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LIPSCHITZ CRITERIA FOR BI-QUADRATIC FUNCTIONAL EQUATIONS

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Abstract. In this paper, we establish approximation of bi-quadratic functional equations in Lipschitz spaces.

1. Introduction

Let $\mathcal{W}$ be an abelian group and $\mathcal{V}$ a vector space. A family of subsets of $\mathcal{V}$ is called linearly invariant if it is closed under the addition and scalar multiplication defined as usual sense and translation invariant. For example, $CB(\mathcal{V})$ the family of all closed balls with center at zero is a linearly invariant family in a normed vector space. Let $S(\mathcal{V})$ be a linearly invariant family of subsets of $\mathcal{V}$. By $B(\mathcal{W}, S(\mathcal{V}))$ we denote the family of all functions $\mathcal{F} : \mathcal{W} \longrightarrow \mathcal{V}$ such that $\mathtt{Im}\, \mathcal{F} \subset B$ for some $B \in S(\mathcal{V})$.

The notion of the left invariant mean (briefly LIM) is defined in [2, 8]. For our purpose, we need to generalize this notion associated to $(x, y) \in \mathcal{W}\times\mathcal{W}$ with a symmetric form. We say that $B(\mathcal{W}, S(\mathcal{V}))$ admits a symmetric left invariant mean (briefly SLIM), if the family $S(\mathcal{V})$ is linearly invariant and there exists a linear operator $M : B(\mathcal{W}, S(\mathcal{V})) \longrightarrow \mathcal{V}$ such that

(i) if $\mathcal{F}_{x,y} \in B(\mathcal{W}, S(\mathcal{V}))$ and $(x, y) \in \mathcal{W} \times \mathcal{W}$, then $M[\mathcal{F}_{x,y}] = M[\mathcal{F}_{y,x}]$,
(ii) if $\mathtt{Im}\mathcal{F} \subset A$ for some $A \in S(\mathcal{V})$, then $M[\mathcal{F}] \in A$,
(iii) if $\mathcal{F} \in B(\mathcal{W}, S(\mathcal{V}))$ and $a \in \mathcal{W}$, then $M[\mathcal{F}^a] = M[\mathcal{F}]$, where $\mathcal{F}^a(x) = \mathcal{F}(x + a)$.

Remark 1.1. Let $\Delta : (\mathcal{W} \times \mathcal{W}) \times (\mathcal{W} \times \mathcal{W}) \longrightarrow S(\mathcal{V})$ be a set-valued function such that

$$\begin{aligned}\Delta((x + a, y + b), (w + a, z + b)) &= \Delta((a + x, b + y), (a + w, b + z))\\ &= \Delta((x, y), (w, z))\end{aligned}$$

for all $(a, b), (x, y), (w, z) \in \mathcal{W} \times \mathcal{W}$. A function $\mathcal{F} : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{V}$ is said to be Δ-Lipschitz if $\mathcal{F}(x, y)-\mathcal{F}(w, z) \in \Delta((x, y), (w, z))$ for all $(x, y), (w, z) \in \mathcal{W}\times\mathcal{W}$ (cf. [8]).

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We considered in [8] the Lipschitz functions of order one. In this paper, we consider the Lipschitz functions of order $\alpha > 0$ and verify their approximation properties. Let $(\mathcal{W} \times \mathcal{W}, \rho)$ be a metric group and $\mathcal{V}$ a normed space. A function $m_{\mathcal{F}} : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a module of continuity of $\mathcal{F} : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{V}$ if $\rho((x,y),(w,z)) \leq \delta$ implies $||\mathcal{F}(x,y) - \mathcal{F}(w,z)|| \leq m_{\mathcal{F}}(\delta)$ for every $\delta > 0$ and every $(x,y),(w,z) \in \mathcal{W} \times \mathcal{W}$ (cf. [8]). A function $\mathcal{F} : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{V}$ is called Lipschitz function of order $\alpha > 0$ if there exists a constant $L > 0$ such that

$$||\mathcal{F}(x,y) - \mathcal{F}(w,z)|| \leq L\rho((x,y),(w,z))^{\alpha} \tag{1.1}$$

for every $(x,y),(w,z) \in \mathcal{W} \times \mathcal{W}$.

