = \frac{1}{(2n+1)\alpha} \delta_{nm}.$$

### 3.3. Function approximation

A function  $y(x)$  defined over the interval  $(0, 1]$  may be expressed in terms of fractional shifted Legendre polynomials as

$$(8) \quad y(x) = \sum_{i=0}^{\infty} c_i FL_i^\alpha(x),$$

where the coefficients  $c_i$  are given by

$$c_i = \alpha(2i+1) \int_0^1 FL_i^\alpha(x) y(x) w(x) dx, \quad i = 0, 1, 2, \dots$$

In practice, only the first  $m$  terms of FLFs are considered. If the infinite series in Eq. (8) is truncated, then it can be written as

$$(9) \quad y(x) \simeq y_m(t) = \sum_{i=0}^{m-1} c_i FL_i^\alpha(t) = C^T FL^\alpha(t)$$

with  $C = [c_0, c_1, \dots, c_{m-1}]^T$ ,  $FL^\alpha(t) = [FL_0^\alpha(t), FL_1^\alpha(t), \dots, FL_{m-1}^\alpha(t)]^T$ .

### 3.4. Description of the method

In order to solve Eq. (1),  $y(t)$  is replaced as given in Eq. (9) and  $D^\alpha y(t)$ ,  $y^2(t)$  and  $y \int_0^t y(s)ds$  by the following expressions.

$$(10) \quad D^\alpha y(t) \simeq \sum_{i=0}^{m-1} c_i D^\alpha FL_i^\alpha(t) = C^T D^\alpha FL^\alpha(t),$$

$$(11) \quad y^2(t) \simeq \left( \sum_{i=0}^{m-1} c_i FL_i^\alpha(t) \right)^2 = E^T FL^\alpha(t),$$

$$(12) \quad y(t) = \int_0^t y(x)dx \simeq \sum_{i=0}^{m-1} c_i FL_i^\alpha(t) \int_0^t \sum_{i=0}^{m-1} c_i FL_i^\alpha(x)dx \simeq H^T FL^\alpha(t),$$

where  $C^T$ ,  $E^T$ ,  $H^T$  are the unknown vectors in terms of  $c_i$ 's.

By substituting Eqs (9)-(12) in Eq. (1) we have  $(C^T D^\alpha - \alpha C^T + E^T + H^T) FL^\alpha(t) = 0$ .

The required  $m$  nonlinear equations are generated by

(i)  $y(0) = \beta$  (one equation)

(ii)  $M^T \int_0^1 FL^\alpha(t) FL_i^\alpha(t) t^{\alpha-1} dt = 0$ ,  $i = 0, 1, 2, \dots, m-2$ ,

where  $M^T = (C^T D^\alpha - \alpha C^T + E^T + H^T)$ . The solution of the generated nonlinear system leads to find the unknown vector  $C$  and the required approximate solution  $y(t)$ .

In case the exact solution to a problem is known, the accuracy and efficiency of the new method based on maximum absolute error  $e_m$  defined as

$$e_m = \max\{|y_{exact}(t) - y_m(t)|\}, \quad a \leq x \leq b, \quad 0 < t < \tau.$$

## 4. Theoretical analysis

In this section, we will discuss the convergence analysis and error estimation for the proposed technique.

**Theorem 4.1** (Convergence Theorem). *A continuous function  $y(t)$  with bounded second derivative, say  $M$ , can be expressed as  $\sum_{i=0}^{\infty} c_i FL_i^\alpha(t)$  and the truncated series given in Eq. (9) converges towards the exact solution  $y(t)$ .*

*Proof.* Let  $y(t) = \sum_{i=0}^{m-1} c_i FL_i^\alpha(t)$ , consider  $c_i = \alpha(2i+1) \int_0^1 y(t) FL_i^\alpha(t) t^{\alpha-1} dt$ . Here  $FL_i^\alpha(t) = P_i(2t^\alpha - 1)$ , where  $P_i$ 's are Legendre polynomials in  $x$  and

the fractional shifted Legendre polynomials are orthogonal with respect to the weight function  $w(t) = t^{\alpha-1}$  in the interval  $(0, 1]$ .

