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DIFFERENTIABILITY AND NON-DIFFERENTIABILITY POINTS OF THE MINKOWSKI QUESTION MARK FUNCTION

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Abstract. Using the periodic continued fraction, we give concrete examples of the points at which the derivatives of the Minkowski question mark function does not exist. We also give examples of the differentiability points which show that recent apparently independent results are consistent and closely related.

1. Introduction

Many authors ([1, 2, 3, 6, 7, 8]) have studied the differentiability and non-differentiability of the Minkowski question mark function. Recently the multifractal spectrum of the non-differentiability points of the Minkowski question mark function was investigated in [4], without giving the concrete example of its non-differentiability point. More recently, the concrete non-trivial examples of the points at which the derivatives of the Minowski question mark function are 0 or infinity were studied ([5]). The differentiability points and non-differentiability points of the Minkowski question mark function are closely related to the Stern-Brocot intervals ([3]). For the study of Stern-Brocot intervals, they used the n-th convergent quotient related ratio ([3])

$$l_1(x) = \lim_{n\to\infty} \frac{2\log q_n(x)}{\sum_{i=1}^{n} a_i(x)} \tag{1}$$

for the continued fraction $x \in (0,1)$, where the n-th convergent quotient ([6])

$$p_n(x)/q_n(x) = [a_1(x), a_2(x), a_3(x), \ldots, a_n(x)] = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \cdots + \cfrac{1}{a_n(x)}}}},$$

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for

$$x = [a_1(x), a_2(x), a_3(x), \ldots] = \cfrac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots}}}.$$

We recall from [6] that the Minkowski question mark function Q is defined by

$$Q(x) = \frac{1}{2^{a_1(x)-1}} - \frac{1}{2^{a_1(x)+a_2(x)-1}} + \cdots + \frac{(-1)^{n+1}}{2^{a_1(x)+a_2(x)+\cdots+a_n(x)-1}} + \cdots$$

for

$$x = [a_1(x), a_2(x), \ldots, a_n(x), \ldots].$$

Lee ([5]) gave the non-trivial examples of differentiability points of the Minkowski question mark function using the simple periodic continued fractions. We note that the end points of the Stern-Brocot intervals are the differentiability points at which the derivative of the Minkowski question mark function is 0. In this paper, using the properties of the eventually simple periodic continued fractions, we give some examples of the differentiability points and the non-differentiability points of the Minkowski question mark function. Finally, we show that our non-trivial examples of the differentiability points give the evidence that recent results [4] are consistent with the earlier results [6]. Further we show that the results ([4, 6]) are closely related comparing the conditions for the differentiability.

2. Preliminaries

From now on, $\mathbb{N}$ denotes the set of the positive integers. It is well-known ([6]) that for each $n \in \mathbb{N}$

$$q_n(x) = a_n q_{n-1}(x) + q_{n-2}(x) \tag{2}$$

with $q_0 = 1$ and $q_{-1} = 0$ for the n-th convergent quotient

$$p_n(x)/q_n(x) = [a_1(x), a_2(x), a_3(x), \ldots, a_n(x)] = \cfrac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \cdots + \frac{1}{a_n(x)}}}},$$

for the continued fraction $x = [a_1(x), a_2(x), a_3(x), \ldots]$ where $a_i(x) \in \mathbb{N}$ for each $i \in \mathbb{N}$. We recall the definition of periodic continued fraction([5])

$$[\overline{a_1, \ldots, a_n}] = [a_1, \ldots, a_n, a_1, \ldots, a_n, a_1, \ldots, a_n, \ldots],$$

satisfying $a_{kn+i} = a_i$ for every non-negative integer k with $1 \leq i \leq n-1$ $n \in \mathbb{N}$. In particular, we call the periodic continued fraction $[\overline{a}]$ the simple periodic continued fraction.

