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ON θ-MODIFICATIONS OF GENERALIZED TOPOLOGIES VIA HEREDITARY CLASSES

Ahmad Al-Omari, Shyamapada Modak, and Takashi Noiri

Abstract. Let (X, μ) be a generalized topological space (GTS) and $\mathcal{H}$ be a hereditary class on X due to Császár [8]. In this paper, we define an operator $()^\circ : \mathcal{P}(X) \to \mathcal{P}(X)$. By setting $c^\circ(A) = A \cup A^\circ$ for every subset A of X, we define the family $\mu^\circ = \{M \subseteq X : X - M = c^\circ(X - M)\}$ and show that μ° is a GT on X such that $\mu(\theta) \subseteq \mu^\circ \subseteq \mu^*$, where μ^* is a GT in [8]. Moreover, we define and investigate μ°-codense and strongly μ°-codense hereditary classes.

1. Introduction

Let X be a nonempty set and let $\mathcal{P}(X)$ be the power set of X. Then $\mu \subseteq \mathcal{P}(X)$ is called a generalized topology (briefly GT) [3] on X if $\emptyset \in \mu$ and $G_i \in \mu$ for $i \in I \neq \emptyset$ implies $G = \cup_{i \in I} G_i \in \mu$. We call the pair (X, μ) a generalized topological space (briefly GTS) on X.

For a GTS (X, μ), the elements of μ are called μ-open sets and the complements of μ-open sets are called μ-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all μ-closed sets containing A, i.e., the smallest μ-closed set containing A; and by $i_\mu(A)$ the union of all μ-open sets contained in A, i.e., the largest μ-open set contained in A.

We recall some notions defined in [8]. Let μ be a GT on nonempty set X and $\mathcal{P}(X)$ the power set of X. Let us define the collection $\mu(\theta) \subseteq \mathcal{P}(X)$ as follows: $A \in \mu(\theta)$ if and only if for each $x \in A$, there exists $U \in \mu$ such that $x \in U \subseteq c_\mu(U) \subseteq A$. Then $\mu(\theta)$ is also a GT included in μ. The elements of $\mu(\theta)$ are called $\mu(\theta)$-open sets and the complement of a $\mu(\theta)$-open set is said to be $\mu(\theta)$-closed. The family of all $\mu(\theta)$-open subsets of a GTS (X, μ) is denoted by $\mu(\theta)$.

Let (X, μ) be a GTS and $A \subseteq X$. Then the notions are defined as follows:

(1) [4] $\gamma_\theta(A) = \{x \in X : c_\mu(U) \cap A \neq \emptyset$ for all μ-open set U containing $x\}$.
(2) [10] $c_\theta(A) = \cap\{F \subseteq X : A \subseteq F,\ F$ is $\mu(\theta)$-closed in $X\}$.

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(3) [10] $i_\theta(A) = \cup\{V \subseteq X : V \subseteq A,\ V$ is $\mu(\theta)$-open in $X\}$.
(4) [10] $\iota_\theta(A) = \{x \in X : c_\mu(U) \subseteq A$ for some μ-open set U containing $x\}$.

Theorem 1.1 ([8]). *Let (X, μ) be a GTS and $A \subseteq X$. Then the following hold*:

(1) $i_\theta(A) \subseteq \iota_\theta(A) \subseteq i_\mu(A) \subseteq A \subseteq c_\mu(A) \subseteq \gamma_\theta(A) \subseteq c_\theta(A)$.
(2) *If A is μ-open, then $c_\mu(A) = \gamma_\theta(A)$.*

Theorem 1.2 ([10]). *Let (X, μ) be a GTS and $A \subseteq X$. Then the following hold*:

(1) $\gamma_\theta(X - A) = X - \iota_\theta(A)$.
(2) $\iota_\theta(X - A) = X - \gamma_\theta(A)$.

The idea of hereditary classes was introduced by Császár [6]. A nonempty family $\mathcal{H} \subseteq \mathcal{P}(X)$ is called a hereditary class if it satisfies the following condition: $A \subseteq B$, $B \in \mathcal{H}$ implies that $A \in \mathcal{H}$. For more details see [1, 2, 9].

2. Properties of the operation A°

In [8], Császár introduced an operator $()^* : \mathcal{P}(X) \to \mathcal{P}(X)$ by using GT μ and a hereditary class $\mathcal{H}$ by $x \in A^* \subseteq X$ if and only if $M \cap A \notin \mathcal{H}$ for every $M \in \mu(x)$. If $x \notin \mathcal{M}_\mu$, then by definition $x \in A^*$ where $\mu(x) = \{M \in \mu : x \in M\}$ and $\mathcal{M}_\mu = \cup\{M : M \in \mu\}$.

And, $x \notin A^*$ if and only if there exists $x \in M \in \mu$ such that $M \cap A \in \mathcal{H}$.

Definition 2.1. Let μ be a *GT* in X and $\mathcal{H}$ a hereditary class on X. For a subset A of X, we define the set $A^\circ = \{x \in X : c_\mu(M) \cap A \notin \mathcal{H}$ for every $M \in \mu(x)\}$ and $x \notin \mathcal{M}_\mu$ then by definition $x \in A^\circ$.

According to definition, $x \notin A^\circ$ if and only if there exists $x \in M \in \mu$ such that $c_\mu(M) \cap A \in \mathcal{H}$.

Lemma 2.2. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then $A^* \subseteq A^\circ$ for every subset A of X.*

Proof. Let $x \in A^*$. Then, $M \cap A \notin \mathcal{H}$ for every μ-open set M containing x. Since $M \cap A \subseteq c_\mu(M) \cap A$, we have $c_\mu(M) \cap A \notin \mathcal{H}$ and hence $x \in A^\circ$. □

Example 2.3. Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a, c\}, \{d\}, \{a, c, d\}\}$, and $\mathcal{H} = \{\phi, \{c\}\}$. Let $A = \{b, c, d\}$. Then $A^\circ = \{a, b, c, d\}$ and $A^* = \{b, d\}$.

