Commun. Korean Math. Soc. **31** (2016), No. 4, pp. 845–855
http://dx.doi.org/10.4134/CKMS.c150114
pISSN: 1225-1763 / eISSN: 2234-3024

# THE PROXIMAL POINT ALGORITHM IN UNIFORMLY CONVEX METRIC SPACES

Byoung Jin Choi and Un Cig Ji

Abstract. We introduce the proximal point algorithm in a $p$-uniformly convex metric space. We first introduce the notion of $p$-resolvent map in a $p$-uniformly convex metric space as a generalization of the Moreau-Yosida resolvent in a CAT(0)-space, and then we secondly prove the convergence of the proximal point algorithm by the $p$-resolvent map in a $p$-uniformly convex metric space.

## 1. Introduction

Let $H$ be a Hilbert space. It is well-known that the parallelogram law holds: for any $x, y \in H$

$$\|x-y\|^2 + \|x+y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right),$$

which is rewritten as that for any $x, y, z \in H$

$$d(x,y)^2 + 4d\left(z, \frac{x+y}{2}\right)^2 = 2(d(z,x)^2 + d(z,y)^2), \tag{1.1}$$

where $d(x,y) = \|x-y\|$. The identity (1.1) is generalized to the semiparallelogram law in a metric space. More precisely, let $(M,d)$ be a complete metric space. Then the semiparallelogram law is stated as following: for any $x, y \in M$, there exists a point $w \in M$ such that for each $z \in M$,

$$d(x,y)^2 + 4d\left(z,w\right)^2 \le 2(d(z,x)^2 + d(z,y)^2)$$

holds, of which the inequality can be rephrased in a $p$-uniformly convex metric space [16], which is introduced as a generalization of the notion of $p$-uniformly Banach space without using the modulus of convexity for $1 < p < \infty$ (see [4, 19, 23] or Example 2.2), as following: for any $z \in M$ and any geodesic

Received June 16, 2015.
2010 *Mathematics Subject Classification.* Primary 51F99, 47J25.
*Key words and phrases.* $p$-uniformly convex metric space, $p$-resolvent map, proximal point algorithm.
The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (No. NRF-2013R1A1A2013712).

$\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = x$, $\gamma(1) = y$, there exists a constant $C > 0$ such that

$$d\left(z, \gamma(1/2)\right)^p \leq \frac{1}{2} d(z,x)^p + \frac{1}{2} d(z,y)^p - \frac{C}{4} d(x,y)^p$$

holds.

The notion of convexity in metric spaces is studied many authors e.g., [2, 7, 8, 13, 14, 16, 17, 19, 20, 22], etc. In particular, in [22], the author proved that any normed spaces and their convex subsets are convex metric spaces. In fact, there are many examples of convex metric spaces which are not embedded in any normed or Banach space. Also, in [16] (resp. [17]), the author have been introduced the notion of $p$-uniformly (resp. 2-uniformly) convexity and in [14, 17] the authors proved that CAT($\kappa$)-spaces are 2-uniformly convex metric spaces. Then the fixed point theorem in a convex metric space has been studied extensively by many authors [2, 8, 13, 16, 19, 20, 22].

Let $(N, d)$ be a CAT(0)-space, and $f : N \longrightarrow (-\infty, \infty]$ be a convex lower semicontinuous function. Then the *Moreau-Yosida resolvent* $J_\lambda^f(x)$ of $f$ is defined by

$$J_\lambda^f(x) := \operatorname*{argmin}_{z \in N} \left\{ f(z) + \frac{1}{2\lambda} d(z,x)^2 \right\}, \qquad \lambda > 0, \quad x \in N, \tag{1.2}$$

and $J_0^f(x) = x$ for $\lambda = 0$ and $x \in N$. The Moreau-Yosida resolvent of $f$, which is also called the *Moreau envelope function* of $f$, in Hilbert spaces has been introduced by Moreau [15], and then studied in case of metric spaces (see [9]). The mapping $J_\lambda^f$ is also well-defined for all $\lambda \geq 0$ in a CAT(0)-space (see Lemma 2 in [9]). In fact, the Moreau-Yosida resolvent is essential for the proof of existence of harmonic maps. For more details, we refer to [9, 10, 11, 12]. Now, for an arbitrary fixed starting point $x_0$, we put

$$x_n := J_{\lambda_n}^f(x_{n-1}), \quad n \geq 1,$$

which is called the *proximal point algorithm* [1, 9, 18], where $\{\lambda_n\}$ is a sequence of positive real numbers. The convergence of the proximal point algorithm in a metric space (e.g. CAT(0)-spaces and Alexandrov spaces) have been studied by several authors [1, 9, 18], etc.