For a metric group $(\mathcal{W} \times \mathcal{W}, \rho)$, a normed space $\mathcal{V}$, and $\alpha \in (0,1]$, let $Lip_{\alpha}(\mathcal{W} \times \mathcal{W}, \mathcal{V})$ to be the Lipschitz space consisting of all bounded Lipschitz functions of order $\alpha > 0$ with the norm

$$||\mathcal{F}||_{\alpha} := ||\mathcal{F}||_{\sup} + P_{\alpha}(\mathcal{F}),$$

where $|| \cdot ||_{\sup}$ is the supremum norm and

$$P_{\alpha}(\mathcal{F}) = \sup\Big\{\frac{||\mathcal{F}(x,y) - \mathcal{F}(u,v)||}{\rho((x,y),(u,v))^{\alpha}} : (x,y),(u,v) \in \mathcal{W} \times \mathcal{W}, (x,y) \neq (u,v)\Big\}.$$

For more details about Lipschitz spaces and Lipschitz algebras we refer the readers to see [4, 12, 13]. The algebra of Lipschitz functions on a complete metric space plays a role in noncommutative metric theory similar to that played by the algebra of continuous functions on a compact space in noncommutative topology.

A generalized stability problem for the following quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

was proved by Skof [14] for mappings from a normed space to a Banach space. Cholewa [1] noticed that the theorem of Skof is still true if the relevant domain is replaced by an Abelian group. Park [9] proved the generalized stability of the quadratic functional equation in Banach modules over a C^*-algebra. Czerwik et al. [2] established the stability of the quadratic functional equations in Lipschitz spaces. The Lipschitz stability type problems for some functional equations were also studied by Tabor [15, 16]. In Lipschitz spaces we investigated the stability of cubic functional equations [7] (see also [3]) and the stability of quartic functional equations [8]. On the other hand, the stability problem for the quadratic and quartic functional equation has been studied by many mathematicians under various degrees of generality imposed on the equation or on the underlying space; see, for example, [5], [6], [11] and the references therein. Park et al. [10] verified the generalized Hyers-Ulam stability of the bi-quadratic functional equations in quasinormed spaces. In this paper we prove a stability result for bi-quadratic functional equations in Lipschitz spaces.

2. Lipschitz stability of bi-quadratic functional equations

A function $\mathcal{F} : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{V}$ is called bi-quadratic if $\mathcal{F}$ satisfies the system of equations

$$\mathcal{F}(x+y,z)+\mathcal{F}(x-y,z)=2\mathcal{F}(x,z)+2\mathcal{F}(y,z), \tag{2.1}$$

$$\mathcal{F}(x,y+z)+\mathcal{F}(x,y-z)=2\mathcal{F}(x,y)+2\mathcal{F}(x,z) \tag{2.2}$$

for all $x, y, z \in \mathcal{W}$, that is, $\mathcal{F}$ is quadratic in each variable.

For a given function $\mathcal{F} : \mathcal{W}\times\mathcal{W} \longrightarrow \mathcal{V}$ we now define its quadratic difference as follows

$$Q\mathcal{F}(x,y,z) := 2\mathcal{F}(x,z)+2\mathcal{F}(y,z)-\mathcal{F}(x+y,z)-\mathcal{F}(x-y,z)$$

for all $x, y, z \in \mathcal{W}$. The function $\mathcal{F}$ is called symmetric if $\mathcal{F}(x, y) = \mathcal{F}(y, x)$ for all $(x, y) \in \mathcal{W} \times \mathcal{W}$.