Lemma 2.4. *Let μ be a GT in X, $\mathcal{H}$ and $\mathcal{J}$ be two hereditary classes on X. Let A and B be subsets of X. Then the following properties hold*:

(1) *If $A \subseteq B$, then $A^\circ \subseteq B^\circ$.*
(2) *If $\mathcal{H} \subseteq \mathcal{J}$, then $A^\circ(\mathcal{H}) \supseteq A^\circ(\mathcal{J})$.*
(3) *$A^\circ = c_\mu(A^\circ) \subseteq \gamma_\theta(A)$ and A° is μ-closed.*
(4) *If $A \subseteq A^\circ$ and A° is μ-open, then $A^\circ = \gamma_\theta(A)$.*
(5) *If $A \subseteq A^\circ$ and A is μ-open, then $A^\circ = \gamma_\theta(A) = c_\mu(A)$.*

Proof. (1) Suppose that $x \notin B^\circ$. Then there exists $M \in \mu(x)$ such that $c_\mu(M) \cap B \in \mathcal{H}$. Since $c_\mu(M) \cap A \subseteq c_\mu(M) \cap B$, $c_\mu(M) \cap A \in \mathcal{H}$. Hence $x \notin A^\circ$. Thus $X \setminus B^\circ \subseteq X \setminus A^\circ$ or $A^\circ \subseteq B^\circ$.

(2) Suppose that $x \notin A^\circ(\mathcal{H})$. There exists $M \in \mu(x)$ such that $c_\mu(M) \cap A \in \mathcal{H}$. Since $\mathcal{H} \subseteq \mathcal{J}$, $c_\mu(M) \cap A \in \mathcal{J}$ and $x \notin A^\circ(\mathcal{J})$. Therefore, $A^\circ(\mathcal{J}) \subseteq A^\circ(\mathcal{H})$.

(3) We have $A^\circ \subseteq c_\mu(A^\circ)$ in general. Let $x \in c_\mu(A^\circ)$. Then $A^\circ \cap M \neq \emptyset$ for every $M \in \mu(x)$. Therefore, there exists some $y \in A^\circ \cap M$ and $M \in \mu(y)$. Since $y \in A^\circ$, $A \cap c_\mu(M) \notin \mathcal{H}$ and hence $x \in A^\circ$. Hence we have $c_\mu(A^\circ) \subseteq A^\circ$ and hence $A^\circ = c_\mu(A^\circ)$, and A° is μ-closed. Again, let $x \in c_\mu(A^\circ) = A^\circ$, then $A \cap c_\mu(M) \notin \mathcal{H}$ for every $M \in \mu(x)$. This implies $A \cap c_\mu(M) \neq \emptyset$ for every $M \in \mu(x)$. Therefore, $x \in \gamma_\theta(A)$. This shows that $A^\circ = c_\mu(A^\circ) \subseteq \gamma_\theta(A)$.

(4) For any subset A of X, by (3) we have $A^\circ = c_\mu(A^\circ) \subseteq \gamma_\theta(A)$. Since $A \subseteq A^\circ$, $\gamma_\theta(A) \subseteq \gamma_\theta(A^\circ) = c_\mu(A^\circ) = A^\circ \subseteq \gamma_\theta(A)$ and hence $A^\circ = \gamma_\theta(A)$.

(5) For any subset A of X, by (3) we have $A^\circ = c_\mu(A^\circ) \subseteq \gamma_\theta(A)$. Since $A \subseteq A^\circ$, $\gamma_\theta(A) = c_\mu(A) \subseteq c_\mu(A^\circ) = A^\circ$ and hence $A^\circ = \gamma_\theta(A) = c_\mu(A)$. □

According to [7], a GT μ is called a quasi-topology (briefly QT) if $M, M' \in \mu$ implies $M \cap M' \in \mu$.

Lemma 2.5. *Let μ be a quasi-topology in X and $\mathcal{H}$ a hereditary class on X. If M is $\mu(\theta)$-open, then $M \cap A^\circ = M \cap (M \cap A)^\circ \subseteq (M \cap A)^\circ$ for any subset A of X.*

Proof. Suppose that M be $\mu(\theta)$-open and $x \in M \cap A^\circ$. Then $x \in M$ and $x \in A^\circ$. Since $M \in \mu(\theta)$, then there exists $W \in \mu$ such that $x \in W \subseteq c_\mu(W) \subseteq M$. Let V be any μ-open set containing x. Then $V \cap W \in \mu(x)$ and $c_\mu(V \cap W) \cap A \notin \mathcal{H}$ and hence $c_\mu(V) \cap (M \cap A) \notin \mathcal{H}$. This shows that $x \in (M \cap A)^\circ$ and hence we obtain $M \cap A^\circ \subseteq (M \cap A)^\circ$. Moreover, $M \cap A^\circ \subseteq M \cap (M \cap A)^\circ$ and by Lemma 2.4 $(M \cap A)^\circ \subseteq A^\circ$ and $M \cap (M \cap A)^\circ \subseteq M \cap A^\circ$. Therefore, $M \cap A^\circ = M \cap (M \cap A)^\circ$. □

Proposition 2.6. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. If $M \in \mu$, $c_\mu(M) \cap A \in \mathcal{H}$ imply $M \cap A^\circ = \emptyset$. Hence $A^\circ = X - \mathcal{M}_\mu$ if $A \in \mathcal{H}$.*

Proof. Suppose $x \in M \cap A^\circ$, $M \in \mu(x)$ and $x \in A^\circ$ would imply $c_\mu(M) \cap A \notin \mathcal{H}$. Now $A \in \mathcal{H}$ imply $c_\mu(M) \cap A \in \mathcal{H}$ for every $M \in \mu$ and $x \notin A^\circ$ when $x \in \mathcal{M}_\mu$, thus $A^\circ \subseteq X - \mathcal{M}_\mu$ on the other hand we know $X - \mathcal{M}_\mu \subseteq A^\circ$. □

Proposition 2.7. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. If $A \in \mathcal{H}$, then $i_\mu(A^\circ) = \emptyset$ and hence $c_\mu(X - A^\circ) = X$.*

Proof. Since $A \in \mathcal{H}$, by Proposition 2.6, $A^\circ = X - \mathcal{M}_\mu$. Since $\mathcal{M}_\mu$ is the largest μ-open subset of X, it follows that $i_\mu(A^\circ) = \emptyset$ and hence $c_\mu(X - A^\circ) = X$. □

Proposition 2.8. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then $A \subseteq X$ implies that $(A \cup A^\circ)^* \subseteq A^\circ$.*

Proof. Let $x \notin A^\circ$. Then there exists $M \in \mu$ such that $x \in M$ and $c_\mu(M) \cap A \in \mathcal{H}$. By Proposition 2.6, $M \cap A^\circ = \emptyset$. Hence $M \cap (A \cup A^\circ) = M \cap A \in \mathcal{H}$. Therefore, $x \notin (A \cup A^\circ)^*$. □

Definition 2.9. Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. We say the μ is μ°-compatible with a hereditary class $\mathcal{H}$, denoted $\mu \sim^\circ \mathcal{H}$, if the following holds for every $A \subseteq X$: if for every $x \in A$ there exists $M \in \mu(x)$ such that $c_\mu(M) \cap A \in \mathcal{H}$, then $A \in \mathcal{H}$.

