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CONVOLUTION THEOREMS FOR FRACTIONAL FOURIER COSINE AND SINE TRANSFORMS AND THEIR EXTENSIONS TO BOEHMIANS

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Abstract. By introducing two fractional convolutions, we obtain the convolution theorems for fractional Fourier cosine and sine transforms. Applying these convolutions, we construct two Boehmian spaces and then we extend the fractional Fourier cosine and sine transforms from these Boehmian spaces into another Boehmian space with desired properties.

1. Introduction

Let $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of all natural, real and complex numbers respectively. The Banach space of all Lebesgue measurable complex valued functions f on $[0,\infty)$ satisfying $\|f\|_p = \left(\int_0^\infty |f(t)|^p\, dt\right)^{\frac{1}{p}} < \infty$, is denoted by L^p_+, where $p = 1, 2$. After the introduction of fractional Fourier transform [11], many integral transforms have been generalized as the corresponding fractional integral transforms. In particular, fractional Fourier cosine transform (frfct), fractional Fourier sine transform (frfst) and Fractional Hartley transform were defined and used extensively in signal processing. See [2, 16]. We now recall the definitions of frfct and frfst of $f \in L^1_+$ from [2].

$$(F^\alpha_C(f))(u) = c_\alpha \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) e^{ia_\alpha(x^2+u^2)} \cos(b_\alpha ux) dx,\ u \in [0,\infty),$$

$$(F^\alpha_S(f))(u) = c_\alpha \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) e^{ia_\alpha(x^2+u^2)} \sin(b_\alpha ux) dx,\ u \in [0,\infty),$$

where $a_\alpha = \frac{\cot\alpha}{2}$, $b_\alpha = \frac{1}{\sin\alpha}$ and $c_\alpha = \frac{e^{i\alpha/2}}{\sqrt{i\sin\alpha}}$.

Analogous to the Plancherel theorem for Fourier transform [22, p. 186], we can define the fractional Fourier cosine (sine) transform of $f \in L^2_+$ by L^2-$\lim_{n\to\infty} F^\alpha_C(f_n)$ (L^2-$\lim_{n\to\infty} F^\alpha_S(f_n)$), where (f_n) is a sequence from $L^1_+ \cap L^2_+$, such that $f_n \to f$ in L^2_+ as $n \to \infty$. The existence of (f_n) is possible by the

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fact that $L_+^1 \cap L_+^2$ is dense in L_+^2 and the existence of L^2-$\lim_{n\to\infty} F_C^\alpha(f_n)$ (L^2-$\lim_{n\to\infty} F_S^\alpha(f_n)$) follows from the identity $\|f\|_2 = \|F_C^\alpha f\|_2$ ($\|f\|_2 = \|F_S^\alpha f\|_2$), $\forall f \in L_+^1 \cap L_+^2$. We refer the reader to [2, Eqn. (19)], for the Parseval's identity for FRFCT and FRFST, which implies the above identities. Thus fractional Fourier cosine and sine transforms become isometries from L_+^2 onto itself with self inverse.

The convolution theorems for Fourier sine and cosine transforms were first studied in [23] and then they are generalized by various researchers in [4, 24, 25, 26, 27]. Motivated by the convolutions discussed in [23, 28], we introduce two convolutions, denoted by $*_c^\alpha$ and $*_{sc}^\alpha$, which are suitable for discussing the convolution theorems for the fractional Fourier cosine and sine transforms. Using these convolutions, we construct suitable Boehmian spaces $\mathscr{B}_C^\alpha$ and $\mathscr{B}_S^\alpha$, which are properly larger than L_+^2. Further, we extend the FRFCT and FRFST to these Boehmian spaces and we also prove that the extended fractional Fourier cosine and sine transforms are well-defined, consistent with classical FRFCT and FRFST, linear, one-to-one and continuous. Thus, this work generalizes the convolution theorems for Fourier cosine and sine transforms in [23], extends the FRFCT and FRFST in [2] to the context of Boehmians, and also generalizes the Fourier cosine and sine transforms on Boehmians in [21, Section 3].

The concept of Boehmian space was first introduced by J. Mikusiński and P. Mikusiński [5], which is, in general, a generalization of the space of distributions. This generalization motivates many researchers to extend the theory of integral transforms to the context of Boehmians (see [1, 3, 7, 9, 10, 12, 13, 14, 15, 17, 18, 19, 20, 21, 28, 29]).

Before ending this section, we briefly recall the construction of Boehmians from [6, 8]. An abstract Boehmian space is, in general, denoted by $\mathscr{B} = \mathscr{B}(G, (S, \cdot), \odot, \Delta)$, where G is a topological vector space over $\mathbb{C}$, $(S, \cdot)$ is a commutative semi-group, $\odot : G \times S \to G$ satisfies the following conditions:

- $(g_1 + g_2) \odot s = g_1 \odot s + g_2 \odot s$, $\forall g_1, g_2 \in G$ and $\forall s \in S$.
- $(cg) \odot s = c(g \odot s)$, $\forall c \in \mathbb{C}$, $\forall g \in G$ and $\forall s \in S$.
- $g \odot (s \cdot t) = (g \odot s) \odot t$, $\forall g \in G$ and $\forall s, t \in S$.
- If $g_n \to g$ as $n \to \infty$ in G and $s \in S$, then $g_n \odot s \to g \odot s$ as $n \to \infty$,

and Δ is a collection of sequences from S with the following properties:

- If $(s_n), (t_n) \in \Delta$, then $(s_n \cdot t_n) \in \Delta$.
- If $g_n \to g$ as $n \to \infty$ in G and $(s_n) \in \Delta$, then $g_n \odot s_n \to g$ as $n \to \infty$ in G.

If $g_n \in G$, $\forall n \in \mathbb{N}$ and $(s_n) \in \Delta$ are such that $g_n \odot s_m = g_m \odot s_n$, $\forall m, n \in \mathbb{N}$, then the pair of sequences $((g_n), (s_n))$ is called a quotient and is denoted by $\frac{g_n}{s_n}$. The equivalence class $\left[\frac{g_n}{s_n}\right]$ containing $\frac{g_n}{s_n}$ induced by the equivalence relation $\sim$, which is defined on the collection of all quotients by

$$\frac{g_n}{s_n} \sim \frac{h_n}{t_n} \text{ if } g_n \odot t_m = h_m \odot s_n, \ \forall m, n \in \mathbb{N}$$

is called a Boehmian and the collection $\mathscr{B}$ of all Boehmians is a vector space with respect to the following addition and scalar multiplication:

$$\left[\frac{g_n}{s_n}\right]+\left[\frac{h_n}{t_n}\right]=\left[\frac{g_n\odot t_n+h_n\odot s_n}{s_n\cdot t_n}\right],\ c\left[\frac{g_n}{s_n}\right]=\left[\frac{cg_n}{s_n}\right].$$

Every member $g\in G$ can be uniquely identified as a member of $\mathscr{B}$ by $\left[\frac{g\odot s_n}{s_n}\right]$, where $(s_n)\in\Delta$ is arbitrary and the operation $\odot$ is also extended to $\mathscr{B}\times S$ by $\left[\frac{g_n}{s_n}\right]\odot t=\left[\frac{g_n\odot t}{s_n}\right]$. There are two notions of convergence on $\mathscr{B}$ namely δ-convergence and Δ-convergence, which are defined as follows.

Definition ([6])**.** We write that $X_m \overset{\delta}{\to} X$ as $m\to\infty$ in $\mathscr{B}$, if there exist $g_{m,n}, g_n\in G$, $m,n\in\mathbb{N}$ and $(s_n)\in\Delta$ such that $X_m=\left[\frac{g_{m,n}}{s_n}\right]$, $X=\left[\frac{g_n}{s_n}\right]$ and for each $n\in\mathbb{N}$, $g_{m,n}\to g_n$ as $m\to\infty$ in G.

Definition ([6])**.** We write that $X_m \overset{\Delta}{\to} X$ as $m\to\infty$ in $\mathscr{B}$, if there exists $(s_n)\in\Delta$ such that $(X_m-X)\odot s_m\in G$ $\forall m\in\mathbb{N}$ and $(X_m-X)\odot s_m\to 0$ as $m\to\infty$ in G. This means that there exist $g_m\in G$, $\forall m\in\mathbb{N}$ and $(s_n)\in\Delta$ such that $(X_m-X)\odot s_m=\left[\frac{g_m\odot s_n}{s_n}\right]$ and $g_m\to 0$ as $m\to\infty$ in G.

2. Convolution for fractional Fourier cosine and sine transforms

In this section, we introduce two special convolutions and prove all the preliminary results required for constructing the Boehmian spaces $\mathscr{B}_C^\alpha$ and $\mathscr{B}_S^\alpha$.

Definition. For $f,g\in L_+^1$ and $x\in[0,\infty)$,

(i) The convolution $*_c^\alpha$ is defined by
$(f*_c^\alpha g)(x)=\frac{c_\alpha}{\sqrt{2\pi}}\int_0^\infty g(y)e^{\beta y^2}[f(x+y)e^{\beta xy}+f(|x-y|)e^{-\beta xy}]dy.$

(ii) The convolution $*_{sc}^\alpha$ is defined by
$(f*_{sc}^\alpha g)(x)=\frac{c_\alpha}{\sqrt{2\pi}}\int_0^\infty f(y)e^{\beta y^2}[g(|x-y|)e^{-\beta xy}-g(x+y)e^{\beta xy}]dy,$

where $\beta=2ia_\alpha$.