Consider an Abelian group $(\mathcal{W} \times \mathcal{W}, +)$ with a metric ρ invariant under translation, i.e., satisfying the condition

$$\begin{aligned}\rho((x+a,y+b),(w+a,z+b)) &= \rho((a+x,b+y),(a+w,b+z))\\ &= \rho((x,y),(w,z))\end{aligned}$$

for all $(a, b), (x, y), (w, z) \in \mathcal{W} \times \mathcal{W}$. We say that a metric D on $\mathcal{W}\times\mathcal{W}\times\mathcal{W}$ is a product metric if it is an invariant metric and the following condition holds

$$\begin{aligned}D((a,x,y),(a,w,z)) &= D((x,y,a),(w,z,a))\\ &= \rho((x,y),(w,z))\end{aligned}$$

for all $a \in \mathcal{W}, (x, y), (w, z) \in \mathcal{W} \times \mathcal{W}$.

Lemma 2.1. *If $\mathcal{F} : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{V}$ is bi-quadratic, then $\mathcal{F}(0, 0) = 0$, $\mathcal{F}(0, z) = 0$, and $\mathcal{F}(z, 0) = 0$ for all $z \in \mathcal{W}$.*

Theorem 2.2. *Let $\mathcal{W}$ be an Abelian group and let $\mathcal{V}$ be a vector space. Assume that the family $B(\mathcal{W}, S(\mathcal{V}))$ admits SLIM. If $\mathcal{F} : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{V}$ is a function and $Q\mathcal{F}(t, \cdot, \cdot) : \mathcal{W}\times\mathcal{W} \longrightarrow \mathcal{V}$ is Δ-Lipschitz for every $t \in \mathcal{W}$, then there exists a bi-quadratic function $\mathcal{G} : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{V}$ such that $\mathcal{F} - \mathcal{G}$ is $\frac{1}{2}\Delta$-Lipschitz. Furthermore, if $\mathtt{Im}\, Q\mathcal{F} \subset A$ for some $A \in S(\mathcal{V})$, then $\mathtt{Im}\,(\mathcal{F} - \mathcal{G}) \subset \frac{1}{2}A$.*

Proof. Since $Q\mathcal{F}(t, \cdot, \cdot) : \mathcal{W} \times \mathcal{W} \longrightarrow \mathcal{V}$ is Δ-Lipschitz for every $t \in \mathcal{W}$,

$$Q\mathcal{F}(t,x,y)-Q\mathcal{F}(t,z,w)\in\Delta((x,y),(z,w)) \tag{2.3}$$

for all $(x, y), (z, w) \in \mathcal{W} \times \mathcal{W}$. So,

$$\mathtt{Im}(\frac{1}{2}Q\mathcal{F}(\cdot,x,y)-\frac{1}{2}Q\mathcal{F}(\cdot,z,w))\subseteq\frac{1}{2}\Delta((x,y),(z,w)). \tag{2.4}$$

For every $(x, y) \in \mathcal{W} \times \mathcal{W}$ we define $\theta_x(\cdot, y) : \mathcal{W} \longrightarrow \mathcal{V}$ by

$$\theta_x(\cdot,y) := \frac{1}{2}\mathcal{F}(\cdot+x,y)+\frac{1}{2}\mathcal{F}(\cdot-x,y)-\mathcal{F}(\cdot,y).$$

We show that $\theta_x(\cdot,y) \in B(\mathcal{W}, S(\mathcal{V}))$. Indeed, we prove that $\mathtt{Im}\,\theta_x(\cdot,y) \subseteq A$ for some $A \in S(\mathcal{V})$. We have for $(x,y) \in \mathcal{W}\times\mathcal{W}$,

$$\begin{aligned}\theta_x(\cdot,y) &= \frac{1}{2}\mathcal{F}(\cdot+x,y) + \frac{1}{2}\mathcal{F}(\cdot-x,y) - \mathcal{F}(\cdot,y) - \mathcal{F}(x,y)\\ &\quad - \frac{1}{2}\mathcal{F}(\cdot,y) - \frac{1}{2}\mathcal{F}(\cdot,y) + \mathcal{F}(\cdot,y) + \mathcal{F}(0,y)\\ &\quad + \mathcal{F}(x,y) - \mathcal{F}(0,y)\\ &= \frac{1}{2}Q\mathcal{F}(\cdot,0,y) - \frac{1}{2}Q\mathcal{F}(\cdot,x,y) + \mathcal{F}(x,y) - \mathcal{F}(0,y).\end{aligned}$$