Theorem 2.10. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then the implications* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *hold.*

(1) $\mu \sim^\circ \mathcal{H}$;
(2) *If a subset A of X has a cover of μ-open sets each of whose c_μ-closure intersection with A is in $\mathcal{H}$, then $A \in \mathcal{H}$*;
(3) *For every $A \subseteq X$, $A \cap A^\circ = \emptyset$ implies that $A \in \mathcal{H}$*;
(4) *For every $A \subseteq X$, $A - A^\circ \in \mathcal{H}$.*

Proof. (1) $\Rightarrow$ (2): The proof is obvious.

(2) $\Rightarrow$ (3): Let $A \subseteq X$ and $x \in A$. Then $x \notin A^\circ$ and there exists $V_x \in \mu(x)$ such that $c_\mu(V_x) \cap A \in \mathcal{H}$. Therefore, we have $A \subseteq \cup\{V_x : x \in A\}$ and $V_x \in \mu(x)$ and by (2) $A \in \mathcal{H}$.

(3) $\Rightarrow$ (4): For any $A \subseteq X$, $A - A^\circ \subseteq A$ and $(A - A^\circ) \cap (A - A^\circ)^\circ \subseteq (A - A^\circ) \cap A^\circ = \emptyset$. By (3), $A - A^\circ \in \mathcal{H}$. □

Theorem 2.11. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Let μ be μ°-compatible with a hereditary class $\mathcal{H}$. If $A \cap A^\circ = \emptyset$ for $A \subseteq X$, then*

(1) $A^\circ = X - \mathcal{M}_\mu$,
(2) $(A - A^\circ)^\circ = X - \mathcal{M}_\mu$.

Proof. We show that (1) holds if μ is μ°-compatible with $\mathcal{H}$. Let A be any subset of X and $A \cap A^\circ = \emptyset$. By Theorem 2.10, $A \in \mathcal{H}$ and by Proposition 2.6 $A^\circ = X - \mathcal{M}_\mu$.

(2) Let $B = A - A^\circ$, then

$$\begin{aligned} B \cap B^\circ &= (A - A^\circ) \cap (A - A^\circ)^\circ \\ &= (A \cap (X - A^\circ)) \cap (A \cap (X - A^\circ))^\circ \\ &\subseteq [A \cap (X - A^\circ)] \cap [A^\circ \cap ((X - A^\circ)^\circ)] = \emptyset. \end{aligned}$$

By (1), we have $B^\circ = X - \mathcal{M}_\mu$. Hence $(A - A^\circ)^\circ = X - \mathcal{M}_\mu$. □

3. Properties of the operator γ_μ°

Definition 3.1. Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. An operator $\gamma_\mu^\circ : \mathcal{P}(X) \to \mu$ is defined as follows: for every $A \in X$, $\gamma_\mu^\circ(A) = \{x \in X :$ there exists $M \in \mu(x)$ such that $c_\mu(M) - A \in \mathcal{H}\}$.

Theorem 3.2. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then $\gamma_\mu^\circ(A) = X - (X - A)^\circ$.*

Proof. Suppose $x \in X - (X - A)^\circ$. Then $x \notin (X - A)^\circ$ and so there exists $M \in \mu(x)$ such that $c_\mu(M) \cap (X - A) \in \mathcal{H}$ which implies that $c_\mu(M) - A \in \mathcal{H}$. Therefore, $X - (X - A)^\circ \subseteq \{x \in X : \text{there exists } M \in \mu(x) \text{ such that } c_\mu(M) - A \in \mathcal{H}\} = \gamma_\mu^\circ(A)$. Conversely, assume that $y \in \gamma_\mu^\circ(A)$. Then there exists $M \in \mu(y)$ such that $c_\mu(M) - A \in \mathcal{H}$. Since $c_\mu(M) - A \in \mathcal{H}$, $c_\mu(M) \cap (X - A) \in \mathcal{H}$ which implies that $y \notin (X - A)^\circ$. Therefore, $y \in X - (X - A)^\circ$. Thus, $\gamma_\mu^\circ(A) = X - (X - A)^\circ$. □

Let μ be a GT in X and $\mathcal{H}$ a hereditary class. Császár [6] introduced the operator $c^* : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: $c^*(A) = A \cup A^*$ for $A \subseteq X$. He showed that there exists a GT μ^* such that $M \in \mu^*$ if and only if $X - M = c^*(X - M)$ and $\mu(\theta) \subseteq \mu \subseteq \mu^*$. Clearly $c^*(A)$ is the intersection of all μ^*-closed supersets of A. Let $i^*(A)$ denote the union of all μ^*-open sets contained in A.

Analogously the operator $c^\circ : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined as follows: $c^\circ(A) = A \cup A^\circ$ for $A \subseteq X$. Then there exists a GT μ° such that $M \in \mu^\circ$ if and only if $X - M = c^\circ(X - M)$ and $\mu(\theta) \subseteq \mu^\circ \subseteq \mu^*$. Clearly $c^\circ(A)$ is the intersection of all μ°-closed supersets of A. Let $i^\circ(A)$ denote the union of all μ°-open sets contained in A.