It is easy to verify the following two inequalities:

$$\|f*_c^\alpha g\|_1\le c_\alpha\sqrt{\frac{2}{\pi}}\|f\|_1\|g\|_1 \text{ and } \|f*_{sc}^\alpha g\|_1\le c_\alpha\sqrt{\frac{2}{\pi}}\|f\|_1\|g\|_1.$$

Lemma 2.1. *If $f,g\in L_+^1$, then $f*_c^\alpha g=g*_c^\alpha f$.*

Proof. Let $f,g\in L_+^1$ and let $x\in[0,\infty)$. If $\beta=2ia_\alpha$, then

$$\begin{aligned}&\frac{\sqrt{2\pi}}{c_\alpha}(f*_c^\alpha g)(x)\\ &=\int_0^\infty g(y)e^{\beta y^2}[f(x+y)e^{\beta xy}+f(|x-y|)e^{-\beta xy}]dy\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty g(y)f(x+y)e^{\beta(y^2+xy)}\,dy + \int_0^\infty g(y)f(|x-y|)e^{\beta(y^2-xy)}dy\\
&= \int_x^\infty g(z-x)f(z)e^{\beta[(z-x)^2+x(z-x)]}\,dz\\
&\quad + \int_{-\infty}^x g(x-z)f(|z|)e^{\beta[(x-z)^2-x(x-z)]}dz\\
&= \int_x^\infty g(z-x)f(z)e^{\beta[z^2-zx]}\,dz + \int_0^\infty g(x+z)f(z)e^{\beta[z^2+zx]}dz\\
&\quad + \int_0^x g(x-z)f(z)e^{\beta[z^2-zx]}dz\\
&= \int_0^\infty f(z)e^{\beta z^2}[g(|z-x|)e^{-\beta zx} + g(x+z)e^{\beta zx}]dz = \frac{\sqrt{2\pi}}{c_\alpha}(g *_c^\alpha f)(x)
\end{aligned}
$$

and hence $f *_c^\alpha g = g *_c^\alpha f$. □

Lemma 2.2. *If f, g and $h \in L_+^1$, then $f *_c^\alpha (g *_c^\alpha h) = (f *_c^\alpha g) *_c^\alpha h$.*

Proof. For $x \in [0,\infty)$,

(1)

$$
\begin{aligned}
&[(f *_c^\alpha g) *_c^\alpha h](x)\\
&= \int_0^\infty [(f *_c^\alpha g)(x+z)e^{\beta xz} + (f *_c^\alpha g)(|x-z|)e^{-\beta xz}]h(z)e^{\beta z^2}dz\\
&= \int_0^\infty h(z)e^{\beta(z^2+xz)}(f *_c^\alpha g)(x+z)dz + \int_0^\infty h(z)e^{\beta(z^2-xz)}(f *_c^\alpha g)(|x-z|)dz\\
&= \int_0^\infty h(z)e^{\beta(z^2+xz)}I(x,z)dz + \int_0^\infty h(z)e^{\beta(z^2-xz)}J(x,z)dz,
\end{aligned}
$$

where

$$
\begin{aligned}
I(x,z) &= \int_0^\infty g(u)e^{\beta u^2}[f(x+z+u)e^{\beta(x+z)u} + f(|x+z-u|)e^{-\beta(x+z)u}du;\\
J(x,z) &= \int_0^\infty g(u)e^{\beta u^2}[f(|x-z|+u)e^{\beta|x-z|u} + f(||x-z|-u|)e^{-\beta|x-z|u}du.
\end{aligned}
$$

Since

$$
\begin{aligned}
&\int_0^\infty h(z)e^{\beta(z^2-xz)}J(x,z)dz\\
&= \int_0^\infty h(z)e^{\beta(z^2-xz)}\int_0^\infty g(u)e^{\beta u^2}[f(|x-z+u|)e^{\beta(x-z)u}\\
&\quad + f(|x-z-u|)e^{-\beta(x-z)u}du\,dz,
\end{aligned}
$$

the equation (1) becomes

$$[(f *_c^\alpha g) *_c^\alpha h](x)$$

$$= \int_0^\infty h(z)e^{\beta(z^2+xz)} \int_0^\infty g(u)e^{\beta u^2}[f(x+z+u)e^{\beta(x+z)u}$$
$$+ f(|x+z-u|)e^{-\beta(x+z)u} du\, dz$$
$$+ \int_0^\infty h(z)e^{\beta(z^2-xz)} \int_0^\infty g(u)e^{\beta u^2}[f(|x-z+u|)e^{\beta(x-z)u}$$
$$+ f(|x-z-u|)e^{-\beta(x-z)u} du\, dz$$
$$= \int_0^\infty h(z)e^{\beta z^2} \Big\{ e^{-\beta xz} \int_z^\infty f(|x+u-z|)g(u)e^{\beta(u^2+xu-uz)} du$$
$$+ \Big[e^{\beta xz} \Big(\int_0^z f(|x+z-u|)g(u)e^{\beta(u^2-xu-uz)} du$$
$$+ \int_0^\infty f(x+u+z)g(u)e^{\beta(u^2+xu+uz)} du \Big) \Big]$$
$$+ e^{\beta xz} \int_z^\infty f(|x+z-u|)g(u)e^{\beta(u^2-xu-uz)} du$$
$$+ \Big[e^{-\beta xz} \Big(\int_0^\infty f(|x-u-z|)g(u)e^{\beta(u^2+uz-xu)} du$$
$$+ \int_0^z f(|x+u-z|)g(u)e^{\beta(u^2+xu-uz)} du \Big) \Big] \Big\}$$
$$= \int_0^\infty e^{\beta y^2}[f(x+y)e^{\beta xy} + f(|x-y|)e^{-\beta xy}] \int_0^\infty h(z)e^{\beta z^2}[g(y+z)e^{\beta yz}$$
$$+ g(|y-z|)e^{-\beta yz}] dz\, dy$$
$$= [f *_c^\alpha (g *_c^\alpha h)](x).$$

Since $x \in [0, \infty)$ is arbitrary, the proof follows. □

In the following sequel, the well known inequality $(a+b)^2 \le 2(a^2+b^2)$, $\forall a, b \ge 0$, will be used at many places, without quoting it, explicitly.

Lemma 2.3. *If $f \in L_+^2$ and $g \in L_+^1$, then $\|f *_c^\alpha g\|_2 \le |c_\alpha|\sqrt{\frac{2}{\pi}}\|f\|_2\, \|g\|_1$, and hence $f *_c^\alpha g \in L_+^2$.*

Proof. By using Jensen's inequality and Fubini's theorem, we obtain that

$$\|f *_c^\alpha g\|_2^2 = \int_0^\infty \left|(f *_c^\alpha g)(x)\right|^2\, dx$$
$$= \int_0^\infty \left| \frac{c_\alpha}{\sqrt{2\pi}} \int_0^\infty g(y)e^{\beta y^2}[f(x+y)e^{\beta xy} + f(|x-y|)e^{-\beta xy}]dy \right|^2 dx$$
$$\le \frac{\|g\|_1^2|c_\alpha|^2}{2\pi} \int_0^\infty \left(\int_0^\infty |g(y)||f(x+y)e^{\beta xy} + f(|x-y|)e^{-\beta xy}|^2 \frac{dy}{\|g\|_1} \right) dx$$
$$\le \frac{\|g\|_1|c_\alpha|^2}{\pi} \int_0^\infty |g(y)| \left(\int_0^\infty (|f(x+y)|^2 + |f(|x-y|)|^2) dx \right) dy,$$

$$
\begin{aligned}
&\text{(using the convexity of } \zeta \mapsto \zeta^2 \text{ on } [0,\infty))\\
&\le \frac{\|g\|_1 |c_\alpha|^2}{\pi} \int_0^\infty |g(y)| \left(\int_y^\infty |f(z)|^2 dz + \int_{-y}^\infty |f(|z|)|^2 dz \right) dy\\
&\le \frac{2\|g\|_1 |c_\alpha|^2}{\pi} \int_0^\infty |g(y)|\, \|f\|_2^2 \, dy \le \frac{2|c_\alpha|^2}{\pi} \|g\|_1^2 \, \|f\|_2^2.
\end{aligned}
$$

Thus $\|f *_c^\alpha g\|_2 \le |c_\alpha| \sqrt{\frac{2}{\pi}} \|g\|_1 \, \|f\|_2$, which completes the proof. □

The following theorem is an immediate consequence of the inequality proved in the previous theorem.

Theorem 2.4. *If $f_n \to f$ as $n \to \infty$ in L^2_+ and $g \in L^1_+$, then $f_n *_c^\alpha g \to f *_c^\alpha g$ as $n \to \infty$ in L^2_+.*

Lemma 2.5. *If $f \in L^2_+$ and if $g, h \in L^1_+$, then $f *_c^\alpha (g *_c^\alpha h) = (f *_c^\alpha g) *_c^\alpha h$.*

Proof. Choose a sequence $f_n \in L^1_+ \cap L^2_+$ such that $f_n \to f$ as $n \to \infty$ in L^2_+. By using Lemma 2.2 and Theorem 2.4, we obtain that

$$
f *_c^\alpha (g *_c^\alpha h) = L^2\text{-}\lim f_n *_c^\alpha (g *_c^\alpha h) = L^2\text{-}\lim (f_n *_c^\alpha g) *_c^\alpha h = (f *_c^\alpha g) *_c^\alpha h.
$$

Hence the theorem follows. □

Theorem 2.6 (Convolution theorems)**.** *If $f, g \in L^1_+$ and $u \in [0,\infty)$, then*

(i) $[F_C^\alpha(f *_c^\alpha g)](u) = e^{-ia_\alpha u^2} [F_C^\alpha(f)](u) [F_C(g)](u)$.
(ii) $[F_S^\alpha(f *_{sc}^\alpha g)](u) = e^{-ia_\alpha u^2} [F_S^\alpha(f)](u) [F_C^\alpha(g)](u)$.