Main purpose of this paper is to introduce the proximal point algorithm in a $p$-uniformly convex metric space. For our purpose, we first introduce the $p$-resolvent map in a $p$-uniformly convex metric space as a generalization of the Moreau-Yosiha resolvent in a CAT(0)-space, and then by using the $p$-resolvent map we prove that the convergence of the proximal point algorithm in a $p$-uniformly convex metric space.

This paper is organized as follows. In Section 2, we recall a notion of $p$-uniformly convex metric spaces. In Section 3, we first define the $p$-resolvent map in a $p$-uniformly convex metric space, and secondly we prove the convergence of the proximal point algorithm induced by the $p$-resolvent map in a $p$-uniformly convex metric space.

## 2. $p$-uniformly convex metric spaces

A nonempty complete metric space $(M, d)$ is said to be a *geodesic* (*metric*) *space* if for any pair of points $x, y \in M$, there exists a point $m \in M$ such that

$$d(x, m) = d(m, y) = \frac{1}{2} d(x, y),$$

which is called a *midpoint* of $x$ and $y$. For a positive real number $D > 0$, a *geodesic of speed* $D$ in $M$ is a map $\gamma : [0, 1] \to M$ with the property that $d(\gamma(t_1), \gamma(t_2)) = D|t_1 - t_2|$ for all $t_1, t_2 \in [0, 1]$. A map $\gamma$ is said to be a (*minimal*) *geodesic* if it is a geodesic of some speed $D$.

For any fixed $2 \le p < \infty$, a geodesic space $(M, d)$ is said to be *$p$-uniformly convex with parameter $C$* [14, 16, 17] if there exists a constant $0 < C \le 1$ such that for any $x, y, z \in M$ and any geodesic $\gamma : [0, 1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$,

$$d(z, \gamma(t))^p \le (1 - t) d(z, x)^p + t d(z, y)^p - C t (1 - t) d(x, y)^p, \quad t \in [0, 1]. \tag{2.1}$$

*Remark* 2.1. For $1 < p < 2$, a $p$-uniformly convex metric space $(M, d)$ with parameter $C$ is also defined as a geodesic space $(M, d)$ satisfying that there exists a constant $0 < C \le 1$ such that (2.1) holds. Then Kuwae in [14] proved that, for $1 < p < 2$, the $p$-uniformly convex metric spaces $(M, d)$ with parameter $C$ can be considered as 2-uniformly convex metric spaces with parameter $C_0$ for some constant $C_0 > 0$ (see Proposition 2.5 in [14]). Therefore, we can consider only the case of $2 \le p < \infty$. For more detailed study of $p$-uniformly convexity, we refer to [14, 16, 17], etc.

**Example 2.2.** A Banach space $B$ with a norm $\|\cdot\|$ is said to be *uniformly convex* if the modulus $\delta_B$ of convexity of $B$ defined on $(0, 2]$ by

$$\delta_B(\epsilon) := \inf \left\{ 1 - \left\| \frac{x + y}{2} \right\| \;\middle|\; \|x\| = \|y\| = 1, \|x - y\| \ge \epsilon, x, y \in B \right\}$$

is strictly positive, i.e., $\delta_B(\epsilon) > 0$ for all $0 < \epsilon \le 2$. For $p > 1$, a Banach space $B$ with a norm $\|\cdot\|$ is said to be *$p$-uniformly convex* if there exists a constant $c > 0$ such that $\delta_B(\epsilon) \ge c \cdot \epsilon^p$ for all $0 < \epsilon \le 2$. In particular, for any $p \ge 2$, $L^p$-space is a $p$-uniformly convex Banach space, in fact, $\delta_B(\epsilon) \ge \epsilon^p / (p 2^p)$ (see [4, 19]). If a Banach space $B$ is $p$-uniformly convex for $p \ge 2$, then $B$ is $p$-uniformly convex with parameter $4C/2^p$, where $C = C(c, p) > 0$ (see [14] for details).