Theorem 3.3. *Let μ be a GT in X, $\mathcal{H}$ a hereditary class on X and $\mu^\circ = \{M \subset X : M = i^\circ(M)\} = \{M \subset X : X - M = c^\circ(X - M)\}$. Then*

(1) *μ° is a GT containing $\mu(\theta)$,*
(2) *$\mu^\circ \subseteq \mu^*$,*
(3) *$\mu^\circ = \{A \subseteq X : A \subseteq \gamma_\mu^\circ(A)\}$.*

Proof. (1) If $M \in \mu(\theta)$, $M = i^\circ(M)$ and hence $M \in \mu^\circ$. Therefore, $\mu(\theta) \subseteq \mu^\circ$. Let $M_\alpha \in \mu^\circ$ for each $\alpha \in \Delta$. Then $M_\alpha = i^\circ(M_\alpha) \subseteq i^\circ(\cup M_\alpha)$ for each $\alpha \in \Delta$. Hence $\cup M_\alpha \subseteq i^\circ(\cup M_\alpha)$ and $\cup M_\alpha = i^\circ(\cup M_\alpha)$. Therefore, $\cup M_\alpha \in \mu^\circ$. And μ° is a GT.

(2) Let $M \in \mu^\circ$. Then, by Lemma 2.2, $X - M = c^\circ(X - M) \supseteq c^*(X - M) \supseteq X - M$. Therefore, $X - M = c^*(X - M)$ and $M \in \mu^*$. Hence $\mu^\circ \subseteq \mu^*$.

(3) Let $A \subseteq X$ and $A \subseteq \gamma_\mu^\circ(A)$. By Theorem 3.2, $A \subseteq X - (X - A)^\circ$ and $X - A \supseteq (X - A)^\circ$. Therefore, $X - A = c^\circ(X - A)$ and hence $A \in \mu^\circ$. The converse is obvious. □

Now we give an example of a set A which is not $\mu(\theta)$-open but satisfies $A \subseteq \gamma_\mu^\circ(A)$ i.e μ°-open set.

Example 3.4. Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a, c\}, \{d\}, \{a, c, d\}\}$, and $\mathcal{H} = \{\phi, \{b\}, \{c\}, \{b, c\}, \{a\}\}$ be a hereditary class on X. Let $A = \{a\}$. Then $\gamma_\mu^\circ(\{a\}) = X - (X - \{a\})^\circ = X - (\{b, c, d\})^\circ = X - \{b, d\} = \{a, c\}$. Therefore, $A \subseteq \gamma_\mu^\circ(A)$ and hence A is μ°-open set, but A is not μ-open.

Example 3.5. Let $X = \{a, b, c\}$ with $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}\}$ be a hereditary class on X. We observe that $\{a\}$ is μ-open but it is not μ° open, since $\gamma_\mu^\circ(\{a\}) = X - (\{b, c\})^\circ = X - X = \emptyset$.

By Theorem 3.3 the following relation holds:

$$\begin{array}{ccc} \mu(\theta)\text{-open} & \longrightarrow & \mu\text{-open} \\ \downarrow & & \downarrow \\ \mu^\circ\text{-open} & \longrightarrow & \mu^*\text{-open} \end{array}$$

Lemma 3.6. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. F is μ°-closed if and only if $F^\circ \subseteq F$.*

Proof. F is μ°-closed if and only if $F = c^\circ(F) = F \cup F^\circ$ if and only if $F^\circ \subseteq F$. □

Corollary 3.7. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then*

(1) *$U \subseteq \gamma_\mu^\circ(U)$ for every $U \in \mu^\circ$.*
(2) *$U \subseteq \gamma_\mu^\circ(U)$ for every $U \in \mu(\theta)$.*

Proof. (1) If $U \in \mu^\circ$, then $X - U$ is μ°-closed. Therefore, $(X - U)^\circ \subseteq X - U$ which implies that $X - (X - U) \subseteq X - (X - U)^\circ$ and so that $U \subseteq \gamma_\mu^\circ(U)$.

(2) We know that $\gamma_\mu^\circ(U) = X - (X - U)^\circ$. Now $(X - U)^\circ \subseteq \gamma_\theta(X - U) = X - U$, since $X - U$ is $\mu(\theta)$-closed. Therefore, $U = X-(X-U) \subseteq X-(X-U)^\circ = \gamma_\mu^\circ(U)$. □

Several basic facts concerning the behavior of the operator γ_μ° are included in the following theorem.

Theorem 3.8. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then the following properties hold*:

(1) *If $A \subseteq X$, then $\gamma_\mu^\circ(A)$ is μ-open.*
(2) *If $A \subseteq B$, then $\gamma_\mu^\circ(A) \subseteq \gamma_\mu^\circ(B)$.*
(3) *If $A, B \subseteq X$, then $\gamma_\mu^\circ(A \cap B) \subseteq \gamma_\mu^\circ(A) \cap \gamma_\mu^\circ(B)$.*
(4) *If $A \subseteq X$, then $\gamma_\mu^\circ(A) = \gamma_\mu^\circ(\gamma_\mu^\circ(A))$ if and only if $(X - A)^\circ = ((X - A)^\circ)^\circ$.*
(5) *$\gamma_\mu^\circ(X) = X$ (if X is strong) or $\mathcal{M}_\mu$ (otherwise).*
(6) *For $X - K \in \mathcal{H}$, $\gamma_\mu^\circ(K) = \mathcal{M}_\mu$.*
(7) *$\gamma_\mu^\circ(\phi) = \mathcal{M}_\mu - X^\circ$.*

Proof. (1) This follows from Lemma 2.4(3) and Theorem 3.2.

(2) This follows from Lemma 2.4(1) and Theorem 3.2.

(3) This follows from (2).

(4) This follows from the facts:

(1) $\gamma_\mu^\circ(A) = X - (X - A)^\circ$.
(2) $\gamma_\mu^\circ(\gamma_\mu^\circ(A)) = X - [X - (X - (X - A)^\circ)]^\circ = X - ((X - A)^\circ)^\circ$.

(5) Since $\phi \in \mathcal{H}$, by Proposition 2.6 we have $\phi^\circ = X - \mathcal{M}_\mu$. If μ is strong, then $\mathcal{M}_\mu = X$, and $\gamma_\mu^\circ(X) = X - \phi^\circ = X - (X - \mathcal{M}_\mu) = X$. Otherwise $\gamma_\mu^\circ(X) = X - \phi^\circ = X - (X - \mathcal{M}_\mu) = \mathcal{M}_\mu$.

(6) For $X - K \in \mathcal{H}$, by Proposition 2.6 $\gamma_\mu^\circ(K) = X - (X - K)^\circ = X - (X - \mathcal{M}_\mu) = \mathcal{M}_\mu$.