Proof. For $f, g \in L^1_+$ and $u \in [0,\infty)$,
(i)

$$
\begin{aligned}
&[F_C^\alpha(f *_c^\alpha g)](u)\\
&= c_\alpha \sqrt{\frac{2}{\pi}} \int_0^\infty (f *_c^\alpha g)(x) e^{ia_\alpha (x^2+u^2)} \cos(b_\alpha u x) dx\\
&= \frac{c_\alpha^2}{\pi} \int_0^\infty g(y) e^{ia_\alpha (2y^2+u^2)} \{ \int_0^\infty [f(x+y) e^{ia_\alpha (x^2+2xy)}\\
&\quad + f(|x-y|) e^{ia_\alpha (x^2-2xy)}] \cos(b_\alpha u x) dx \} dy\\
&= \frac{c_\alpha^2}{\pi} \int_0^\infty g(y) e^{ia_\alpha (u^2+y^2)} \{ \int_0^\infty [f(x+y) e^{ia_\alpha (x+y)^2}\\
&\quad + f(|x-y|) e^{ia_\alpha (x-y)^2}] \cos(b_\alpha u x) dx \} dy\\
&= \frac{c_\alpha^2}{\pi} \int_0^\infty g(y) e^{ia_\alpha (u^2+y^2)} \{ \int_y^\infty f(z) e^{ia_\alpha z^2} \cos(b_\alpha u (z-y)) dz\\
&\quad + \int_{-y}^\infty f(|z|) e^{ia_\alpha z^2} \cos(b_\alpha u (z+y)) dz \} dy
\end{aligned}
$$

$$= \frac{c_\alpha^2}{\pi}\int_0^\infty g(y)e^{ia_\alpha(u^2+y^2)}\{\int_0^\infty f(z)e^{ia_\alpha z^2}\cos(b_\alpha u(z-y))dz$$
$$+\int_0^\infty f(z)e^{ia_\alpha z^2}\cos(b_\alpha u(z+y))dz\}dy$$
$$= c_\alpha\sqrt{\frac{2}{\pi}}\int_0^\infty g(y)e^{ia_\alpha(u^2+y^2)}\int_0^\infty f(z)e^{ia_\alpha z^2}\cos(b_\alpha uz)\cos(b_\alpha uy)dzdy$$
$$= e^{-ia_\alpha u^2}[F_C^\alpha(f)](u)\cdot[F_C^\alpha(g)](u).$$

(ii)

$$[F_S^\alpha(f *_{sc}^\alpha g)](u)$$
$$= c_\alpha\sqrt{\frac{2}{\pi}}\int_0^\infty (f *_{sc}^\alpha g)(x)e^{ia_\alpha(x^2+u^2)}\sin(b_\alpha ux)dx$$
$$= \frac{c_\alpha^2}{2}\frac{2}{\pi}\int_0^\infty f(y)e^{ia_\alpha(u^2+2y^2)}\{\int_0^\infty [g(|x-y|)e^{ia_\alpha(x^2-2xy)}$$
$$- g(x+y)e^{ia_\alpha(x^2+2xy)}]\sin(b_\alpha ux)dx\}dy$$
$$= \frac{c_\alpha^2}{2}\frac{2}{\pi}\int_0^\infty f(y)e^{ia_\alpha(u^2+y^2)}\{\int_0^\infty [g(|x-y|)e^{ia_\alpha(x-y)^2}$$
$$- g(x+y)e^{ia_\alpha(x+y)^2}]\sin(b_\alpha ux)dx\}dy$$
$$= \frac{c_\alpha^2}{2}\frac{2}{\pi}\int_0^\infty f(y)e^{ia_\alpha(u^2+y^2)}\{\int_{-y}^\infty g(|z|)e^{ia_\alpha z^2}\sin(b_\alpha u(z+y))dz$$
$$- \int_y^\infty g(z)e^{ia_\alpha z^2}\sin(b_\alpha u(z-y))dz\}dy$$
$$= \frac{c_\alpha^2}{2}\frac{2}{\pi}\int_0^\infty f(y)e^{ia_\alpha(u^2+y^2)}\{\int_0^\infty g(z)e^{ia_\alpha z^2}\sin(b_\alpha(uz+uy))dz$$
$$- \int_0^\infty g(z)e^{ia_\alpha z^2}\sin(b_\alpha(uz-uy))dz\}dy$$
$$= c_\alpha^2\frac{2}{\pi}\int_0^\infty f(y)e^{ia_\alpha y^2}\sin(b_\alpha uy)\int_0^\infty g(z)e^{ia_\alpha(z^2+u^2)}\cos(b_\alpha uz)dzdy$$
$$= e^{-ia_\alpha u^2}[F_S^\alpha(f)](u)[F_C^\alpha(g)](u).$$

Hence the theorem follows. □

Theorem 2.7 (Convolution theorems on L_+^2)**.** *If $f \in L_+^2$, $g \in L_+^1$ and $u \in [0,\infty)$, then*

(i) $[F_C^\alpha(f *_c^\alpha g)](u) = e^{-ia_\alpha u^2}[F_C^\alpha(f)](u)[F_C(g)](u)$.
(ii) $[F_S^\alpha(f *_{sc}^\alpha g)](u) = e^{-ia_\alpha u^2}[F_S^\alpha(f)](u)[F_C^\alpha(g)](u)$.

Proof. Using Theorem 2.6, this proof follows from the facts that both FRFCT and FRFST are continuous from L_+^2 onto L_+^2 and $L_+^1 \cap L_+^2$ is dense in L_+^2. □

Definition. A sequence (ϕ_n) in L^2_+ is called a δ-sequence if it satisfies the following conditions:

($\Delta 1$) $c_\alpha\sqrt{\frac{2}{\pi}}\int_0^\infty e^{ia_\alpha y^2}\phi_n(y)\,dy = 1$, $\forall n \in \mathbb{N}$.
($\Delta 2$) $\int_0^\infty |\phi_n(y)|\,dy \leq M$, $\forall n \in \mathbb{N}$, for some $M > 0$.
($\Delta 3$) Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that support of $\phi_n \subseteq (0,\epsilon)$, $\forall n \geq N$.

The collection of all δ-sequences is denoted by $\Delta(\alpha)$.

Lemma 2.8. *If $(\delta_n), (\psi_n) \in \Delta(\alpha)$, then $(\delta_n *_c^\alpha \psi_n) \in \Delta(\alpha)$.*

Proof. Let $(\delta_n), (\psi_n) \in \Delta(\alpha)$. By a routine calculation, we obtain that

$$\int_0^\infty e^{ia_\alpha y^2}[\delta_n *_c^\alpha \psi_n](y)\,dy = c_\alpha\sqrt{\frac{2}{\pi}}\int_0^\infty \delta_n(u)e^{ia_\alpha u^2}du \int_0^\infty \psi_n(u)e^{ia_\alpha u^2}du,$$

which implies that $c_\alpha\sqrt{\frac{2}{\pi}}\int_0^\infty e^{ia_\alpha y^2}[\delta_n *_c^\alpha \psi_n](y)\,dy = 1$, using the property ($\Delta 1$) of $\Delta(\alpha)$. It is easy to very that $\|\delta_n *_c^\alpha \psi_n\|_1 \leq c_\alpha\sqrt{\frac{2}{\pi}}\|\delta_n\|_1\|\psi_n\|_1 < P$, $\forall n \in \mathbb{N}$ and for some $P > 0$. For a given $\epsilon > 0$, we choose $N \in \mathbb{N}$ such that supp δ_n, supp $\psi_n \subset [0, \frac{\epsilon}{2})$ for all $n \geq N$. Using the fact that supp $(\delta_n *_c^\alpha \psi_n) \subset$ [supp δ_n + supp ψ_n] $\cup$ $\{[0,\infty) \cap$ [supp $\delta_n -$ supp ψ_n]$\}$ $\cup$ $\{[0,\infty) \cap$ [supp $\psi_n -$ supp δ_n]$\}$, we get that supp $(\delta_n *_c^\alpha \psi_n) \subset [0, \frac{\epsilon}{2}) + [0, \frac{\epsilon}{2}) = [0, \epsilon)$ for all $n \geq N$. Hence it follows that $(\delta_n *_c^\alpha \psi_n) \in \Delta(\alpha)$. □

Lemma 2.9. *If $f \in L^2_+$ and $(\phi_n) \in \Delta(\alpha)$, then $f *_c \phi_n \to f$ as $n \to \infty$ in L^2_+.*