**Example 2.3.** Let $(Y, d)$ be a CAT($\kappa$)-space with diameter $\frac{\pi}{2\sqrt{\kappa}} - \epsilon$ for $\epsilon \in (0, \pi/(2\sqrt{\kappa}))$ and $\kappa > 0$ (i.e., for any $x, y \in Y$, $d(x, y) \le \frac{\pi}{2\sqrt{\kappa}} - \epsilon$). Then for any geodesic $\gamma : [0, 1] \to Y$ with $\gamma(0) = x$ and $\gamma(1) = y$, $z \in Y$ and $t \in [0, 1]$ the following inequality

$$d(z, \gamma(t))^2 \le (1 - t) d(z, x)^2 + t d(z, y)^2 - C_Y t (1 - t) d(x, y)^2,$$

holds, where $C_Y = [(\pi - 2\sqrt{\kappa}\epsilon)\tan(\sqrt{\kappa}\epsilon)]/2$ (see [14, 17]). Therefore, CAT($\kappa$)-space with diameter $\frac{\pi}{2\sqrt{\kappa}} - \epsilon$ is a 2-uniformly convex metric space with parameter $0 < C_Y < 1$.

**Example 2.4.** Let $(N, d)$ be a Hadamard space (or CAT(0)-space), i.e., for two elements $x, y \in N$, there exists an element $m \in N$ such that

$$d(z,m)^2 \leq \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2 \quad \text{for all } z \in N.$$

For any $x, y \in N$, there exists a unique geodesic $\gamma : [0,1] \to N$ of speed $D = d(x,y)$ with $\gamma(0) = x$ and $\gamma(1) = y$. Furthermore, for any $z \in N$ and $t \in [0,1]$,

$$d(z,\gamma(t)) \leq (1-t)d(z,x) + td(z,y)$$

and

$$d(z,\gamma(t))^2 \leq (1-t)d(z,x)^2 + td(z,y)^2 - t(1-t)d(x,y)^2$$

(see [21]). Therefore, a Hadamard space is a 2-uniformly convex metric space with parameter $C = 1$.

*Remark* 2.5. In [14], Kuwae has proved that if $(M, d)$ is a $p$-uniformly convex metric space for $p \geq 2$ with parameter $C > 0$, then the positive constant $C$ in (2.1), satisfies $C \in (0, 4/2^p]$. Therefore, if $C = 1$, then $4/2^p \geq 1$, (i.e., $p \leq 2$), which implies that $(M, d)$ becomes a Hadamard space.

**Example 2.6.** Let $\mathcal{B}(H)$ be the Banach space of all bounded linear operators on a separable Hilbert space $H$ equipped with the operator norm, $\mathcal{B}(H)_{sa}$ be the set of all self-adjoint elements in $\mathcal{B}(H)$ and $\mathcal{P}$ be the set of all positive invertible elements in $\mathcal{B}(H)_{sa}$. For $1 \leq p < \infty$, the *p-Schatten class* $\mathcal{S}_p(H)$ of $\mathcal{B}(H)$ is defined by

$$\mathcal{S}_p(H) := \Big\{ x \in \mathcal{B}(H) \;\Big|\; x \text{ is a compact operator and } \|x\|_p := \left(\sum_j [s_j(x)]^p\right)^{1/p} = (\mathrm{Tr}|x|^p)^{1/p} < \infty \Big\},$$

where $s_j(x)$ is the sequence of singular values of $x$ with decreasing order, $|x| = (x^*x)^{1/2}$ and Tr is the usual trace on $\mathcal{B}(H)$.

On the $p$-Schatten class $\mathcal{S}_p(H)$, we define the norm $\|\cdot\|_{p,b}$ associated with $b \in \mathcal{P}$ by

$$\|a\|_{p,b} = \left\| b^{-1/2} a b^{-1/2} \right\|_p \quad \text{for } a \in \mathcal{S}_p(H).$$

We denote by $\mathcal{S}_{p,b}$ the $p$-Schatten class $\mathcal{S}_p(H)$ equipped with the norm $\|\cdot\|_{p,b}$, that is, $\mathcal{S}_{p,b} = (\mathcal{S}_p(H), \|\cdot\|_{p,b})$. For $1 < p < \infty$, let $\Delta_p := \{I + a \in \mathcal{P} \mid a \in \mathcal{S}_{p,b}\}$ be the positive cone of the operator algebra that is obtained by adjoining the unit to the ideal of compact $p$-Schatten class operators.