(7) By Theorem 3.2 $\gamma_\mu^\circ(\phi) = X - X^\circ = (\mathcal{M}_\mu \cup (X - \mathcal{M}_\mu)) - X^\circ = (\mathcal{M}_\mu - X^\circ) \cup ((X - \mathcal{M}_\mu) - X^\circ) = \mathcal{M}_\mu - X^\circ$, since X° is μ-closed by Lemma 2.4(3) and $X - \mathcal{M}_\mu$ is the smallest μ-closed set contained in every μ-closed set. □

Theorem 3.9. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. If $\sigma = \{A \subseteq X : A \subseteq \gamma_\mu^\circ(A)\}$. Then σ is a generalized topological space and $\sigma = \mu^\circ$.*

Proof. Let $A \in \sigma$. Then $A \subseteq \gamma_\mu^\circ(A) = X - (X - A)^\circ$ which implies that $(X - A)^\circ \subseteq X - A$. Therefore, $X - A$ is μ°-closed and hence A is μ°-open. Therefore, $\sigma \subseteq \mu^\circ$. Conversely, Let $A \in \mu^\circ$, then by Corollary 3.7, $A \subseteq \gamma_\mu^\circ(A)$ and $A \in \sigma$. Therefore, $\mu^\circ \subseteq \sigma$. Hence $\sigma = \mu^\circ$. Since μ° is a generalized topology, it follows that σ is a generalized topology. □

Theorem 3.10. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then the following properties hold*:

(1) $\gamma_\mu^\circ(A) = \cup\{M \in \mu : c_\mu(M) - A \in \mathcal{H}\}$.

(2) $\gamma_\mu^\circ(A) = \cup\{M \in \mu : (c_\mu(M) - A) \cup (A - c_\mu(M)) \in \mathcal{H}\}$ *if A is μ-clopen.*

Proof. (1) For $x \in X$, $x \in \gamma_\mu^\circ(A)$ if and only if $x \in X - (X - A)^\circ$ if and only if $x \notin (X - A)^\circ$ if and only if there exists $M \in \mu$ containing x such that $c_\mu(M) \cap (X - A) = c_\mu(M) - A \in \mathcal{H}$.

(2) Let $\rho = \cup\{M \in \mu : (c_\mu(M) - A) \cup (A - c_\mu(M)) \in \mathcal{H}\}$. Since $\mathcal{H}$ is heredity, it is obvious that $\rho \subseteq \cup\{M \in \mu : c_\mu(M) - A \in \mathcal{H}\} = \gamma_\mu^\circ(A)$ for every $A \subseteq X$. Conversely, assume A is μ-clopen and $x \in \gamma_\mu^\circ(A)$. Then there exists $U \in \mu$ such that $x \in U$ and $c_\mu(U) - A \in \mathcal{H}$. If $M = U \cup A$, then $M \in \mu$, $x \in M$ and $c_\mu(M) = c_\mu(U) \cup A$. Now $(c_\mu(M) - A) \cup (A - c_\mu(M)) = (c_\mu(U) - A) \cup \emptyset = c_\mu(U) - A \in \mathcal{H}$ implies that $(c_\mu(M) - A) \cup (A - c_\mu(M)) \in \mathcal{H}$ and hence $x \in \rho$. Therefore, $\gamma_\mu^\circ(A) \subseteq \rho$. Hence $\gamma_\mu^\circ(A) = \rho$. □

Theorem 3.11. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then for $A \subseteq X$, $\iota_\theta(A) \subseteq \gamma_\mu^\circ(A)$.*

Proof. Let $x \in \iota_\theta(A)$, then there exists $M \in \mu$ containing x such that $c_\mu(M) \subseteq A$. This implies that $c_\mu(M) - A = \phi \in \mathcal{H}$ and hence by Definition 3.1 $x \in \gamma_\mu^\circ(A)$. □

The reverse inclusion of the above theorem may not hold as shown in the next example.

Example 3.12. Let $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{a, c\}, \{d\}, \{a, c, d\}\}$, and $\mathcal{H} = \{\phi, \{b\}, \{c\}, \{b, c\}, \{a\}\}$ be a hereditary class on X. Let $A = \{a\}$. Then $\gamma_\mu^\circ(\{a\}) = X - (X - \{a\})^\circ = X - (\{b, c, d\})^\circ = X - \{b, d\} = \{a, c\}$ and $\iota_\theta(A) = \emptyset$, hence $\gamma_\mu^\circ(A) \nsubseteq \iota_\theta(A)$.

Lemma 3.13. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. For $A \subseteq X$,*

(1) $i^\circ(A) = X - c^\circ(X - A)$.

(2) $c^\circ(A) = X - i^\circ(X - A)$.

Proposition 3.14. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. For $A \subseteq X$,*

(1) $\iota_\theta(A) \subseteq i^\circ(A) \subseteq \gamma_\mu^\circ(A)$.

(2) $A^\circ \subseteq c^\circ(A) \subseteq \gamma_\theta(A)$.

Proof. Since the GT μ° is finer than $\mu(\theta)$, obviously $\iota_\theta(A) \subseteq i^\circ(A)$ and $c^\circ(A) \subseteq \gamma_\theta(A)$. And from $c^\circ(A) = A \cup A^\circ$, obviously $A^\circ \subseteq c^\circ(A)$. Let $x \in i^\circ(A)$, there exists a μ°-open set G containing x such that $G \subseteq A$. Hence there exists $c_\mu(M) - H \subseteq G$ such that $M \in \mu$ and $H \in \mathcal{H}$. Since $c_\mu(M) - A \subseteq H$, then $c_\mu(M) - A \in \mathcal{H}$. This implies $x \notin (X - A)^\circ$ and hence $x \in \gamma_\mu^\circ(A)$. □

Theorem 3.15. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then $\mu \sim^\circ \mathcal{H}$ if and only if $\gamma_\mu^\circ(A) - A \in \mathcal{H}$ for every $A \subseteq X$.*

Proof. Necessity. Assume $\mu \sim^\circ \mathcal{H}$ and let $A \subseteq X$. Observe that $x \in \gamma_\mu^\circ(A) - A$ if and only if $x \notin A$ and $x \notin (X - A)^\circ$ if and only if $x \notin A$ and there exists $U_x \in \mu(x)$ such that $c_\mu(U_x) - A \in \mathcal{H}$ if and only if there exists $U_x \in \mu(x)$ such that $x \in c_\mu(U_x) - A \in \mathcal{H}$. Now, for each $x \in \gamma_\mu^\circ(A) - A$ and $U_x \in \mu(x)$, $c_\mu(U_x) \cap (\gamma_\mu^\circ(A) - A) \in \mathcal{H}$ by heredity and hence $\gamma_\mu^\circ(A) - A \in \mathcal{H}$ by assumption that $\mu \sim^\circ \mathcal{H}$.