Also we recall the definition of the eventually periodic continued fraction ([5])

$$[b_1, \ldots, b_m, \overline{c_1, \ldots, c_k}] = [b_1, \ldots, b_m, c_1, \ldots, c_k, c_1, \ldots, c_k, c_1, \ldots, c_k, \ldots]$$

for $m, k \in \mathbb{N}$. We generalize the definition of (1) as follows:

$$\underline{l}_1(x) = \liminf_{n\to\infty} \frac{2\log q_n(x)}{\sum_{i=1}^n a_i(x)},$$

$$\overline{l}_1(x) = \limsup_{n\to\infty} \frac{2\log q_n(x)}{\sum_{i=1}^n a_i(x)}.$$

3. Main results

Proposition 1 ([5]). *For each $a \in \mathbb{N}$,*

$$l_1([\overline{a}]) = \frac{2}{a}\log\frac{a+\sqrt{a^2+4}}{2}.$$

It was shown ([5]) that $l_1([\overline{a}]) > \log 2$ for $a = 1, 2, 3, 4$ and $l_1([\overline{a}]) < \log 2$ for the positive integer $a \geq 5$. We have more exact values for the n-th convergent quotient related ratio $l_1(x)$ as follows.

Proposition 2.

$$l_1([\overline{1}]) = \log 2 + 0.269276469559\cdots,$$
$$l_1([\overline{2}]) = \log 2 + 0.18822640646\cdots,$$
$$l_1([\overline{3}]) = \log 2 + 0.103361630965\cdots,$$
$$l_1([\overline{4}]) = \log 2 + 0.0286705570295\cdots.$$

On the other hand,

$$l_1([\overline{5}]) = \log 2 - 0.0342547220115\cdots,$$
$$l_1([\overline{6}]) = \log 2 - 0.0869983608159\cdots,$$
$$l_1([\overline{7}]) = \log 2 - 0.131512760089\cdots.$$

Proof. It follows immediately from the above proposition. □

We recall the Binet's formula ([5]) and give its revised form.

Proposition 3. *Let q_{m+n} satisfy the recurrence relation $q_{m+n} = aq_{m+n-1} + bq_{m+n-2}$ for every $m, n \in \mathbb{N}$ with $q_{m-1} = E$, $q_m = C$ and $q_{m+1} = D$. Assume that the equation $r^2 - ar - b = 0$ has the distinct solutions. Then $q_{m+n} = A\lambda_1^n + B\lambda_2^n$ where λ_1, λ_2 are the distinct solutions of the equation $r^2 - ar - b = 0$ and A, B satisfy the initial conditions $q_m = C$ and $q_{m+1} = D$.*

Proof. It follows from the same arguments of the proof of the Binet's formula. □

The following theorem is essential for our main result.

Theorem 4. *For each $a \in \mathbb{N}$,*

$$l_1([b_1, \ldots, b_m, \overline{a}]) = l_1([\overline{a}]).$$

Proof. Let $x = [b_1, \dots, b_m, \overline{a}]$. From (1) and (2),

$$l_1(x) = \lim_{n\to\infty} \frac{2\log q_{m+n}(x)}{b_1 + \cdots + b_m + \sum_{i=1}^n a_i(x)} = \lim_{n\to\infty} \frac{2\log q_{m+n}(x)}{b_1 + \cdots + b_m + na}.$$

From the above proposition, we have $q_{m+n}(x) = A\lambda_1^n + B\lambda_2^n$ where A, B are constants with $\lambda_1 = \frac{a+\sqrt{a^2+4}}{2}$, $\lambda_2 = \frac{a-\sqrt{a^2+4}}{2}$.

Noting $\lim_{n\to\infty}(\frac{a-\sqrt{a^2+4}}{2})^n = 0$, we have

$$l_1(x) = \lim_{n\to\infty} \frac{2\log(\frac{a+\sqrt{a^2+4}}{2})^n}{b_1 + \cdots + b_m + na} = \lim_{n\to\infty} \frac{2\log(\frac{a+\sqrt{a^2+4}}{2})^n}{na} = l_1([\overline{a}]). \qquad \square$$

We recall the following fundamental theorem for the information of the differentiability and non-differentiability of the Minkowski question mark function.