We define the *geodesic distance* $d_p(x,y)$ between two points $x,y \in \Delta_p$ as follows [5, 6]:

$$\begin{aligned} & d_p(x,y) \\ = & \inf\Big\{L(\gamma) \,\Big|\, \gamma \text{ is a } \Delta_p\text{-valued smooth curve on } [0,1],\ \gamma(0)=x, \gamma(1)=y\Big\} \\ = & \|\log(x^{-1/2}yx^{-1/2})\|_p, \end{aligned}$$

where $L(\gamma) = \int_0^1 \|\gamma'(t)\|_{p,\gamma(t)}\,dt = \int_0^1 \|\gamma(t)^{-1/2}\gamma'(t)\gamma(t)^{-1/2}\|_p\,dt$. Let $x,y \in \Delta_p$ for $1 < p \le 2$. Then for any geodesic $\gamma : [0,1] \to \Delta_p$ joining $x$ to $y$ in $\Delta_p$, we have that

$$d_p(z,\gamma(t))^2 \le (1-t)d_p(z,x)^2 + td_p(z,y)^2 - t(1-t)\alpha_p d_p(x,y)^2,$$

where $\alpha_p = p-1$ (see [6]). Therefore, $(\Delta_p, d_p)$ is a 2-uniformly convex metric space with parameter $\alpha_p$ for $1 < p \le 2$.

**Example 2.7.** Let $(M,d)$ be a complete metric space. A map $\gamma : M \times M \times [0,1] \longrightarrow M$ is said to be a *convex structure* [22] on $M$ if for each $(x,y,t) \in M \times M \times [0,1]$ and $z \in M$,

$$d\left(z, \gamma(x,y,t)\right) \le td(z,x) + (1-t)d(z,y).$$

A complete metric space $M$ is said to be a *convex metric space* if $M$ has a convex structure $\gamma$ on $M$. Note that a convex metric space is a geodesic space since $\gamma(x,y,1/2)$ is a midpoint of $x$ and $y$.

A convex metric space $M$ is said to be have *property* (B) [2] if any convex structure $\gamma$ on $M$ satisfies

$$d\left(\gamma(x,a,t), \gamma(y,a,t)\right) = td(x,y),$$

and is said to have *property* (G) if for any convex structure $\gamma$ on $M$ satisfies

$$d\left(\gamma(x,y,t), \gamma(x,y,t')\right) = |t-t'|d(x,y) \quad \text{for any } t,t' \in [0,1].$$

A convex metric space $M$ is said to be *uniformly convex* [2] if for all $x,y,z \in M$,

$$d\left(z,\gamma(x,y,1/2)\right)^2 \le \frac{1}{2}\left(1 - \delta\left(\frac{d(x,y)}{\max\{d(z,x), d(z,y)\}}\right)\right)\left(d(z,x)^2 + d(z,y)^2\right),$$

where $\gamma$ is a convex structure on $M$ and $\delta$ is a strictly increasing function on $\mathbb{R}_+$ with $\delta(0) = 0$.

A uniformly convex metric space $M$ is said to be 2-*uniformly convex in the sense of Beg* [2] if there exists a constant $c > 0$ such that $\delta(\epsilon) \ge c\epsilon^2$.

Let $(M,d)$ be a 2-uniformly convex space in the sense of Beg with properties (B) and (G) and let $\gamma$ be a convex structure on $M$. Then for any $x,y,z \in M$ and any $t \in [0,1]$

$$d(z,\gamma(x,y,t))^2 \le (1-t)d(z,x)^2 + td(z,y)^2 - t(1-t)Cd(x,y)^2,$$

where $C$ is a positive constant (see [3]). Therefore, a 2-uniformly convex space in the sense of Beg with properties (B) and (G) is a 2-uniformly convex metric space with parameter $C$.

## 3. Convergence of proximal point algorithm

Let $(M, d)$ be a $p$-uniformly convex metric space with parameter $C$ and $\{x_i\}$ be a sequence of elements in $(M, d)$.