Sufficiency. Let $A \subseteq X$ and assume that for each $x \in A$ there exists $U_x \in \mu(x)$ such that $c_\mu(U_x) \cap A \in \mathcal{H}$. Observe that $\gamma_\mu^\circ(X - A) - (X - A) = A - A^\circ = \{x : \text{there exists } U_x \in \mu(x) \text{ such that } x \in c_\mu(U_x) \cap A \in \mathcal{H}\}$. Thus we have $A \subseteq \gamma_\mu^\circ(X - A) - (X - A) \in \mathcal{H}$ and hence $A \in \mathcal{H}$ by heredity of $\mathcal{H}$. □

Proposition 3.16. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X with $\mu \sim^\circ \mathcal{H}$, $A \subseteq X$. If N is a nonempty μ-open subset of $A^\circ \cap \gamma_\mu^\circ(A)$, then $N - A \in \mathcal{H}$ and $c_\mu(N) \cap A \notin \mathcal{H}$.*

Proof. If $N \in \mu - \{\emptyset\}$ and $N \subseteq A^\circ \cap \gamma_\mu^\circ(A)$, then $N - A \subseteq \gamma_\mu^\circ(A) - A \in \mathcal{H}$ by Theorem 3.15 and hence $N - A \in \mathcal{H}$ by heredity. Since $N \in \mu - \{\emptyset\}$ and $N \subseteq A^\circ$, we have $c_\mu(N) \cap A \notin \mathcal{H}$ by the definition of A°. □

4. μ°-codense

Proposition 4.1. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. $X = X^\circ$ if and only if $c_\mu(\mu) \cap \mathcal{H} = \emptyset$, where $c_\mu(\mu) = \{c_\mu(V) : V \in \mu\}$.*

Proof. Assume $X = X^\circ$. Then $M \in \mu$, $M \neq \emptyset$ would imply the existence of $x \in M$ and $x \in X^\circ$ would furnish $c_\mu(M) \cap X = c_\mu(M) \notin \mathcal{H}$ so that $c_\mu(\mu) \cap \mathcal{H} = \emptyset$. Conversely, $c_\mu(\mu) \cap \mathcal{H} = \emptyset$ implies $c_\mu(M) = c_\mu(M) \cap X \notin \mathcal{H}$ whenever $x \in M \in \mu$ so that $x \in X^\circ$ for $x \in X$. Hence $X = X^\circ$. □

Theorem 4.2. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then the implications* (1) ⇒ (2) *and* (3) ⇒ (4) ⇒ (1) *hold. If μ is a quasi-topology in X, then the following properties are equivalent*:

(1) $c_\mu(\mu) \cap \mathcal{H} = \emptyset$;
(2) *If* $H \in \mathcal{H}$, *then* $\iota_\theta(H) = \emptyset$;
(3) *For every* μ-*clopen* G, $G \subseteq G^\circ$;
(4) $X = X^\circ$.

Proof. (1) $\Rightarrow$ (2): Let $c_\mu(\mu) \cap \mathcal{H} = \emptyset$ and $H \in \mathcal{H}$. Suppose that $x \in \iota_\theta(H)$. Then there exists $M \in \mu$ such that $x \in M \subseteq c_\mu(M) \subseteq H$. Since $H \in \mathcal{H}$ and hence $\emptyset \neq \{x\} \subseteq c_\mu(M) \in c_\mu(\mu) \cap \mathcal{H}$. This is contrary that $c_\mu(\mu) \cap \mathcal{H} = \emptyset$. Therefore, $\iota_\theta(H) = \emptyset$.

(2) $\Rightarrow$ (3): Let G be μ-clopen and $x \in G$. Assume $x \notin G^\circ$ then there exists $M_x \in \mu(x)$ such that $G \cap c_\mu(M_x) \in \mathcal{H}$ and hence $G \cap M_x \in \mathcal{H}$. By (2), $x \in G \cap M_x = i_\mu(G \cap M_x) \subseteq i_\mu(G \cap c_\mu(M_x)) = \iota_\theta(G \cap c_\mu(M_x)) = \emptyset$. This is a contradiction. Hence $x \in G^\circ$ and $G \subseteq G^\circ$.

(3) $\Rightarrow$ (4): Since X is μ-clopen, then $X = X^\circ$.

(4) $\Leftrightarrow$ (1): This is obvious by Proposition 4.1. □

Proposition 4.3. *Let* μ *be a* GT *in* X *and* $\mathcal{H}$ *a hereditary class on* X. $M \in \mu$ *implies* $M \subseteq M^\circ$ *if and only if* $N, M \in \mu$, $c_\mu(N) \cap M \in \mathcal{H}$ *implies* $N \cap M = \emptyset$.

Proof. Assume $M \subseteq M^\circ$ whenever $M \in \mu$. If $x \in M \cap N$ and $M, N \in \mu$, then $x \in M^\circ$, hence $c_\mu(N) \cap M \notin \mathcal{H}$. Consequently $M, N \in \mu$ and $c_\mu(N) \cap M \in \mathcal{H}$ implies $M \cap N = \emptyset$. Conversely, if the latter statement is true and $x \in M \in \mu$, then $x \in N \in \mu$ implies $M \cap N \neq \emptyset$. Hence $c_\mu(N) \cap M \notin \mathcal{H}$, so that $x \in M^\circ$. Therefore, $M \subseteq M^\circ$ whenever $M \in \mu$. □

Definition 4.4. Let μ be a GT in X. A hereditary class $\mathcal{H}$ on X is said to be
(1) μ°-codense if $c_\mu(\mu) \cap \mathcal{H} = \emptyset$,
(2) strongly μ°-codense if $N, M \in \mu$, $c_\mu(N) \cap M \in \mathcal{H}$ implies $N \cap M = \emptyset$.