Proof. Let $\epsilon > 0$ be given. Since $C_c([0,\infty))$ is dense in L^2_+, we can find $g \in C_c([0,\infty))$ such that $\|g - f\|_2 < \epsilon$. If $g_y(x) = g(|x-y|), \forall x \in [0,\infty)$, then the mapping $y \mapsto g_y$ is continuous on $[0,\infty)$ and hence for $0 < \delta < \min\{1, \epsilon^2\}$, we have $\|g_y - g_0\|_2 < \epsilon, \forall y \in [0,\delta)$. Therefore, for each $y \in (0,\delta)$, we have

$$\int_0^\infty |g(x+y) - g(x)|^2 dx < \epsilon^2. \tag{2}$$

Indeed,

$$\begin{aligned}\int_0^\infty |g(x+y) - g(x)|^2 dx &= \int_y^\infty |g(z) - g(z-y)|^2 dz \\ &\leq \int_0^y |g(z) - g(y-z)|^2 dz + \int_0^\infty |g(z) - g(z-y)|^2 dz \\ &= \int_0^\infty |g(z) - g(|z-y|)|^2 dz = \|g_y - g_0\|_2^2 < \epsilon^2.\end{aligned}$$

We choose $N \in \mathbb{N}$ such that $\operatorname{supp}\phi_n \subset [0,\delta)\,\forall n \geq N$. Applying Jensen's inequality and Fubini's theorem, for $n \geq N$, we get that

$$\|(g *_c^\alpha \phi_n) - g\|_2^2$$

$$= \int_0^\infty \left| \frac{c_\alpha}{\sqrt{2\pi}} \int_0^\infty \phi_n(y) e^{\beta y^2} [g(x+y)e^{\beta xy} + g(|x-y|)e^{-\beta xy}] \right.$$
$$\left. -2g(x) e^{i a_\alpha y^2} \phi_n(y)\, dy \right|^2 dx$$
$$\le \int_0^\infty \frac{|c_\alpha^2|}{2\pi} \left(\int_0^\infty \left| g(x+y) e^{i a_\alpha (2xy+y^2)} + g(|x-y|) e^{-i a_\alpha (2xy-y^2)} \right. \right.$$
$$\left. \left. -2g(x) \right| |\phi_n(y)|\, dy \right)^2 dx$$
$$\le \frac{|c_\alpha^2| \|\phi_n\|_1}{2\pi} \int_0^\infty |\phi_n(y)| \int_0^\infty |g(x+y)e^{i a_\alpha (2xy+y^2)} - g(x)$$
$$+ g(|x-y|) e^{-i a_\alpha (2xy-y^2)} - g(x)|^2\, dx\, dy. \tag{3}$$

Now for $0 \le y < \delta$, we have

$$\int_0^\infty |g(x+y)e^{i a_\alpha (2xy+y^2)} - g(x)|^2\, dx$$
$$< 2\epsilon^2 + 2\int_0^\infty |g(x)|\, |e^{i a_\alpha (2xy+y^2)} - 1|^2\, dx, \text{ (using (2))}$$
$$\le 2\epsilon^2 + 2\int_0^\infty |g(x)|\, (|a_\alpha|(2xy+y^2))^2\, dx$$
$$< \epsilon^2 C_1, \quad \text{since } y^2 < y < \delta < \epsilon^2,$$

where $C_1 = 1 + 2|a_\alpha|^2 \int_0^\infty (2x+\delta)^2 |g(x)|\, dx < \infty$. Similarly, we can prove that

$$\int_0^\infty |g(|x-y|)e^{-\beta xy} - g(x)|^2\, dx < \epsilon^2 C_2 \text{ for some } 0 < C_2 < \infty.$$

Using these estimates in (3), we get that

$$\|(g *_c^\alpha \phi_n) - g\|_2^2 < \frac{1}{\pi} M^2 |c_\alpha|^2 \epsilon^2 (C_1 + C_2), \tag{4}$$

where $M > 0$ is such that $\int_0^\infty |\phi_n(x)|\, dx \le M$, $\forall n \in \mathbb{N}$. Thus, using (4), Lemma 2.3 and property $(\Delta 2)$ of (ϕ_n), we have

$$\|f *_c^\alpha \phi_n - f\|_2 \le \|f *_c^\alpha \phi_n - g *_c^\alpha \phi_n\|_2 + \|g *_c^\alpha \phi_n - g\|_2 + \|g - f\|_2 < K\epsilon$$

for some $K > 0$. Hence the lemma follows. □

Theorem 2.10. *If $f_n \to f$ as $n \to \infty$ in L_+^2 and $(\delta_n) \in \Delta(\alpha)$, then $f_n *_c^\alpha \delta_n \to f$ as $n \to \infty$ in L_+^2.*

Proof. Let $f_n, f \in L_+^2$ be such that $f_n \to f$ as $n \to \infty$ in L_+^2 and let $(\delta_n) \in \Delta(\alpha)$. Using Lemma 2.9 and the property $(\Delta 2)$ of (δ_n), we get that

$$\|f_n *_c^\alpha \delta_n - f\|_2 = \|f_n *_c^\alpha \delta_n - f *_c^\alpha \delta_n + f *_c^\alpha \delta_n - f\|_2$$
$$\le M|c_\alpha| \sqrt{\frac{2}{\pi}}\, \|f_n - f\|_2 + \|f *_c^\alpha \delta_n - f\|_2 \to 0$$

as $n \to \infty$. □

Thus, we have proved all auxiliary results required to construct the Boehmian space $\mathscr{B}_C^\alpha = \mathscr{B}_C^\alpha(L^2_+, (L^1_+, *_c^\alpha), *_c^\alpha, \Delta(\alpha))$. We shall denote a typical element of $\mathscr{B}_C^\alpha$ by $X = [(f_n), (\delta_n)]$.

In the following sequel, we obtain some lemmas which are required to construct the Boehmian space $\mathscr{B}_S^\alpha = \mathscr{B}_S^\alpha(L^2_+, (L^1_+, *_c^\alpha), *_{sc}^\alpha, \Delta(\alpha))$.

Lemma 2.11. *If f, g and $h \in L^1_+$, then $(f *_{sc}^\alpha g) *_{sc}^\alpha h = f *_{sc}^\alpha (g *_c^\alpha h)$.*

Proof. For arbitrary $x \in [0, \infty)$,

$$\begin{aligned}
&\frac{2\pi}{c_\alpha^2}[f *_{sc}^\alpha (g *_c^\alpha h)](x)\\
&= \int_0^\infty f(y)e^{\beta(y^2-xy)}\{\int_0^\infty g(z)e^{\beta z^2}[h(|x-y|+z)e^{\beta|x-y|z}\\
&\quad + h(||x-y|-z|)e^{-\beta|x-y|z}]dz\}dy\\
&\quad - \int_0^\infty f(y)e^{\beta(y^2+xy)}\{\int_0^\infty g(z)e^{\beta z^2}[h(x+y+z)e^{\beta(x+y)z}\\
&\quad + h(|x+y-z|)e^{-\beta(x+y)z}]dz\}dy\\
&= \int_0^x f(y)e^{\beta(y^2-xy)}\{\int_0^\infty g(z)e^{\beta z^2}[h(|x-y+z|)e^{\beta(x-y)z}\\
&\quad + h(|x-y-z|)e^{-\beta(x-y)z}]dz\}dy\\
&\quad + \int_x^\infty f(y)e^{\beta(y^2-xy)}\{\int_0^\infty g(z)e^{\beta z^2}[h(|x-y-z|)e^{\beta(y-x)z}\\
&\quad + h(|x-y+z|)e^{-\beta(y-x)z}]dz\}dy\\
&\quad - \int_0^\infty f(y)e^{\beta(y^2+xy)}\{\int_0^\infty g(z)e^{\beta z^2}[h(x+y+z)e^{\beta(x+y)z}\\
&\quad + h(|x+y-z|)e^{-\beta(x+y)z}]dz\}dy\\
&= \int_0^\infty f(y)e^{\beta(y^2-xy)}\{\int_0^\infty g(z)e^{\beta z^2}h(|x-y+z|)e^{\beta(x-y)z}dz\\
&\quad + \int_0^\infty g(z)e^{\beta z^2}h(|x-y-z|)e^{-\beta(x-y)z}dz\}dy\\
&\quad - \int_0^\infty f(y)e^{\beta(y^2+xy)}\{\int_0^\infty g(z)e^{\beta z^2}h(x+y+z)e^{\beta(x+y)z}dz\\
&\quad + \int_0^\infty g(z)e^{\beta z^2}h(|x+y-z|)e^{-\beta(x+y)z}dz\}dy\\
&= \int_0^\infty f(y)e^{\beta y^2}\{\int_0^\infty g(u+y)e^{\beta u(u+y)}e^{\beta xu}h(x+u)du\\
&\quad + \int_0^\infty g(|u-y|)e^{\beta u(u-y)}e^{-\beta xu}h(|x-u|)du\}
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^\infty f(y)e^{\beta y^2}\{\int_0^\infty g(|y-u|)e^{\beta u(u-y)}e^{\beta ux}h(x+u)du\\
&+\int_0^\infty g(u+y)e^{\beta u(u+y)}e^{-\beta ux}h(|x-u|)du\}dy\\
=\ &\int_0^\infty f(y)e^{\beta y^2}\int_0^\infty e^{\beta u^2}\{g(|u-y|)e^{-\beta uy}-g(u+y)e^{\beta uy}\}\\
&\times\{h(|x-u|)e^{-\beta ux}-h(x+u)e^{\beta xu}\}du\,dy\\
=\ &\int_0^\infty\left(\int_0^\infty f(y)e^{\beta y^2}\{g(|u-y|)e^{-\beta uy}-g(u+y)e^{\beta uy}\}dy\right)\\
&\times e^{\beta u^2}\{h(|x-u|)e^{-\beta ux}-h(x+u)e^{\beta xu}\}du\\
=\ &\frac{2\pi}{c_\alpha^2}[(f*_{sc}^\alpha g)*_{sc}^\alpha h](x).
\end{aligned}
$$

Hence the theorem follows. □

Remark 2.12. For $f\in L_+^2, g\in L_+^1$ and $x\in[0,\infty)$, we have $(f*_{sc}^\alpha g)(x)=(f*_c^\alpha g)(x)-c_\alpha\sqrt{\frac{2}{\pi}}\int_x^\infty f(u-x)g(u)e^{\beta(u-x)u}du$.