For a given real-valued function $\varphi$ on $M$, if there exists a point $x \in M$ such that $\varphi(x) = \inf_{z\in M} \varphi(z)$, then $x$ is called a *minimizer* and denoted by $x := \underset{z\in M}{\operatorname{argmin}}\varphi(z)$. A function $\varphi : M \to (-\infty, \infty]$ is said to be *convex* if for any geodesic $\gamma : [0, 1] \to M$, with $\gamma(0) = x$ and $\gamma(1) = y$,

$$\varphi(\gamma(t)) \leq (1 - t)\varphi(x) + t\varphi(y),$$

and said to be *uniformly convex* if there exists a strictly increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any geodesic $\gamma : [0, 1] \to M$, with $\gamma(0) = x$ and $\gamma(1) = y$,

$$\varphi(\gamma(1/2)) \leq \frac{1}{2}\big[\varphi(x) + \varphi(y)\big] - \omega\big(d(x, y)\big).$$

**Proposition 3.1** ([21]). *Let $(M, d)$ be a geodesic space. If $\varphi : M \to (-\infty, \infty]$ is a uniformly convex, lower semicontinuous function not identically $\infty$ on $M$, then there exists a unique minimizer $x \in M$, i.e., $x = \underset{z\in M}{\operatorname{argmin}}\,\varphi(z)$.*

**Theorem 3.2.** *Let $(M, d)$ be a $p$-uniformly convex metric space with parameter $C$ and $x$ be an element in $(M, d)$. Let $f : M \longrightarrow (-\infty, \infty]$ be a convex, lower semi-continuous function not identically $\infty$. Then, there exists a unique element $w \in M$ which minimizes the real-valued function on $M$ given by $z \mapsto f(z) + \frac{1}{2\lambda}d(z, x)^p$.*

*Proof.* Put $F^f_\lambda(z) = f(z) + \frac{1}{2\lambda}d(z, x)^p$. Then since $f$ is lower semi-continuous, $F^f_\lambda(z)$ is a lower semi-continuous function. Let $z_0, z_1 \in M$ and let $\gamma$ be a geodesic joining $z_0$ and $z_1$. Then since $f$ is convex, by using the inequality (2.1) for $d$, we obtain that for all $t \in [0, 1]$

$$\begin{aligned}
&F^f_\lambda(\gamma(1/2))\\
&= f(\gamma(1/2)) + \frac{1}{2\lambda}d(\gamma(1/2), x)^p\\
&\leq \frac{1}{2}\left(f(z_0) + \frac{1}{2\lambda}d(z_0, x)^p\right) + \frac{1}{2}\left(f(z_1) + \frac{1}{2\lambda}d(z_1, x)^p\right) - \frac{C}{8\lambda}d(z_0, z_1)^p\\
&= \frac{1}{2}\big[F^f_\lambda(z_0) + F^f_\lambda(z_1)\big] - \frac{C}{8\lambda}d(z_0, z_1)^p.
\end{aligned}$$

Therefore, $F^f_\lambda$ is uniformly convex and lower semi-continuous, so that by applying Proposition 3.1, we prove the existence and uniqueness of minimizer. $\square$

The *p-resolvent map* $J_\lambda^f$ on $(M,d)$ is defined by

$$J_\lambda^f(x) := \operatorname*{argmin}_{z\in M}\left\{f(z)+\frac{1}{2\lambda}d(z,x)^p\right\}, \tag{3.1}$$

where $f$ is a convex, lower semi-continuous function not identically $\infty$ and $\lambda$ is a positive number.

**Lemma 3.3.** *Let $f : M \longrightarrow (-\infty,\infty]$ be a convex, lower semicontinuous function on a p-uniformly convex metric space with parameter $C$. Then for each $z \in M$ we have*

$$d\left(z, J_\lambda^f(x)\right)^p \le \frac{1}{C}\left[d(z,x)^p - 2\lambda\left(f(J_\lambda^f(x)) - f(z)\right)\right] \tag{3.2}$$

*for any $x \in M$.*

*Proof.* Let $z \in M$ be given. For each given $x \in M$ and any $w \in M$, from the definition of $J_\lambda^f(x)$, we have

$$f(J_\lambda^f(x)) + \frac{1}{2\lambda}d(J_\lambda^f(x),x)^p \le f(w) + \frac{1}{2\lambda}d(w,x)^p,$$

and so for any geodesic $\gamma : [0,1] \longrightarrow M$ with $\gamma(0) = z$ and $\gamma(1) = J_\lambda^f(x)$ and any $t \in [0,1)$,

$$f(J_\lambda^f(x)) + \frac{1}{2\lambda}d(J_\lambda^f(x),x)^p \le f(\gamma(t)) + \frac{1}{2\lambda}d(\gamma(t),x)^p.$$