From the fact that $Q\mathcal{F}(t,\cdot,\cdot)$ is Δ-Lipschitz for every $t \in \mathcal{W}$, it follows that $\mathtt{Im}\,\theta_x(\cdot,y) \subseteq A$, where $A := \frac{1}{2}\Delta((0,y),(x,y)) + \mathcal{F}(x,y) - \mathcal{F}(0,y)$. The family $B(\mathcal{W}, S(\mathcal{V}))$ admits SLIM, so there exists a linear operator $M : B(\mathcal{W}, S(\mathcal{V})) \longrightarrow \mathcal{V}$ such that the following requirements are fulfilled:

(i) $M[\theta_x(\cdot,y)] = M[\theta_y(\cdot,x)]$ for every $(x,y) \in \mathcal{W}\times\mathcal{W}$,
(ii) $M[\theta_x(\cdot,y)] \in A$ for some $A \in S(\mathcal{V})$ and every $(x,y) \in \mathcal{W}\times\mathcal{W}$,
(iii) if $z \in \mathcal{W}$ and $\theta^z_x(\cdot,y) : \mathcal{W} \longrightarrow \mathcal{V}$ defined by $\theta^z_x(\cdot,y) := \theta_x(\cdot+z,y)$ for every $(x,y) \in \mathcal{W}\times\mathcal{W}$, then $\theta^z_x(\cdot,y) \in B(\mathcal{W}, S(\mathcal{V}))$ and $M[\theta^z_x(\cdot,y)] = M[\theta_x(\cdot,y)]$.

Define the function $\mathcal{G} : \mathcal{W}\times\mathcal{W} \longrightarrow \mathcal{V}$ by $\mathcal{G}(x,y) := M[\theta_x(\cdot,y)]$. It follows from property (i) of M that $\mathcal{G}$ is symmetric.

We know that $B(\mathcal{W}\times\mathcal{W}, S(\mathcal{V}))$ contains constant functions. Property (ii) of M shows that for a constant function $R : \mathcal{W}\times\mathcal{W} \longrightarrow \mathcal{V}$, $M[R] = R$. We now show that $\mathcal{F}-\mathcal{G}$ is $\frac{1}{2}\Delta$-Lipschitz. Let for any $(x,y) \in \mathcal{W}\times\mathcal{W}$ the constant function $R_{x,y} : \mathcal{W}\times\mathcal{W} \longrightarrow \mathcal{V}$ be a function $R_{x,y}(\cdot,\cdot) := \mathcal{F}(x,y)$. We have

$$\begin{aligned}&(\mathcal{F}(x,y) - \mathcal{G}(x,y)) - (\mathcal{F}(z,w) - \mathcal{G}(z,w))\\ &= (M[R_{x,y}(\cdot,\cdot)] - M[\theta_x(\cdot,y)]) - (M[R_{z,w}(\cdot,\cdot)] - M[\theta_z(\cdot,w)])\\ &= M[R_{x,y}(\cdot,\cdot) - \theta_x(\cdot,y)] - M[R_{z,w}(\cdot,\cdot) - \theta_z(\cdot,w)]\\ &= M[\frac{1}{2}Q\mathcal{F}(\cdot,x,y) - \frac{1}{2}Q\mathcal{F}(\cdot,z,w)]\end{aligned}$$

for all $(x,y),(z,w) \in \mathcal{W}\times\mathcal{W}$. In view of property (ii) of M and (2.4) we realize that

$$M[\frac{1}{2}Q\mathcal{F}(\cdot,x,y) - \frac{1}{2}Q\mathcal{F}(\cdot,z,w)] \in \frac{1}{2}\Delta((x,y),(z,w))$$

for all $(x,y),(z,w) \in \mathcal{W}\times\mathcal{W}$. This shows that

$$(\mathcal{F}(x,y) - \mathcal{G}(x,y)) - (\mathcal{F}(z,w) - \mathcal{G}(z,w)) \in \frac{1}{2}\Delta((x,y),(z,w))$$

for all $(x,y),(z,w) \in \mathcal{W}\times\mathcal{W}$, i.e., $\mathcal{F}-\mathcal{G}$ is a $\frac{1}{2}\Delta$-Lipschitz function. We now have

$$2\mathcal{G}(x,y) + 2\mathcal{G}(z,y) = 2M[\theta_x(\cdot,y)] + 2M[\theta_z(\cdot,y)].$$