Lemma 2.13. *For $f\in L_+^2$ and $g\in L_+^1$, $\|f*_{sc}^\alpha g\|_2\le 2|c_\alpha|\sqrt{\frac{2}{\pi}}\|f\|_2\,\|g\|_1$.*

Proof. By previous remark, we have

$$
\|f*_{sc}^\alpha g\|_2\le\|f*_c^\alpha g\|_2+|c_\alpha|\sqrt{\frac{2}{\pi}}\left(\int_0^\infty\left|\int_x^\infty f(u-x)g(u)e^{\beta(u-x)u}du\right|^2dx\right)^{\frac{1}{2}}.
$$

Using Jensen's inequality and Fubini's theorem, we obtain that

$$
\begin{aligned}
&\int_0^\infty|\int_x^\infty f(u-x)g(u)e^{\beta(u-x)u}du|^2dx\\
\le\ &\int_0^\infty\left(\int_x^\infty|f(u-x)g(u)|du\right)^2dx\le\|g\|_1\int_0^\infty\int_x^\infty f(u-x)|^2|g(u)|dudx\\
\le\ &\|g\|_1\int_0^\infty|g(u)|\int_0^u|f(u-x)|^2dx\,du\le\|g\|_1^2\|f\|_2^2.
\end{aligned}
$$

Therefore, using Lemma 2.3 and the above estimate, we obtain that $\|f*_{sc}^\alpha g\|_2\le 2|c_\alpha|\sqrt{\frac{2}{\pi}}\|f\|_2\,\|g\|_1$, and hence $f*_{sc}^\alpha g\in L_+^2$. □

Lemma 2.14. *If $f\in C_c([0,\infty))$ and $(\delta_n)\in\Delta(\alpha)$, then $f*_{sc}^\alpha\delta_n\to f$ as $n\to\infty$ in L_+^2.*

Proof. In view of Remark 2.12 and Lemma 2.9, we have

$$
f*_{sc}^\alpha\delta_n=(f*_c^\alpha\delta_n-f)+c_\alpha\sqrt{\frac{2}{\pi}}\int_x^\infty f(u-x)\delta_n(u)e^{\beta(u-x)u}du
$$

and $f *_c^\alpha \delta_n - f \to 0$ as $n \to \infty$. Therefore, to conclude this proof, we shall show that $\int_x^\infty f(u-x)\delta_n(u)e^{\beta(u-x)u}du \to 0$ in L_+^2 as $n \to \infty$. Let $\epsilon > 0$ be given. We choose $N \in \mathbb{N}$ such that supp $\delta_n \subset [0,\epsilon), \forall n \geq N$. For any $n \geq N$, we have

$$\begin{aligned}
&\int_0^\infty \Big| \int_x^\infty f(u-x)\delta_n(u)e^{\beta(u-x)u}du\Big|^2 dx \\
\leq\ & \int_0^\infty \left(\int_x^\infty |f(u-x)||\delta_n(u)|du \right)^2 dx \\
\leq\ & \int_0^\infty M \int_x^\infty |f(u-x)|^2 |\delta_n(u)| du\, dx, \text{(by Jensen's inequality)} \\
&\quad \text{(Here } M > 0 \text{ is as in the property } (\Delta 2) \text{ of } (\delta_n)) \\
\leq\ & M \int_0^\infty |\delta_n(u)| \int_0^u |f(u-x)|^2 dx\, du, \ \text{(by Fubini's theorem)} \\
\leq\ & M^2 \|f\|_\infty^2 \epsilon, \ \text{where } \|f\|_\infty = \sup_{t\geq 0}|f(t)|.
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the proof follows. □

Lemma 2.15. *If $f \in L_+^2$ and $(\delta_n) \in \Delta(\alpha)$, then $f *_{sc}^\alpha \delta_n \to f$ as $n \to \infty$ in L_+^2.*

Proof. Let $f \in L_+^2$ and $(\delta_n) \in \Delta(\alpha)$. For $\epsilon > 0$, choose $g \in C_c([0,\infty))$ such that $\|f-g\|_2 < \epsilon$. By Lemma 2.14, there is a positive integer N with $\|g *_{sc}^\alpha \delta_n - g\|_2 < \epsilon$ for all $n \geq N$. For any $n \geq N$, we get that

$$\begin{aligned}
&\|f *_{sc}^\alpha \delta_n - f\|_2 \\
\leq\ & |c_\alpha|\sqrt{\frac{2}{\pi}}\|f-g\|_2\ \|\delta_n\|_1 + \|g *_{sc}^\alpha \delta_n - g\|_2 + \|g-f\|_2, \quad \text{(by Lemma 2.13)} \\
\leq\ & M|c_\alpha|\sqrt{\frac{2}{\pi}}\|f-g\|_2 + \|g *_{sc}^\alpha \delta_n - g\|_2 + \|g-f\|_2, \\
&\quad \text{(by property } (\Delta 2) \text{ of } (\delta_n)) \\
<\ & \epsilon M|c_\alpha|\sqrt{\frac{2}{\pi}} + \epsilon + \epsilon = \epsilon\left(M|c_\alpha|\sqrt{\frac{2}{\pi}} + 2 \right).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $f *_{sc}^\alpha \delta_n \to f$ as $n \to \infty$ in L_+^2. □

Theorem 2.16. *If $f_n \to f$ as $n \to \infty$ in L_+^2 and $(\delta_n) \in \Delta(\alpha)$, then $f_n *_{sc}^\alpha \delta_n \to f$ as $n \to \infty$ in L_+^2.*

Proof. As a consequence of Lemma 2.15 and Lemma 2.13, it follows that

$$\|f_n *_{sc}^\alpha \delta_n - f\|_2 \leq M|c_\alpha|\sqrt{\frac{2}{\pi}}\ \|f_n - f\|_2 + \|f *_{sc}^\alpha \delta_n - f\|_2 \to 0$$

as $n \to \infty$, which completes the proof. □

Lemma 2.17. *If $f \in L_+^2$ and if $g, h \in L_+^1$, then $f *_{sc}^\alpha (g *_c^\alpha h) = (f *_{sc}^\alpha g) *_{sc}^\alpha h$.*

Proof. It follows immediately, by using the same technique applied in the proof of Lemma 2.5. $\square$

Now, let $\mathscr{B}_S^\alpha = \mathscr{B}(L_+^2, (L_+^1, *_c^\alpha), *_{sc}^\alpha, \Delta(\alpha))$ and denote a typical element of $\mathscr{B}_S^\alpha$ by $\mathcal{U} = [(f_n)/(\delta_n)]$.

3. Fractional Fourier cosine transform on Boehmians

In this section, first we extend the FRFCT as a map from $\mathscr{B}_C^\alpha$ onto the Boehmian space $\hat{\mathscr{B}}_C^\alpha = \mathscr{B}(L_+^2, (C_0^+ \cap L_+^1, \cdot), \cdot, \Delta_C^\alpha)$, where C_0^+ is the Banach space of complex-valued continuous functions on $[0,\infty)$ vanishing at infinity, with the norm $\|\psi\|_\infty = \sup_{x\geq 0}|\psi(x)|$, '$\cdot$' denotes the usual point-wise multiplication of functions and $\Delta_C^\alpha = \left\{(e^{-ia_\alpha u^2} F_C^\alpha(\phi_n)) : (\phi_n) \in \Delta(\alpha)\right\}$.