Therefore, by applying (2.1) and the convexity of $f$, we obtain that

$$\begin{aligned}&-\frac{(1-t)}{2\lambda}d(z,x)^p + \frac{(1-t)}{2\lambda}d(x,J_\lambda^f(x))^p + \frac{C}{2\lambda}(1-t)td(z,J_\lambda^f(x))^p\\ &\le (1-t)[f(z) - f(J_\lambda^f(x))].\end{aligned}$$

Since $t \neq 1$, we obtain that

$$\begin{aligned}f(J_\lambda^f(x)) - f(z) &\le \frac{1}{2\lambda}d(z,x)^p - \frac{1}{2\lambda}d(x,J_\lambda^f(x))^p - \frac{C}{2\lambda}td(z,J_\lambda^f(x))^p\\ &\le \frac{1}{2\lambda}d(z,x)^p - \frac{C}{2\lambda}td(z,J_\lambda^f(x))^p.\end{aligned}$$

By taking the limit $t \to 1$, we have (3.2). ☐

Now, fix an arbitrary starting element $x_0 \in M$ and put

$$x_n := J_{\lambda_n}^f(x_{n-1}), \quad n \ge 1, \tag{3.3}$$

which is called proximal point algorithm, where $\{\lambda_n\}$ is a sequence of positive real numbers.

**Lemma 3.4.** *Let $(M,d)$ be a p-uniformly convex metric space with parameter $C$, $f : M \longrightarrow (-\infty,\infty]$ be a convex, lower semi-continuous function not* *identically* $\infty$ *and* $\{\lambda_n\}$ *be a sequence of positive real numbers. Let* $\{x_i\}$ *be a sequence of elements in* $(M,d)$ *given by* (3.3)*. Then for any* $z \in M$ *we have*

$$f(x_n) - f(z) \le \frac{\sum_{i=0}^{n-1} d(z,x_i)^p - C\sum_{i=1}^{n} d(z,x_i)^p}{2\sum_{i=1}^{n}\lambda_i}. \tag{3.4}$$

*Proof.* By the definition of $J_\lambda^f$, we have

$$f\left(J_{\lambda_n}^f(x_{n-1})\right) + \frac{1}{2\lambda_n} d(J_\lambda^f(x_{n-1}), x_{n-1})^p \le f(x_{n-1}) + \frac{1}{2\lambda_n} d(x_{n-1}, x_{n-1})^p,$$

which implies that $\{f(x_n)\}$ is a monotone non-increasing sequence. By taking $x = x_{n-1}$ and $\lambda = \lambda_n$ in (3.2), for any $z \in M$, we have

$$Cd\left(z, x_n\right)^p \le d(z, x_{n-1})^p - 2\lambda_n \left(f(x_n) - f(z)\right).$$

Therefore, by the monotonicity of $\{f(x_n)\}$, we obtain that

$$\begin{aligned} 2\left(f(x_n) - f(z)\right)\sum_{i=1}^{n}\lambda_i &\le 2\sum_{i=1}^{n}\lambda_i\left(f(x_i) - f(z)\right) \\ &\le d(z,x_0)^p + (1-C)\sum_{i=1}^{n-1} d(z,x_i)^p - Cd(z,x_n)^p, \end{aligned}$$

which implies

$$f(x_n) - f(z) \le \frac{\sum_{i=0}^{n-1} d(z,x_i)^p - C\sum_{i=1}^{n} d(z,x_i)^p}{2\sum_{i=1}^{n}\lambda_i},$$

which is the assertion. □

**Theorem 3.5.** *Let* $(M,d)$ *be a* $p$*-uniformly convex metric space with parameter* $C$ *and diameter* $\alpha > 0$ *(i.e., for any* $x, y \in M$*,* $d(x,y) \le \alpha$*). Let* $f : M \longrightarrow (-\infty, \infty]$ *be a convex, lower semi-continuous function not identically* $\infty$ *and* $\{\lambda_n\}$ *be a sequence of positive real numbers such that* $\lim_{n\to\infty} n/(\sum_{i=1}^{n}\lambda_i) = 0$*. Suppose that* $f$ *has a minimizer in* $M$*. Let* $\{x_i\}$ *be a sequence of elements in* $(M,d)$ *given by* (3.3)*. Then we have*