Applying property (iii) of M, we find that

$$\begin{aligned}2\mathcal{G}(x,y)+2\mathcal{G}(z,y) &= 2M[\theta_x(\cdot,y)]+2M[\theta_z(\cdot,y)]\\ &= M[\theta_x^z(\cdot,y)]+M[\theta_x^{-z}(\cdot,y)]+2M[\theta_z(\cdot,y)].\end{aligned}$$

On the other hand we have

$$\begin{aligned}&M[\theta_x^z(\cdot,y)]+M[\theta_x^{-z}(\cdot,y)]+2M[\theta_z(\cdot,y)]\\ =\ & M[\frac{1}{2}\mathcal{F}(\cdot+x+z,y)+\frac{1}{2}\mathcal{F}(\cdot-x+z,y)-\mathcal{F}(\cdot+z,y)]\\ &+M[\frac{1}{2}\mathcal{F}(\cdot+x-z,y)+\frac{1}{2}\mathcal{F}(\cdot-x-z,y)-\mathcal{F}(\cdot-z,y)]\\ &+M[\mathcal{F}(\cdot+z,y)+\mathcal{F}(\cdot-z,y)-2\mathcal{F}(\cdot,y)]\\ =\ & M[\theta_{x+z}(\cdot,y)]+M[\theta_{x-z}(\cdot,y)]\\ =\ & \mathcal{G}(x+z,y)+\mathcal{G}(x-z,y).\end{aligned}$$

This shows that $\mathcal{G}$ is quadratic on its first variable. Since $\mathcal{G}$ is symmetric, $\mathcal{G}$ is quadratic on its second variable and hence $\mathcal{G}$ is bi-quadratic. Moreover, we have

$$\mathtt{Im}(\frac{1}{2}Q\mathcal{F}(\cdot,x,y))\subset \mathtt{Im}(\frac{1}{2}Q\mathcal{F})\subset\frac{1}{2}A$$

and so $\frac{1}{2}Q\mathcal{F}(\cdot,x,y)\in B(\mathcal{W},S(\mathcal{V}))$ for all $(x,y)\in\mathcal{W}\times\mathcal{W}$. Thus, property (ii) of M implies

$$\mathcal{F}(x,y)-\mathcal{G}(x,y)=M[\frac{1}{2}Q\mathcal{F}(\cdot,x,y)]\in\frac{1}{2}A$$

for all $(x,y)\in\mathcal{W}\times\mathcal{W}$. Therefore, $\mathtt{Im}\ (\mathcal{F}-\mathcal{G})\subset\frac{1}{2}A$. □

Theorem 2.3. *Let $(\mathcal{W}\times\mathcal{W},+,\rho,D)$ be a product metric, $\mathcal{V}$ a normed space such that $B(\mathcal{W}\times\mathcal{W},CB(\mathcal{V}))$ admits SLIM, and $\mathcal{F}:\mathcal{W}\times\mathcal{W}\longrightarrow\mathcal{V}$ a function. If $Q\mathcal{F}\in Lip_\alpha(\mathcal{W}\times\mathcal{W}\times\mathcal{W},\mathcal{V})$, then there exists a bi-quadratic function $\mathcal{G}$ such that*

$$||\mathcal{F}-\mathcal{G}||_\alpha\le\frac{1}{2}||Q\mathcal{F}||_\alpha.$$

Proof. Assume that $m_{Q\mathcal{F}}:\mathbb{R}^+\to\mathbb{R}^+$ is the module of continuity of $Q\mathcal{F}$ with the product metric D. Define the set-valued function $\Delta:\mathcal{W}\times\mathcal{W}\longrightarrow CB(\mathcal{V})$ by