Lemma 3.1. *Let $f \in L_+^2$ and $\psi \in C_0^+$. Then $f \cdot \psi \in L_+^2$ and $f_n \cdot \psi \to f \cdot \psi$ as $n \to \infty$ in L_+^2, whenever $f_n \to f$ as $n \to \infty$ in L_+^2.*

Proof. Since $\|f\cdot\psi\|_2 \leq \|f\|_2\|\psi\|_\infty$, the proof of this lemma follows immediately. $\square$

Lemma 3.2. *If $(\delta_n) \in \Delta(\alpha)$, then $e^{-ia_\alpha u^2} F_C^\alpha(\delta_n) \to 1$ as $n \to \infty$ uniformly on each compact subset of $[0,\infty)$.*

Proof. Let K be a compact subset of $[0,\infty)$ and let $\epsilon > 0$ be given. Choose a positive integer N such that $supp\,\delta_n \subset [0,\epsilon)$ for all $n \geq N$. Then for $u \in K$ and $n \geq N$, we have

$$
\begin{aligned}
&|e^{-ia_\alpha u^2}[F_C^\alpha(\delta_n)](u) - 1| \\
&\leq |c_\alpha|\sqrt{\frac{2}{\pi}}\int_0^\epsilon |\delta_n(x)|\,|\cos(b_\alpha ux) - 1|\,dx,\ \forall\, n \geq N \\
&\leq |c_\alpha|\sqrt{\frac{2}{\pi}}\int_0^\epsilon |\delta_n(x)|\,|b_\alpha ux||\sin z|\,dx, \\
&\qquad \text{(by mean-value theorem, such a } z \text{ exists in } (0, |b_\alpha|ux)) \\
&\leq \epsilon M|b_\alpha c_\alpha|\sqrt{\frac{2}{\pi}}\sup_{t\in K}|t|.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, the proof follows. $\square$

Lemma 3.3. *If $f \in L_+^2$ and $(\delta_n) \in \Delta(\alpha)$, then $f \cdot e^{-ia_\alpha u^2} F_C^\alpha(\delta_n) \to f$ as $n \to \infty$ in L_+^2.*

Proof. Let $\epsilon > 0$ be arbitrary. Since $C_c([0,\infty))$ is dense in L_+^2, choose $g \in C_c([0,\infty))$ such that $\|f - g\|_2 < \frac{\epsilon}{2}$. By the property $(\Delta 2)$ of (δ_n), we get that $|[F_C^\alpha(\delta_n)](u)| = |e^{-ia_\alpha u^2}[F_C^\alpha(\delta_n)](u)| \leq \int_0^\infty |\delta_n(x)|dx \leq M, \forall n \in \mathbb{N}$ for some $M > 0$. If $K = \operatorname{supp} g$, then K is compact. Then, using Lemma 3.2, we find

$N \in \mathbb{N}$ such that $|e^{-ia_\alpha y^2} F_C^\alpha(\delta_n)(y) - 1| < \epsilon$, $\forall y \in K\, \forall n \geq N$. Therefore, for any $n \geq N$,

$$\begin{aligned}
&\|f \cdot e^{-ia_\alpha u^2} F_C^\alpha(\delta_n) - f\|_2 \\
&\leq \|(f-g) \cdot e^{-ia_\alpha u^2} F_C^\alpha(\delta_n)\|_2 + \|g \cdot [e^{-ia_\alpha u^2} F_C^\alpha(\delta_n) - 1]\|_2 + \|g - f\|_2 \\
&\leq M\|f-g\|_2 + \{\int_K |g(y)|^2 |e^{-ia_\alpha y^2} F_C^\alpha(\delta_n)](y) - 1|^2 dy\}^{\frac{1}{2}} + \|g - f\|_2 \\
&\leq \epsilon(M + \|g\|_2 + 1).
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the proof follows. □

Lemma 3.4. *Let $f_n \to f$ as $n \to \infty$ in L^2_+ and $(\delta_n) \in \Delta(\alpha)$. Then $f_n \cdot e^{-ia_\alpha y^2} F_C^\alpha(\delta_n) \to f$ as $n \to \infty$ in L^2_+.*

Proof. Proof of this lemma is similar to that of Lemma 2.10, which can obtained by using Lemmas 3.1 and 3.3. □

Lemma 3.5. *If $(\delta_n), (\phi_n) \in \Delta(\alpha)$, then $(e^{-ia_\alpha y^2} F_C^\alpha(\delta_n) \cdot e^{-ia_\alpha y^2} F_C^\alpha(\phi_n)) \in \Delta_C^\alpha$.*

Proof. It follows immediately from Lemma 2.8 and Theorem 2.6. □

Thus the Boehmian space $\hat{\mathscr{B}}_C^\alpha$ is constructed and we denote a typical element of $\hat{\mathscr{B}}_C^\alpha$ by $\mathcal{V} = [(g_n)/(e^{-ia_\alpha y^2} F_C^\alpha(\delta_n))]$.

Definition. We define the extended FRFCT $\mathscr{F}_C^\alpha : \mathscr{B}_C^\alpha \to \hat{\mathscr{B}}_C^\alpha$ by $\mathscr{F}_C^\alpha(X) = [(F_C^\alpha f_n)/(e^{-ia_\alpha y^2} F_C^\alpha \delta_n)]$, where $X = [(f_n), (\delta_n)] \in \mathscr{B}_C^\alpha$.

Suppose $[(f_n), (\delta_n)] \in \mathscr{B}_C^\alpha$, then for all $n, m \in \mathbb{N}$, we have $f_n *_c^\alpha \delta_m = f_m *_c^\alpha \delta_n$, which implies that $F_C^\alpha(f_n *_c^\alpha \delta_m)(u) = F_C^\alpha(f_m *_c^\alpha \delta_n)(u)$. Applying the convolution theorem, we get that

$$e^{-ia_\alpha u^2}[F_C^\alpha f_n](u) \cdot [F_C^\alpha \delta_m](u) = e^{-ia_\alpha u^2}[F_C^\alpha f_m](u) \cdot [F_C^\alpha \delta_n](u)$$

and hence $[F_C^\alpha f_n](u) \cdot e^{-ia_\alpha u^2}[F_C^\alpha \delta_m](u) = [F_C^\alpha f_m](u) \cdot e^{-ia_\alpha u^2}[F_C^\alpha \delta_n](u)$. By a similar argument, it is easy to prove that $\mathscr{F}_C^\alpha(X)$ is independent of the choice of the representative of X. Thus, FRFCT is well-defined.

Theorem 3.6. *The* FRFCT *$\mathscr{F}_C^\alpha : \mathscr{B}_C^\alpha \to \hat{\mathscr{B}}_C^\alpha$ is consistent with $F_C^\alpha : L^2_+ \to L^2_+$.*

Proof. If $f \in L^2_+$ then $\mathcal{F} = [(f *_c^\alpha \delta_n), (\delta_n)]$ is the Boehmian representing f in $\mathscr{B}_C^\alpha$. By definition, we have $\mathscr{F}_C^\alpha(\mathcal{F}) = [(F_C^\alpha(f *_c^\alpha \delta_n)), (e^{-ia_\alpha u^2} F_C^\alpha \delta_n)] = [(F_C^\alpha f \cdot e^{-ia_\alpha u^2} F_C^\alpha \delta_n)/(e^{-ia_\alpha u^2} F_C^\alpha \delta_n)]$, which is the Boehmian representing $F_C^\alpha f$ in $\hat{\mathscr{B}}_C^\alpha$. □

Theorem 3.7. *The* FRFCT *$\mathscr{F}_C^\alpha : \mathscr{B}_C^\alpha \to \hat{\mathscr{B}}_C^\alpha$ is a bijective linear map.*

Proof. Let $X = [(f_n),(\delta_n)]$, and $Y = [(g_n),(\phi_n)] \in \mathscr{B}_C^\alpha$ be such that $\mathscr{F}_C^\alpha(X) = \mathscr{F}_C^\alpha(Y)$. Therefore, it follows that

$$[(F_C^\alpha f_n)/(e^{-ia_\alpha u^2} F_C^\alpha \delta_n)] = [(F_C^\alpha g_n)/(e^{-ia_\alpha u^2} F_C^\alpha \phi_n)]$$

and hence for any $n, m \in \mathbb{N}$,

$$e^{-ia_\alpha u^2}[F_C^\alpha f_n](u) \cdot [F_C^\alpha \phi_m](u) = e^{-ia_\alpha u^2}[F_C^\alpha g_m](u) \cdot [F_C^\alpha \delta_n](u).$$

Using the convolution Theorem 2.7, we have $F_C^\alpha(f_n *_c^\alpha \phi_m) = F_C^\alpha(g_m *_c^\alpha \delta_n)$, $\forall m, n \in \mathbb{N}$, which implies that $f_n *_c^\alpha \phi_m = g_m *_c^\alpha \delta_n$, $\forall n, m \in \mathbb{N}$ and hence $X = Y$.

Let $\mathcal{X} = [(g_n)/(e^{-ia_\alpha u^2} F_C^\alpha \delta_n)] \in \hat{\mathscr{B}}_C^\alpha$. Since $F_C^\alpha : L_+^2 \to L_+^2$ is onto, choose $f_n \in L_+^2$ such that $g_n = F_C^\alpha f_n$ for each $n \in \mathbb{N}$. For any $n, m \in \mathbb{N}$, we get that $[F_C^\alpha f_n](u) \cdot e^{-ia_\alpha u^2}[F_C^\alpha \delta_m](u) = [F_C^\alpha f_m](u) \cdot e^{-ia_\alpha u^2}[F_C^\alpha \delta_n](u)$. Therefore, by convolution theorem, we have $[F_C^\alpha(f_n *_c^\alpha \delta_m)](u) = [F_C^\alpha(f_m *_c^\alpha \delta_n)](u)$, and hence $f_n *_c^\alpha \delta_m = f_m *_c^\alpha \delta_n$. Thus $X = [(f_n),(\delta_n)] \in \mathscr{B}_C^\alpha$ and $\mathscr{F}_C^\alpha(X) = [(F_C^\alpha f_n)/(e^{-ia_\alpha u^2} F_C^\alpha \delta_n)] = \mathcal{X}$, which implies that $\mathscr{F}_C^\alpha$ is a surjective map.