$$\begin{aligned} c_i &= \alpha(2i+1) \int_0^1 y(2t^\alpha - 1)P_i(2t^\alpha - 1)t^{\alpha-1} dt \\ &= \frac{1}{2} \int_{-1}^1 y(v)d[P_{i+1}(v) - P_{i-1}(v)] \\ &= \frac{1}{2} \int_{-1}^1 y''(v) \left[ \frac{P_{i+2}(v) - P_i(v)}{2i+3} - \frac{P_i(v) - P_{i-2}(v)}{2i-1} \right] dv. \end{aligned}$$

Consider

$$\begin{aligned} &\left| \frac{1}{2} \int_{-1}^1 y''(v) \left[ \frac{P_{i+2}(v) - P_i(v)}{2i+3} - \frac{P_i(v) - P_{i-2}(v)}{2i-1} \right] dv \right|^2 \\ &< \frac{M^2}{(2i-3)(2i-1)^2} < \frac{M}{(2i-3)^{1/2}(2i-1)}. \end{aligned}$$

Thus we get

$$|c_i| < \frac{M}{(2i-3)^{1/2}(2i-1)}.$$

Hence  $\sum_{i=0}^{\infty} c_i$  is absolutely convergent and thus the expansion of the function given in Eq. (9) converges uniformly [9].  $\square$

**Theorem 4.2** (Error bound). *Let  $y(t)$  be a function defined on  $(0, 1]$  with bounded second derivative say  $M$ . Then we have the following accuracy estimation:*

$$\epsilon \leq \sum_{i=m}^{\infty} \frac{M}{(2i-1)\sqrt{(2i-3)(2i+1)\alpha}},$$

$$\text{where } \epsilon = \left( \int_0^1 \left[ \sum_{i=0}^{\infty} c_i FL_i^\alpha(t) - \sum_{i=0}^{m-1} c_i FL_i^\alpha(t) \right]^2 w(t) dt \right)^{\frac{1}{2}}.$$

*Proof.*

$$\begin{aligned} \epsilon^2 &= \int_0^1 \left[ \sum_{i=0}^{\infty} c_i FL_i^\alpha(t) - \sum_{i=0}^{m-1} c_i FL_i^\alpha(t) \right]^2 w(t) dt. \\ &= \int_0^1 \sum_{i=m}^{\infty} c_i^2 [FL_i^\alpha(t)]^2 t^{\alpha-1} dt. \end{aligned}$$

Hence,

$$\epsilon \leq \sum_{i=m}^{\infty} \frac{M}{(2i-1)\sqrt{(2i-3)(2i+1)\alpha}}.$$

$\square$

### 5. Application of FSLPM for fractional population growth model

In order to show the effectiveness of the FSLPM, we implement FSLPM to the fractional population growth model with real coefficients for small  $k$  and large  $k$ ,  $k = c/ab$ .

Let  $y(t) = c_0 + c_1(2t^\alpha - 1) + c_2(6t^{2\alpha} - 6t^\alpha + 1) + c_3(20t^{3\alpha} - 30t^{2\alpha} + 12t^\alpha - 1) + c_4(70t^{4\alpha} - 140t^{3\alpha} + 90t^{2\alpha} - 20t^\alpha + 1)$  be the approximate solution of Eq. (1).

Case (i) Setting  $\alpha = 1$ ,  $k = 0.1$  (i.e.,  $a = b = c = 10$ ),  $\beta = 0.1$ ; the approximate solution of Eq. (1) is  $y(t) = 0.1 + 0.9t + 3.55t^2 + 6.31666667t^3 - 5.5375t^4 - 63.70916667t^5 - 156.0804167t^6 - 18.47323411t^7 + 1056.288569t^8$  which is the same as in [21].

Case (ii) Setting  $\alpha = 0.5$ ,  $k = 0.1$  (i.e.,  $a = b = c = 10$ ),  $\beta = 0.1$ ,  $y(t) = 0.1 + 1.01554t^{1/2} + 7.2t + 35.4964t^{3/2} + 90.422t^2 - 321.158t^{5/2} - 5346.32t^3 - 32307.8t^{7/2} - 82694.8t^4$ , is the same as in [21].

Case (iii) For  $k = 1$  ( $a = b = c = 1$ ) with  $\alpha = 1$ ,  $y(t) = 0.1 + 0.09t + 0.031t^2 + 0.0010666t^3 - 0.0032275t^4 - 0.001238666t^5 + 0.00068208t^6 + 0.000057765t^7 - 0.0000067618t^8$ , is the same as in [21].