Lemma 4.5. *Let* μ *be a* GT *in* X *and* $\mathcal{H}$ *a hereditary class on* X. $\gamma^\circ_\mu(\emptyset) = \emptyset$ *if and only if a hereditary class* $\mathcal{H}$ *is* μ°*-codense.*

Proof. Since $\gamma^\circ_\mu(\phi) = X - X^\circ$, $\gamma^\circ_\mu(\emptyset) = \emptyset$ if and only if $X = X^\circ$ and hence by Theorem 4.2(4), $\gamma^\circ_\mu(\emptyset) = \emptyset$ if and only if a hereditary class $\mathcal{H}$ is μ°-codense. □

Proposition 4.6. *Let* μ *be a* GT *in* X *and* $\mathcal{H}$ *a hereditary class on* X. *Then the following are equivalent.*
(1) $\mathcal{H}$ *is* μ°*-codense.*
(2) $\mathcal{M}^\circ_\mu = X$.
(3) $\gamma^\circ_\mu(X - \mathcal{M}_\mu) = \emptyset$.

Proof. (1) $\Leftrightarrow$ (2). Suppose $x \in X$ and $x \notin \mathcal{M}^\circ_\mu$. Then there exists $M \in \mu$ such that $x \in M$ and $c_\mu(M) \cap \mathcal{M}_\mu \in \mathcal{H}$ which implies that $c_\mu(M) \in \mathcal{H}$ and hence $c_\mu(M) = \phi$ since $\mathcal{H}$ is μ°-codense which is a contradiction. Therefore, $x \in \mathcal{M}^\circ_\mu$. Hence $\mathcal{M}^\circ_\mu = X$. Conversely, suppose $c_\mu(M) \in c_\mu(\mu) \cap \mathcal{H}$, $M \in \mu$. If $M \neq \emptyset$, then there exists $x \in M$ and hence $x \in \mathcal{M}^\circ_\mu$ which implies that $c_\mu(M) \cap \mathcal{M}_\mu = c_\mu(M) \notin \mathcal{H}$, a contradiction. Therefore, $c_\mu(\mu) \cap \mathcal{H} = \emptyset$.

(2) $\Leftrightarrow$ (3). It is obvious from $\gamma^\circ_\mu(X-\mathcal{M}_\mu) = X-(X-(X-\mathcal{M}_\mu))^\circ = X-\mathcal{M}^\circ_\mu$. Hence (2) and (3) are equivalent. □

Theorem 4.7. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then the following are equivalent.*

(1) *$\mathcal{H}$ is strongly μ°-codense*;
(2) *$M \subseteq M^\circ$ for every $M \in \mu$*;
(3) *$S \subseteq S^\circ$ for every $S \in \sigma(\mu)$, where $\sigma(\mu) = \{A \subset X : A \subset c_\mu(i_\mu(A))\}$ [5]*;
(4) *$\gamma_\theta(M) = M^\circ$ for every $M \in \mu$*;
(5) *$c_\mu(S) \subseteq S^\circ$ for every $S \in \sigma(\mu)$.*

Proof. (1) and (2) are equivalent by the definition of strongly μ°-codense.

(2) $\Rightarrow$ (3). Suppose $M \subseteq M^\circ$ for every $M \in \mu$. Let $S \in \sigma(\mu)$. Then there exists a μ-open set M such that $M \subseteq S \subseteq c_\mu(M) = \gamma_\theta(M)$. Now $S \subseteq \gamma_\theta(M) = M^\circ$ by Lemma 2.4(5) and hence $S \subseteq M^\circ \subseteq S^\circ$.

(3) $\Rightarrow$ (4). It follows from the fact that $\mu \subseteq \sigma(\mu)$ and Lemma 2.4.

(4) $\Rightarrow$ (5). Let $S \in \sigma(\mu)$. Then there exists a μ-open set M such that $M \subseteq S \subseteq c_\mu(M) = \gamma_\theta(M) = M^\circ$, also we have $M^\circ \subseteq S^\circ$ and hence $c_\mu(S) \subseteq c_\mu(M) \subseteq S^\circ$.

(5) $\Rightarrow$ (2). It follows from the fact that $\mu \subseteq \sigma(\mu)$. □

Theorem 4.8. *Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. Then the following are equivalent.*

(1) *$\mathcal{H}$ is strongly μ°-codense*;
(2) *If A is $\sigma(\mu)$-closed, then $\gamma^\circ_\mu(A) \subseteq A$*;
(3) *$\gamma^\circ_\mu(c_\mu(A)) \subseteq i_\mu(c_\mu(A))$ for every $A \subseteq X$*;
(4) *$\gamma^\circ_\mu(A) \subseteq i_\mu(A)$ for every μ-closed set A.*

Proof. (1) $\Rightarrow$ (2). Suppose A is $\sigma(\mu)$-closed. By Theorem 4.7(3), $X - A \subseteq (X - A)^\circ$. Therefore, $X - (X - A)^\circ \subseteq A$ which implies that $\gamma^\circ_\mu(A) \subseteq A$.

(2) $\Rightarrow$ (1). If $A \in \sigma(\mu)$, then $X - A$ is $\sigma(\mu)$-closed. Therefore, by (2) $\gamma^\circ_\mu(X-A) \subseteq X-A$ and hence $A \subseteq A^\circ$ for every $A \in \sigma(\mu)$. Hence by Theorem 4.7 we have $\mathcal{H}$ is strongly μ°-codense.

(1) $\Rightarrow$ (3). If $A \subseteq X$, $\gamma^\circ_\mu(c_\mu(A)) = X - (X - c_\mu(A))^\circ = X - \gamma_\theta(X - c_\mu(A))$ by Theorem 4.7(4) and hence $\gamma^\circ_\mu(c_\mu(A)) = \iota_\theta(c_\mu(A)) \subseteq i_\mu(c_\mu(A))$.

(3) $\Rightarrow$ (2). If A is $\sigma(\mu)$-closed, then by (3) we have $\gamma^\circ_\mu(c_\mu(A)) \subseteq i_\mu(c_\mu(A)) \subseteq A$. Since γ°_μ is monotonic, it follows that $\gamma^\circ_\mu(A) \subseteq \gamma^\circ_\mu(c_\mu(A)) \subseteq A$.