The linearity of the $\mathscr{F}_C^\alpha$ follows from the linearity of F_C^α and Theorem 2.7. □

Theorem 3.8 (Convolution theorem for FRFCT on Boehmians)**.** *If $X \in \mathscr{B}_C^\alpha$ and $h \in L_+^1$, then $\mathscr{F}_C^\alpha(X *_c^\alpha h) = \mathscr{F}_C^\alpha(X) \cdot e^{-ia_\alpha u^2} F_C^\alpha h$.*

Proof. Let $X = [(f_n),(\delta_n)]$ and $h \in L_+^1$. By using Theorem 2.7,

$$\begin{aligned}
\mathscr{F}_C^\alpha(X *_c^\alpha h) &= \mathscr{F}_C^\alpha[(f_n *_c^\alpha h),(\delta_n)] = [F_C^\alpha(f_n *_c^\alpha h)/(e^{-ia_\alpha u^2} F_C^\alpha \delta_n)] \\
&= [(e^{-ia_\alpha u^2} F_C^\alpha f_n \cdot F_C^\alpha h)/(e^{-ia_\alpha u^2} F_C^\alpha \delta_n)] \\
&= [(F_C^\alpha f_n)/(e^{-ia_\alpha u^2} F_C^\alpha \delta_n)] \cdot e^{-ia_\alpha u^2} F_C^\alpha h \\
&= \mathscr{F}_C^\alpha(X) \cdot e^{-ia_\alpha u^2} F_C^\alpha h.
\end{aligned}$$

□

Theorem 3.9. *The* FRFCT *on $\mathscr{B}_C^\alpha$ is continuous with respect to δ-convergence and Δ-convergence.*

Proof. Let $X_n \xrightarrow{\delta} X$ as $n \to \infty$ in $\mathscr{B}_C^\alpha$. By the definition of δ-convergence, $X_n *_c^\alpha \delta_k$, $X *_c^\alpha \delta_k \in L_+^2$ and $X_n *_c^\alpha \delta_k \to X *_c^\alpha \delta_k$ as $n \to \infty$ in L_+^2 for each fixed $k \in \mathbb{N}$ and for some $(\delta_n) \in \Delta(\alpha)$.

In view of Theorems 3.8 and 3.6, we get that

$$\mathscr{F}_C^\alpha(X_n) \cdot e^{-ia_\alpha u^2} F_C^\alpha \delta_k = \mathscr{F}_C^\alpha(X_n *_c^\alpha \delta_k) = F_C^\alpha(X_n *_c^\alpha \delta_k) \in L_+^2$$

and

$$\mathscr{F}_C^\alpha(X) \cdot e^{-ia_\alpha u^2} F_C^\alpha \delta_k = \mathscr{F}_C^\alpha(X *_c^\alpha \delta_k) = F_C^\alpha(X *_c^\alpha \delta_k) \in L_+^2$$

for all $n, k \in \mathbb{N}$. Further, using the continuity of the FRFCT, we obtain

$$\mathscr{F}_C^\alpha(X_n) \cdot e^{-ia_\alpha u^2} F_C^\alpha \delta_k = F_C^\alpha(X_n *_c^\alpha \delta_k) \to F_C^\alpha(X *_c^\alpha \delta_k) = \mathscr{F}_C^\alpha(X) \cdot e^{-ia_\alpha u^2} F_C^\alpha \delta_k$$

as $n \to \infty$ in L_+^2, for each fixed $k \in \mathbb{N}$. Hence $\mathscr{F}_C^\alpha(X_n) \xrightarrow{\delta} \mathscr{F}_C^\alpha(X)$ as $n \to \infty$.

Let $X_n \overset{\Delta}{\to} X$ as $n \to \infty$. Then there exists $(\delta_n) \in \Delta(\alpha)$ such that $(X_n - X) *_c^\alpha \delta_n \in L_+^2, \forall n \in \mathbb{N}$ and $(X_n - X) *_c \delta_n \to 0$ as $n \to \infty$ in L_+^2. Now using Theorems 3.7 and 3.6, we have for each $n \in \mathbb{N}$,

$$\begin{aligned}[\mathscr{F}_C^\alpha(X_n) - \mathscr{F}_C^\alpha(X)] \cdot e^{-ia_\alpha u^2} F_C^\alpha \delta_n &= \mathscr{F}_C^\alpha(X_n - X) \cdot e^{-ia_\alpha u^2} F_C^\alpha \delta_n \\ &= F_C^\alpha((X_n - X) *_c^\alpha \delta_n),\end{aligned}$$

which belongs to L_+^2 and the continuity of F_C^α on L_+^2 yields that

$$[\mathscr{F}_C^\alpha(X_n) - \mathscr{F}_C^\alpha(X)] \cdot e^{-ia_\alpha u^2} F_C^\alpha \delta_n = F_C^\alpha((X_n - X) *_c^\alpha \delta_n) \to 0$$

as $n \to \infty$ in L_+^2. This shows that $\mathscr{F}_C^\alpha(X_n) \overset{\Delta}{\to} \mathscr{F}_C^\alpha(X)$ as $n \to \infty$. □

4. Fractional Fourier sine transform on Boehmians

In this section, we extend the FRFST as a mapping from the Boehmian space $\mathscr{B}_S^\alpha$ onto the Boehmian space $\hat{\mathscr{B}}_C^\alpha$.

Definition. For each $\mathcal{U} = [(f_n), (\delta_n)] \in \mathscr{B}_S^\alpha$, we define the extended FRFST of $\mathcal{U}$ by $\mathscr{F}_S^\alpha(\mathcal{U}) = [(F_S^\alpha f_n)/(e^{-ia_\alpha y^2} F_C^\alpha \delta_n)]$.

If $[(f_n), (\delta_n)], [(g_n), (\epsilon_n)]$ in $\mathscr{B}_S^\alpha$ such that $[(f_n), (\delta_n)] = [(g_n), (\epsilon_n)]$, then we have $f_n *_{sc}^\alpha \epsilon_m = g_m *_{sc}^\alpha \delta_n$, $\forall m, n \in \mathbb{N}$. As in the case of FRFCT on Boehmians, applying Theorem 2.7(ii), we get that $F_S^\alpha(f_n) e^{-ia_\alpha y^2} F_C^\alpha(\epsilon_m) = F_S^\alpha(g_m) e^{-ia_\alpha y^2} F_C^\alpha(\delta_n)$, $\forall m, n \in \mathbb{N}$. This implies that the images of $[(f_n), (\delta_n)]$ and $[(g_n), (\epsilon_n)]$ are same in $\hat{\mathscr{B}}_C^\alpha$. Thus, $\mathscr{F}_S^\alpha : \mathscr{B}_S^\alpha \to \hat{\mathscr{B}}_C^\alpha$ is well-defined.

Theorem 4.1. *The* FRFST $\mathscr{F}_S^\alpha : \mathscr{B}_S^\alpha \to \hat{\mathscr{B}}_C^\alpha$ *is consistent with* $F_S^\alpha : L_+^2 \to L_+^2$.

Proof. Let $f \in L_+^2$ be arbitrary. Then, the Boehmian representing f is of the form $[(f *_{sc}^\alpha \delta_n), (\delta_n)]$, where $(\delta_n) \in \Delta(\alpha)$ is arbitrary. Then, Theorem 2.7(ii) implies that $\mathscr{F}_S^\alpha([(f *_{sc}^\alpha \delta_n), (\delta_n)]) = [(F_S^\alpha(f *_{sc}^\alpha \delta_n))/(e^{-ia_\alpha y^2} F_C^\alpha \delta_n)] = [(F_S^\alpha(f) e^{-ia_\alpha y^2} F_C^\alpha(\delta_n))/(e^{-ia_\alpha y^2} F_C^\alpha \delta_n)]$, which is the representation of $F_S^\alpha(f)$ in $\hat{\mathscr{B}}_C^\alpha$. Hence, $\mathscr{F}_S^\alpha$ is consistent with F_S^α. □

Theorem 4.2. *The* FRFST $\mathscr{F}_S^\alpha : \mathscr{B}_S^\alpha \to \hat{\mathscr{B}}_C^\alpha$ *is a bijective linear map.*

Proof. The linearity $\mathscr{F}_S^\alpha$ is a direct consequence of linearity of the $\mathscr{F}_S^\alpha$ on L_+^2 and the convolution theorem (Theorem 2.7(ii)). To prove the injectivity, let $[(f_n), (\delta_n)]$ and $[(g_n), (\epsilon_n)]$ in $\mathscr{B}_S^\alpha$ be such that

$$[(F_S^\alpha f_n)/(e^{-ia_\alpha y^2} F_C^\alpha \delta_n)] = [(F_S^\alpha g_n)/(e^{-ia_\alpha y^2} F_C^\alpha \epsilon_n)].$$

Then, it follows that $F_S^\alpha(f_n) e^{-ia_\alpha y^2} F_C^\alpha(\epsilon_m) = F_S^\alpha(g_m) e^{-ia_\alpha y^2} F_C^\alpha(\delta_n)$, $\forall m, n \in \mathbb{N}$. Applying Theorem 2.7(ii) and using the invertibility of F_S^α on both sides, we obtain that $f_n *_{sc}^\alpha \epsilon_m = g_m *_{sc}^\alpha \delta_n$, $\forall m, n \in \mathbb{N}$, which implies that $[(f_n), (\delta_n)] = [(g_n), (\epsilon_n)]$. Therefore, $\mathscr{F}_S^\alpha$ is injective on $\mathscr{B}_S^\alpha$. To prove that $\mathscr{F}_S^\alpha : \mathscr{B}_S^\alpha \to \hat{\mathscr{B}}_C^\alpha$ is surjective, let $[(g_n)/(e^{-ia_\alpha y^2} F_C^\alpha(\delta_n))] \in \hat{\mathscr{B}}_C^\alpha$ be arbitrary. If we choose

$f_n \in L^2_+$ such that $F^\alpha_S(f_n) = g_n$ for all $n \in \mathbb{N}$, then adopting the proof of Theorem 3.7, one can show that $[(f_n)/(\delta_n)] \in \mathscr{B}^\alpha_S$ and that $\mathscr{F}^\alpha_S([(f_n)/(\delta_n)]) = [(g_n)/(e^{-ia_\alpha y^2} F^\alpha_C(\delta_n))]$ in $\hat{\mathscr{B}}^\alpha_C$. □

As the proofs of the following properties of $\mathscr{F}^\alpha_S$ are much similar to that of $\mathscr{F}^\alpha_C$ in Section 3, we prefer to leave the details.