$$\lim_{n\longrightarrow\infty} f(x_n) = \inf_{z\in M} f(z). \tag{3.5}$$

*Proof.* By Lemma 3.4, for any $z \in M$ we have

$$f(x_n) - f(z) \le \frac{\sum_{i=0}^{n-1} d(z,x_i)^p - C\sum_{i=1}^{n} d(z,x_i)^p}{2\sum_{i=1}^{n}\lambda_i}.$$

Since for any $x, y \in M$, $d(x,y) \le \alpha$, we have

$$f(x_n) - f(z) \le \frac{n\alpha^p}{2\sum_{i=1}^{n}\lambda_i} \longrightarrow 0$$

as $n \to \infty$. Thus, we obtain that $\lim_{n\longrightarrow\infty} f(x_n) \le f(z)$ for any $z \in M$. Therefore, we have

$$\lim_{n\longrightarrow\infty} f(x_n) = \inf_{z\in M} f(z),$$

which gives the proof. □

**Theorem 3.6.** *Let* $(M,d)$ *be a* $p$*-uniformly convex metric space with parameter* $C$ *and diameter* $\alpha > 0$*. Let* $f : M \longrightarrow (-\infty, \infty]$ *be a uniformly convex, lower semi-continuous function not identically* $\infty$*, and* $\{\lambda_n\}$ *be a sequence of positive real numbers such that* $\lim_{n\to\infty} n/(\sum_{i=1}^{n} \lambda_i) = 0$*. Let* $\{x_i\}$ *be a sequence of elements in* $(M,d)$ *given by* (3.3)*. Then the sequence* $\{x_n\} \subseteq M$ *converges to the minimizer of* $f$*.*

*Proof.* By Proposition 3.1, $f$ has a unique minimizer $x$, and Theorem 3.5, we have

$$\lim_{n\longrightarrow\infty} f(x_n) = f(x).$$

Since $f$ is uniformly convex, there exists a strictly increasing function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any geodesic $\gamma : [0,1] \to M$, with $\gamma(0) = x_n$ and $\gamma(1) = x_m$,

$$\omega\big(d(x_n, x_m)\big) \le \frac{1}{2}\big[f(x_n) + f(x_m)\big] - f(\gamma(1/2))$$

for any $n, m \in \mathbb{N}$. Since $\{x_n\}$ is minimizing sequence of $f$ and $\omega$ vanishes only at 0, we obtain that $\{x_n\}$ is a Cauchy sequence in $M$. Thus, there exists $\widetilde{x} \in M$ such that $x_n$ converges to a point $\widetilde{x} \in M$ which by lower semicontinuity of $f$, should be a minimizer of $f$. The proof is completed. □

*Remark* 3.7. In the proof in Theorems 3.5, if $(M,d)$ is a 2-uniformly convex metric space with parameter $C = 1$ (or Hadamard space given as in Example 2.4), then we have

$$f(x_n) - f(z) \le \frac{\sum_{i=0}^{n-1} d(z,x_i)^p - \sum_{i=1}^{n} d(z,x_i)^p}{2\sum_{i=1}^{n} \lambda_i} \le \frac{d(z,x_0)^p}{2\sum_{i=1}^{n} \lambda_i}.$$

Therefore, we can replace the condition $\lim_{n\to\infty} n/(\sum_{i=1}^{n} \lambda_i) = 0$ by $\sum_{i=1}^{\infty} \lambda_i = \infty$ which is a weaker condition.

**Corollary 3.8** ([1])**.** *Let* $(M,d)$ *be a* 2*-uniformly convex metric space with parameter* $C = 1$ (*or Hadamard space given as in Example 2.4*)*. Let* $\{x_i\}$ *be a sequence of elements in* $(M,d)$ *given by* (3.3) *with* $p = 2$*. Let* $f : M \longrightarrow (-\infty,\infty]$ *be a uniformly convex, lower semi-continuous function not identically* $\infty$ *and* $\{\lambda_n\}$ *be a sequence of positive real numbers such that* $\sum_{n=1}^{\infty} \lambda_n = \infty$*. Then the sequence* $\{x_n\} \subseteq M$ *converges to a minimizer of* $f$*.*

*Proof.* By Theorem 3.6 and Remark 3.7, the proof is obvious. □

## References


[1] M. Bačák, *The proximal point algorithm in metric spaces*, Israel J. Math. **194** (2013), no. 2, 689–701.