Proposition 5 ([4]). *Let $Q(x)$ be the Minkowski question mark function of x. Then we have the following results.*

(i) *If $l_1(x) > \log 2$, then $Q'(x) = \infty$.*

(ii) *If $l_1(x) < \log 2$, then $Q'(x) = 0$.*

(iii) *If $\underline{l}_1(x) < \log 2 < \overline{l}_1(x)$, then $Q'(x)$ does not exist.*

We give some examples of the differentiability points of the Minkowski question mark function. From now on, we assume that $Q(x)$ is the Minkowski question mark function of x.

Example 1. Let $x = [a_1(x), a_2(x), a_3(x), \dots]$ where $a_i(x) \in \mathbb{N}$ with $\sum_{i=1}^n a_i(x) \le 1.3n$ for each $n \in \mathbb{N}$. $q_n(x) \ge q_n([\overline{1}])$ and $\sum_{i=1}^n a_i(x) \le 1.3n$ gives

$$\frac{2\log q_n(x)}{\sum_{i=1}^n a_i(x)} \ge \frac{2\log q_n([\overline{1}])}{1.3n},$$

which also gives

$$\underline{l}_1(x) - \log 2 \ge \lim_{n\to\infty} \frac{2\log(\frac{1+\sqrt{1^2+4}}{2})^n}{1.3n} - \log 2 = 0.0471787041471\cdots.$$

If $\underline{l}_1(x) = \overline{l}_1(x)$, then $Q'(x) = \infty$ from Proposition 5(i).

Example 2. Let $x = [a_1(x), a_2(x), a_3(x), \dots]$ where

$$a_i(x) \in \{1, 2, 3, 4, 5, 6\}$$

with $\sum_{i=1}^n a_i(x) \ge 5.5n$ for each $n \in \mathbb{N}$. $q_n(x) \le q_n([\overline{6}])$ and $\sum_{i=1}^n a_i(x) \ge 5.5n$ gives

$$\frac{2\log q_n(x)}{\sum_{i=1}^n a_i(x)} \le \frac{2\log q_n([\overline{6}])}{5.5n},$$

which also gives

$$\overline{l}_1(x) - \log 2 \le \lim_{n\to\infty} \frac{2\log(\frac{6+\sqrt{6^2+4}}{2})^n}{5.5n} - \log 2 = -0.0318939226574\cdots.$$

If $\underline{l}_1(x) = \overline{l}_1(x)$, then $Q'(x) = 0$ from Proposition 5(ii).

Using the above Proposition 5(iii), we construct the non-differentiability point x as follows, which is our main result.

Example 3. Let $x = [a_1(x), a_2(x), a_3(x), \ldots]$. Let $a \in \{1, 2, 3, 4\}$ and $b \in \mathbb{N} - \{1, 2, 3, 4\}$. Then $l_1([\overline{b}]) < \log 2$ and $l_1([\overline{a}]) > \log 2$ from Proposition 2. Let

$$\epsilon = \frac{\min\{\log 2 - l_1([\overline{b}]), l_1([\overline{a}]) - \log 2\}}{2} > 0.$$

Evidently we see that there is an integer N_1 such that

$$\frac{2 \log q_{N_1}(x)}{\sum_{i=1}^{N_1} a_i(x)} < l_1([\overline{b}]) + \epsilon < \log 2$$

for $x = [\overbrace{b, \ldots, b}^{N_1}, \ldots]$.

Theorem 4 gives an integer N_2 such that

$$\frac{2 \log q_{N_1+N_2}(x)}{\sum_{i=1}^{N_1+N_2} a_i(x)} > l_1([\overline{a}]) - \epsilon > \log 2,$$

for $x = [\overbrace{b, \ldots, b}^{N_1}, \overbrace{a, \ldots, a}^{N_2}, \ldots]$.