$$\Delta((x,y),(w,z)):=\inf_{\rho((x,y),(w,z))\le\delta} m_{Q\mathcal{F}}(\delta)B(0,1),$$

where $B(0,1)$ is the closed unit ball with center at zero. It is immediate that

$$\begin{aligned}||Q\mathcal{F}(t,x,y)-Q\mathcal{F}(t,w,z)|| &\le \inf_{D((t,x,y),(t,w,z))\le\delta} m_{Q\mathcal{F}}(\delta)\\ &= \inf_{\rho((x,y),(w,z))\le\delta} m_{Q\mathcal{F}}(\delta)\end{aligned}$$

for all $t \in \mathcal{W}, (x,y),(w,z) \in \mathcal{W}\times\mathcal{W}$. This means that $Q\mathcal{F}(t,\cdot,\cdot)$ is Δ-Lipschitz and so Theorem 2.2 implies there exists a bi-quadratic function $\mathcal{G}$ such that $\mathcal{F}-\mathcal{G}$ is $\frac{1}{2}\Delta$-Lipschitz. Hence,

$$||(\mathcal{F}-\mathcal{G})(x,y)-(\mathcal{F}-\mathcal{G})(w,z)|| \leq \inf_{\rho((x,y),(w,z))\leq\delta} \frac{1}{2} m_{Q\mathcal{F}}(\delta),$$

which shows that $m_{\mathcal{F}-\mathcal{G}} = \frac{1}{2} m_{Q\mathcal{F}}$. Moreover, $||Q\mathcal{F}||_{\sup} < \infty$ and clearly $\mathtt{Im}\ Q\mathcal{F} \subset ||Q\mathcal{F}||_{\sup} B(0,1)$. Using Theorem 2.2 we get

$$||\mathcal{F}-\mathcal{G}||_{\sup} \leq \frac{1}{2}||Q\mathcal{F}||_{\sup}. \tag{2.5}$$

Define the function $\nabla : \mathbb{R}^+ \to \mathbb{R}^+$ by $\nabla(t) := P_\alpha(Q\mathcal{F})t^\alpha$. Taking into account that $Q\mathcal{F} \in Lip_\alpha(\mathcal{W}\times\mathcal{W}\times\mathcal{W},\mathcal{V})$, we have

$$||Q\mathcal{F}(t,x,y) - Q\mathcal{F}(t,w,z)|| \leq \nabla(D((t,x,y),(t,w,z)),$$

which means that ∇ is the module of continuity of the function $Q\mathcal{F}$ and so we deduce that $m_{\mathcal{F}-\mathcal{G}} = \frac{1}{2}\nabla$. Thus, we have

$$\begin{aligned} ||(\mathcal{F}-\mathcal{G})(x,y)-(\mathcal{F}-\mathcal{G})(w,z)|| &\leq \frac{1}{2}\nabla(\rho((x,y),(w,z))) \\ &= \frac{1}{2}P_\alpha(Q\mathcal{F})\rho((x,y),(w,z))^\alpha. \end{aligned}$$

The last inequality ensures that $\mathcal{F}-\mathcal{G}$ is a Lipschitz function of order α and $P_\alpha(\mathcal{F}-\mathcal{G}) \leq \frac{1}{2}P_\alpha(Q\mathcal{F})$. Applying the inequality (2.5) we get

$$\begin{aligned} ||\mathcal{F}-\mathcal{G}||_\alpha &= ||\mathcal{F}-\mathcal{G}||_{\sup} + P_\alpha(\mathcal{F}-\mathcal{G}) \\ &\leq \frac{1}{2}||Q\mathcal{F}||_{\sup} + \frac{1}{2}P_\alpha(Q\mathcal{F}) \\ &= \frac{1}{2}||Q\mathcal{F}||_\alpha. \end{aligned}$$

□

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