(3) $\Leftrightarrow$ (4). It is clear. □

Definition 4.9. Let μ be a GT in X and $\mathcal{H}$ a hereditary class on X. A hereditary class $\mathcal{H}$ is said to be $*$-strongly μ°-codense if for $M, N \in \mu$, $(c_\mu(M) \cap N) \cap A \in \mathcal{H}$ and $(c_\mu(M) \cap N) - A \in \mathcal{H}$ implies $M \cap N = \emptyset$.

Lemma 4.10. *Let μ be a GT on X. Let a hereditary class $\mathcal{H}$ be $*$-strongly μ°-codense. Then*

(1) $X = X^\circ$;
(2) $\mathcal{H}$ *is strongly* μ°*-codense.*

Proof. (1) If $X - X^\circ \neq \emptyset$, then for some $x \in X - X^\circ$, there exists a nonempty μ-open set M containing x such that $c_\mu(M) \cap X \in \mathcal{H}$. Since $c_\mu(M) - X = \emptyset \in \mathcal{H}$, since $\mathcal{H}$ is $*$-strongly μ°-codense, then $M = \emptyset$. It is a contradiction, and hence $X = X^\circ$. Hence $\mathcal{H}$ is μ°-codense.

(2) For $M, N \in \mu$, if $(c_\mu(M) \cap N) \in \mathcal{H}$, then $(c_\mu(M) \cap N) \cap \emptyset \in \mathcal{H}$ and $(c_\mu(M) \cap N) - \emptyset \in \mathcal{H}$ and by hypothesis, $M \cap N = \emptyset$ and hence $\mathcal{H}$ is strongly μ°-codense. □

Theorem 4.11. *Let μ be a GT on X. Let a hereditary class $\mathcal{H}$ be $*$-strongly μ°-codense. Then $\gamma^\circ_\mu(A) \subseteq A^\circ \subseteq \gamma_\theta(A)$.*

Proof. Suppose there exists an element $x \in \gamma^\circ_\mu(A)$ such that $x \notin A^\circ$. For $x \in \gamma^\circ_\mu(A)$, since $x \notin (X - A)^\circ$, there exists a μ-open set M containing x such that $c_\mu(M) \cap (X - A) \in \mathcal{H}$. For $x \notin A^\circ$, there exists a μ-open set N containing x such that $c_\mu(N) \cap A \in \mathcal{H}$. Form the definition of hereditary class, it satisfies $(c_\mu(M) \cap N) \cap A \in \mathcal{H}$ and $(c_\mu(M) \cap N) \cap (X - A) \in \mathcal{H}$. Since $\mathcal{H}$ is $*$-strongly μ°-codense, $M \cap N = \emptyset$. But this contradicts the fact that both M and N containing x. Hence we have $\gamma^\circ_\mu(A) \subseteq A^\circ$ and by Lemma 2.4 $\gamma^\circ_\mu(A) \subseteq A^\circ \subseteq \gamma_\theta(A)$. □

Theorem 4.12. *Let μ be a GT on X. Let a hereditary class $\mathcal{H}$ be $*$-strongly μ°-codense. Then $\gamma^\circ_\mu(A) \cap \gamma^\circ_\mu(X - A) = \emptyset$.*

Proof. Assume that $z \in \gamma^\circ_\mu(A) \cap \gamma^\circ_\mu(X - A)$ for some $z \in X$, then there exist μ-open sets M, N containing z such that $c_\mu(M) - A \in \mathcal{H}$ and $c_\mu(N) \cap (X - A) \in \mathcal{H}$. Hence $(c_\mu(M) \cap N) - A \in \mathcal{H}$ and $(c_\mu(M) \cap N) \cap A \in \mathcal{H}$, since $\mathcal{H}$ is $*$-strongly μ°-codense, we have $M \cap N = \emptyset$. This is a contradiction. Hence $\gamma^\circ_\mu(A) \cap \gamma^\circ_\mu(X - A) = \emptyset$. □

Corollary 4.13. *Let μ be a GT on X. Let a hereditary class $\mathcal{H}$ be $*$-strongly μ°-codense. Then $A^\circ \cup (X - A)^\circ = X$.*

Theorem 4.14. *Let μ be a GT on X. Let a hereditary class $\mathcal{H}$ be $*$-strongly μ°-codense. For $A \subseteq X$, then*

(1) $\gamma^\circ_\mu(A) \subseteq i_\mu(A^\circ) \subseteq i_\mu(\gamma_\theta(A))$.
(2) *For a μ°-closed set $A \subseteq X$, $\gamma^\circ_\mu(A) \subseteq A^\circ \subseteq A$.*
(3) *If $A = i_\mu(\gamma_\theta(A))$, then $\gamma^\circ_\mu(A) \subseteq A$.*
(4) *If $A \in \mathcal{H}$, then $\gamma^\circ_\mu(A) = \phi$.*
(5) *If $X - A \in \mathcal{H}$, then $A^\circ = X$.*

Proof. (1), (2) It is obvious.

(3) It is obvious from (1).

(4) It follows from Theorem 3.2, Corollary 4.13 and Proposition 2.6 that $\gamma^\circ_\mu(A) = X - (X - A)^\circ \subseteq A^\circ = X - \mathcal{M}_\mu$. But from the fact that $\mathcal{M}_\mu$ is the union of all μ-open sets and $\gamma^\circ_\mu(A)$ is a μ-open set, $\gamma^\circ_\mu(A) = \emptyset$.

(5) By (4) $\gamma_\mu^\circ(X-A)=\emptyset$ and by Theorem 3.2, $\gamma_\mu^\circ(X-A)=X-A^\circ$. Hence $A^\circ=X$. □

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Ahmad Al-Omari
Department of Mathematics
Faculty of Sciences
Al al-Bayt University
P.O. Box 130095, Mafraq 25113, Jordan
E-mail address: omarimutah1@yahoo.com

Shyamapada Modak
Department of Mathematics
University of Gour Banga
P.O. Mokdumpur, 732 103, Malda
E-mail address: spmodak2000@yahoo.co.in

Takashi Noiri
2949-1 Shiokita-cho, Hinagu, Yatsushiro-shi, Kumamoto-ken, 869-5142 Japan
E-mail address: t.noiri@nifty.com