Theorem 4 also gives an integer N_3 such that

$$\frac{2 \log q_{N_1+N_2+N_3}(x)}{\sum_{i=1}^{N_1+N_2+N_3} a_i(x)} < l_1([\overline{b}]) + \epsilon < \log 2,$$

for $x = [\overbrace{b, \ldots, b}^{N_1}, \overbrace{a, \ldots, a}^{N_2}, \overbrace{b, \ldots, b}^{N_3}, \ldots]$.

Similarly we see that there is an integer N_4 such that

$$\frac{2 \log q_{N_1+N_2+N_3+N_4}(x)}{\sum_{i=1}^{N_1+N_2+N_3+N_4} a_i(x)} > l_1([\overline{a}]) - \epsilon > \log 2$$

for $x = [\overbrace{b, \ldots, b}^{N_1}, \overbrace{a, \ldots, a}^{N_2}, \overbrace{b, \ldots, b}^{N_3}, \overbrace{a, \ldots, a}^{N_4}, \ldots]$. Continuing these processes, we get $x = [a_1(x), a_2(x), a_3(x), \ldots]$ satisfying

$$\underline{l}_1(x) \leq l_1([\overline{b}]) + \epsilon < \log 2 < l_1([\overline{a}]) - \epsilon \leq \overline{l}_1(x).$$

Hence $Q'(x)$ does not exist from (iii) of the above proposition.

Proposition 6 ([6]). *Let $Q(x)$ be the Minkowski question mark function of x. Assume that $Q'(x)$ exists for $x = [a_1(x), a_2(x), a_3(x), \ldots]$. Then we have*:

(i) $\limsup_{n\to\infty} \frac{\sum_{i=1}^{n} a_i(x)}{n} < 2\frac{\log \gamma}{\log 2} = 1.38848\cdots$, *where the golden mean $\gamma = (1 + \sqrt{5})/2$ implies $Q'(x) = \infty$.*

(ii) $\liminf_{n\to\infty} \frac{\sum_{i=1}^{n} a_i(x)}{n} > \beta = 5.31972\cdots$, *where β is the solution of the equation $2\frac{\log(1+x)}{\log 2} - x = 0$ implies $Q'(x) = 0$.*

Remark 1. The above results of the examples of differentiability points are consistent with the above proposition in the sense that, for example,

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n a_i([b_1, \dots, b_m, \overline{1}])}{n} = 1 < 1.38848\cdots,$$

implies $Q'([b_1, \dots, b_m, \overline{1}]) = \infty$ and

$$\lim_{n\to\infty} \frac{\sum_{i=1}^n a_i([b_1, \dots, b_m, \overline{6}])}{n} = 6 > 5.31972\cdots$$

implies $Q'([b_1, \dots, b_m, \overline{6}]) = 0$.

Remark 2. Consider $x = [a_1(x), a_2(x), a_3(x), \dots]$ where $a_i(x) \in \mathbb{N}$ with $\sum_{i=1}^n a_i(x) \le 2\frac{\log\gamma}{\log 2} n$ where the golden mean $\gamma = (1+\sqrt{5})/2$ for each $n \in \mathbb{N}$. Then $q_n(x) \ge q_n([\overline{1}])$ and $\sum_{i=1}^n a_i(x) \le 2\frac{\log\gamma}{\log 2} n$ gives

$$\frac{2\log q_n(x)}{\sum_{i=1}^n a_i(x)} \ge \frac{2\log q_n([\overline{1}])}{2\frac{\log\gamma}{\log 2} n},$$

which also gives

$$\underline{l}_1(x) - \log 2 \ge \lim_{n\to\infty} \frac{2\log(\frac{1+\sqrt{1^2+4}}{2})^n}{2\frac{\log\gamma}{\log 2} n} - \log 2 = 0.$$

This implies that we need the constant $0 < C < 2\frac{\log\gamma}{\log 2}$ for example $C = 1.3$ in Example 1, for us to apply Proposition 5(i) to the example to get the information $Q'(x) = \infty$. In this respect, we see that Proposition 5(i) is closely related to Proposition 6(i).

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