The numerical values of  $y_{\max}$  for various values are given in Table 1 for different values of  $k$ . To study the performance of FSLPM, we compare the numerical solutions of Eq. (1) with other methods, reported in the literature, like ADM, Hermite functions collocation method (HFC), Second derivative multi-step method (SDMM) and Block-pulse function (BPF) method are presented in Table 2 in terms of  $y_{\max}$  with  $\alpha = 1$ . We also present the FSLPM-Pade approximants solutions for various  $\alpha$  and  $k$  values in Figure 1 and Figure 2 respectively. We notice from the Figure 1 and Figure 2 that the solution falls slowly when the value of  $\alpha$  and  $k$  increases respectively. From both figures and tables, it is shown that  $y_{\max}$  decreases as  $k$  increases. It confirms that the behaviour of a population depends on the fractional time parameter and  $k$  values. The overall obtained outcome indicates that FSLPM is a powerful and important tool for solving fractional population growth model.

TABLE 1. Comparison of methods and FSLPM with exact values for  $y_{\max}$

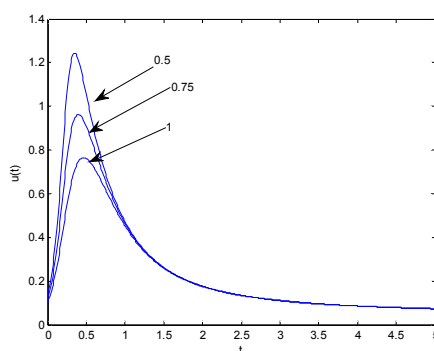
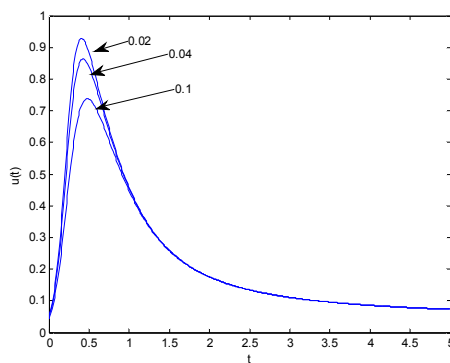
$k$	ADM [12]	HFC [16]	SDMM [15]	BPF[18]	FSLPM	Exact $y_{\max}$
0.02	0.9038380533	0.92342704	0.92342714	0.9234271721	0.9234271721	0.9234271721
0.04	0.8612401770	0.87371998	0.87381998	0.8737199832	0.8737199832	0.8737199832
0.1	0.7651130834	0.76974149	0.76974140	0.7697414907	0.7697414907	0.7697414907
0.2	0.6579123080	0.65905038	0.65905037	0.6590503815	0.6590503815	0.6590503815
0.5	0.4852823482	0.48519030	0.48519029	0.4851902914	0.4851902914	0.4851902914

### 6. Conclusion

This paper develops an effective FSLPM for solving fractional population growth model based on fractional order Legendre function. We have discussed

TABLE 2. A comparison of FSLPM with exact values of  $y_{\max}$  for different fractional order

$k$	$\alpha = 1/2$	$\alpha = 3/4$	$\alpha = 1$	Exact $y_{\max}$
0.02	2.92000000	1.49200000	0.922942037	0.92342717
0.04	2.01600000	1.24600000	0.873725344	0.87371998
0.1	1.23200000	0.94150000	0.765113089	0.76974149
0.2	0.81130000	0.71690000	0.659050432	0.65905038
0.5	0.44320000	0.45740000	0.485190290	0.48519030

FIGURE 1. Approximate solutions of Eq. (1) for different values of  $\alpha$ FIGURE 2. Approximate solutions of Eq. (1) for different values of  $k$ 

the fractional derivatives in the Caputo sense. Error analysis and convergence



analysis have been demonstrated for our proposed method. The solution obtained using the suggested method is in excellent agreement with the already existing ones and show that this approach can solve the problem effectively. Although we have only considered a model problem in this paper, the main idea and the techniques used in this work are also applicable to many other problems. The validity and applicability of the method have been verified by considering a fractional population growth model problem.

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