Theorem 4.3 (Convolution theorem for extended FRFST)**.** *If* $\mathcal{U} \in \mathscr{B}^\alpha_S$ *and* $h \in L^1_+$, *then* $\mathscr{F}^\alpha_S(\mathcal{U} *^\alpha_{sc} h) = \mathscr{F}^\alpha_S(\mathcal{U}) \cdot e^{-ia_\alpha u^2} F^\alpha_C h$.

Theorem 4.4. *The* FRFST *on* $\mathscr{B}^\alpha_S$ *is continuous with respect to* δ*-convergence and* Δ*-convergence.*

5. Concluding remarks

It is interesting to note that the Fourier sine and cosine transforms on Boehmians discussed in [21, Section 3] becomes a particular case of this paper, when $\alpha = \frac{\pi}{2}$. This is a first benefit of this paper, in pure mathematical point of view. Furthermore, we see that the every distribution Λ with compact support in $[0, \infty)$ can be identified as a member of $\mathscr{B}^\alpha_C$ by the identification $\Lambda \mapsto [(\Lambda *^\alpha_c \varphi_n)/(\varphi_n)]$, where $(\varphi_n) \in \Delta(\alpha)$ is such that φ_n is infinitely smooth function on $\mathbb{R}$ having support in $[0, \infty)$, $\forall n \in \mathbb{N}$ and

$$(\Lambda *^\alpha_c \varphi)(x) = \frac{c_\alpha}{\sqrt{2\pi}} \langle \Lambda(y), [e^{\beta(y^2+xy)}\phi(x+y) + e^{\beta(y^2-xy)}\phi(|x-y|)]\rangle, \ \forall x \geq 0.$$

Therefore, the extension of the fractional Fourier cosine transform discussed in this paper properly generalizes the fractional Fourier cosine transform on L^2_+, which is a second benefit of this paper, in view of pure mathematics.

In signal processing, the Dirac's delta function δ and convolution play a vital role, especially to find *the system waiting function* of a linear time-invariant system (LTI system) and the output of the LTI system, respectively. A filer in signal processing is an example of an LTI system. So, if an integral transform is used in signal processing, it is necessary to study the image of δ under the transform and the convolution theorem of the transform.

In particular, it is well known that the fractional Fourier sine and cosine transforms are having many applications in signal processing. We point out that since δ is identified as the Boehmian $[(\delta *^\alpha_c \phi_n)/(\phi_n)]$, using the extended fractional Fourier cosine transform on $\mathscr{B}^\alpha_C$, we can find the image of the same. Since, every Boehmian $[(f_n)/(\phi_n)]$ can be approximated by the sequence of functions (f_n) (see [6]), the image of any Boehmian in $\mathscr{B}^\alpha_C$ under $\mathscr{F}^\alpha_C$ can be approximated by sequence of functions. In particular, we can find the fractional Fourier cosine transform of δ, approximately by sequence of functions.

In addition to this, since convolution theorems of these transforms are obtained as products, one can find the fractional cosine (respectively, sine) convolution of f and g easily by applying the inverse fractional Fourier cosine (respectively, sine) transform on $e^{-ia_\alpha u^2} F_C(f) F_C(g)$ (respectively, $e^{-ia_\alpha u^2} F_C(f) F_S(g)$).

Since all the properties of the fractional Fourier cosine and sine transforms on function space are extended to the context of Boehmian space, we can freely apply the extended fractional Fourier cosine and sine transform on any mathematical expression which involves both functions and generalized functions.

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References

[1] L. Akila and R. Roopkumar, *A natural convolution of quaternion valued functions and its applications*, Appl. Math. Comput. **242** (2014), 633–642.
[2] T. Alieva and M. J. Bastiaans, *Fractional cosine and sine transform in relation to the fractional Fourier and Hartley transforms*, Proceedings of the Seventh International Symposium on Signal Processing and its Applications Paris, France **1** (2003), 561–564.
[3] C. Arteaga and I. Marrero, *The Hankel transform of tempered boehmians via the exchange property*, Appl. Math. Comput. **219** (2012), no. 3, 810–818.
[4] P. J. Miana, *Convolutions, Fourier trigonometric transforms and applications*, Integral Transforms Spec. Funct. **16** (2005), no. 7, 583–585.
[5] J. Mikusiński and P. Mikusiński, *Quotients de suites et leurs applications dans l'anlyse fonctionnelle*, C. R. Acad. Sci. Paris Sér. I Math. **293** (1981), no. 9, 463–464.
[6] P. Mikusiński, *Convergence of boehmians*, Japan. J. Math. **9** (1983), no. 1, 159–179.
[7] ———, *Tempered Boehmians and ultra distributions*, Proc. Amer. Math. Soc. **123** (1995), no. 3, 813–817.
[8] ———, *On flexibility of Boehmians*, Integral Transforms Spec. Funct. **4** (1996), no. 1-2, 141–146.
[9] P. Mikusiński, A. Morse, and, D. Nemzer, *The two sided Laplace transform for Boehmians*, Integral Transforms Spec. Funct. **2** (1994), no. 3, 219–230.
[10] P. Mikusiński and A. I. Zayed, *The Radon transform of Boehmians*, Proc. Amer. Math. Soc. **118** (1993), no. 2, 561–570.
[11] V. Namias, *The fractional order Fourier transform and its application to quantum mechanics*, J. Inst. Math. Appl. **25** (1980), no. 3, 241–265.
[12] D. Nemzer, *The Laplace transform on a class of Boehmians*, Bull. Austral. Math. Soc. **46** (1992), no. 2, 347–352.
[13] ———, *Integrable Boehmians, Fourier transforms, and Poisson's summation formula*, Appl. Anal. Discrete Math. **1** (2007), no. 1, 172–183.
[14] ———, *Extending the Stieltjes transform*, Sarajevo J. Math. **10** (2014), no. 2, 197–208.
[15] ———, *Extending the Stieltjes transform II*, Fract. Calc. Appl. Anal. **17** (2014), no. 4, 1060–1074.
[16] S. C. Pei and J. J. Ding, *Fractional cosine, sine, and Hartley transforms*, IEEE Trans. Signal Process. **50** (2002), no. 7, 1661–1680.
[17] R. Roopkumar, *Generalized Radon transform*, Rocky Mountain J. Math. **36** (2006), no. 4, 1375–1390.
[18] ———, *Ridgelet transform on square integrable Boehmians*, Bull. Korean Math. Soc. **46** (2009), no. 5, 835–844.
[19] ———, *Mellin transform for Boehmians*, Bull. Inst. Math. Acad. Sinica. **4** (2009), no. 1, 75–96.
[20] R. Roopkumar and E. R. Negrin, *Poisson transform on Boehmians*, Appl. Math. Comput. **216** (2010), no. 9, 2740–2748.
[21] R. Roopkumar, E. R. Negrin, and C. Ganesan, *Fourier cosine and sine transforms Boehmian spaces*, Asian-Eur. J. Math. **6** (2013), no. 1, 1350005, 17 pages.

[22] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Inc., New York, 1987.
[23] I. N. Sneddon, *The Use of Integral Transform*, McGraw-Hill Inc., New York, 1972.
[24] N. X. Thao, V. A. Kakichev, and V. K. Tuan, *On the generalized convolution for Fourier cosine and sine transforms*, East-West J. Math. **1** (1998), no. 1, 85–90.
[25] N. X. Thao, V. K. Tuan, and N. T. Hong, *Integral transforms of Fourier cosine and sine generalized convolution type*, Internat. J. Math. Math. Sci. **2007** (2007), 97250, 11 pages.
[26] N. X. Thao, V. K. Tuan, and N. M. Khoa, *On the generalized convolution with a weight function for the Fourier cosine and sine transforms*, Fract. Calc. Appl. Anal. **7** (2004), no. 3, 323–337.
[27] V. K. Tuan, *Integral transforms of Fourier cosine convolution type*, J. Math. Anal. Appl. **229** (1999), no. 2, 519–529.
[28] A. I. Zayed, *Fractional fourier transform of generalized functions*, Integral Transforms Spec. Funct. **7** (1998), no. 3-4, 299–312.
[29] A. I. Zayed and P. Mikusiński, *On the extension of the Zak transform*, Methods Appl. Anal. **2** (1995), no. 2, 160–172.

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