[2] I. Beg, *Inequalities in metric spaces with applications*, Topol. Methods Nonlinear Anal. **17** (2001), no. 1, 183–190.

[3] B. J. Choi, J. Heo, and U. C. Ji, *Asymptotic property for inductive means in uniformly convex metric spaces*, preprint, 2015.

[4] J. A. Clarkson, *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), no. 3, 396–414.
[5] C. Conde, *Geometric interpolation in p-Schatten class*, J. Math. Anal. Appl. **340** (2008), no. 2, 920–931.
[6] ______, *Nonpositive curvature in p-Schatten class*, J. Math. Anal. Appl. **356** (2009), no. 2, 664–673.
[7] T. Foertsch, *Ball versus distance convexity of metric spaces*, Beiträge Algebra Geom. **45** (2004), no. 2, 481–500.
[8] H. Fukharuddin, A. R. Khan, and Z. Akhtar, *Fixed point results for a generalized non-expansive map in uniformly convex metric spaces*, Nonlinear Anal. **75** (2012), no. 13, 4747–4760.
[9] J. Jost, *Convex functionals and generalized harmonic maps into spaces of nonpositive curvature*, Comment. Math. Helv. **70** (1995), no. 4, 659–673.
[10] ______, *Nonpositive Curvature*: *Geometric and Analytic Aspects*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel, 1997.
[11] ______, *Nonlinear Dirichlet forms*, in New Directions in Dirichlet Forms, AMS/IP Stud. Adv. Math. **8**, pp. 1–7, American Mathematical Society, Providence, RI, 1998.
[12] ______, *Riemannian Geometry and Geometric Analysis*, Fifth ed., Universitext, Springer-Verlag, Berlin, 2008.
[13] M. A. Khamsi and A. R. Khan, *Inequalities in metric spaces with applications*, Nonlinear Anal. **74** (2011), no. 12, 4036–4045.
[14] K. Kuwae, *Jensen's inequality on convex spaces*, Calc. Var. Partial Differential Equations **49** (2014), no. 3-4, 1359–1378.
[15] J.-J. Moreau, *Proximité et dualité dans un espace hilbertien*, Bull. Soc. Math. France **93** (1965), 273–299.
[16] A. Naor and L. Silberman, *Poincaré inequalities, embeddings, and wild groups*, Compos. Math. **147** (2011), no. 5, 1546–1572.
[17] S. Ohta, *Convexities of metric spaces*, Geom. Dedicata **125** (2007), 225–250.
[18] S. Ohta and M. Pálfia, *Discrete-time gradient flows and law of large numbers in Alexandrov spaces*, Calc. Var. Partial Differential Equations **54** (2015), no. 2, 1591–1610.
[19] B. Prus and R. Smarzewski, *Strongly unique best approximations and centers in uniformly convex spaces*, J. Math. Anal. Appl. **121** (1987), no. 1, 10–21.
[20] T. Shimizu and W. Takahashi, *Fixed points of multivalued mappings in certain convex metric spaces*, Topol. Methods Nonlinear Anal. **8** (1996), no. 1, 197–203.
[21] K.-T. Sturm, *Probability measures on metric spaces of nonpositive curvature*, Heat kernels and analysis on manifolds, graphs, and metric spaces (Paris, 2002), 357–390, Contemp. Math. **338**, Amer. Math. Soc., Providence, RI, 2003.
[22] W. Takahashi, *A convexity in metric space and nonexpansive mappings. I*, Kōdai Math. Sem. Rep. **22** (1970), 142–149.
[23] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal. **16** (1991), no. 12, 1127–1138.

Byoung Jin Choi
Department of Mathematics
Sungkyunkwan University
Suwon 16419, Korea
*E-mail address*: choibj@skku.edu

Un Cig Ji
Department of Mathematics
Research Institute of Mathematical Finance
Chungbuk National University
Cheongju 28644, Korea
*E-mail address*: uncigji@chungbuk.ac.kr