Commun. Korean Math. Soc. **31** (2016), No. 4, pp. 779–790
http://dx.doi.org/10.4134/CKMS.c150242
pISSN: 1225-1763 / eISSN: 2234-3024

SOME INTEGRAL TRANSFORMS INVOLVING EXTENDED GENERALIZED GAUSS HYPERGEOMETRIC FUNCTIONS

Junesang Choi, Krunal B. Kachhia, Jyotindra C. Prajapati, and Sunil Dutt Purohit

Abstract. Using the extended generalized integral transform given by Luo *et al.* [6], we introduce some new generalized integral transforms to investigate such their (potentially) useful properties as inversion formulas and Parseval-Goldstein type relations. Classical integral transforms including (for example) Laplace, Stieltjes, and Widder-Potential transforms are seen to follow as special cases of the newly-introduced integral transforms.

1. Introduction, preliminaries, and definitions

Throughout this paper let $\mathbb{C}$, $\mathbb{N}$ and $\mathbb{Z}_0^-$ denote the sets of complex numbers, positive and non-positive integers, respectively. Integral transforms have been widely used in various problems of mathematical physics and applied mathematics (for some recent works, see, *e.g.*, [1, 4, 8, 9]). Integral transforms with such special functions as (for example) the hypergeometric functions have been played important roles in solving numerous applied problems. Due mainly to their demonstrated applications, several generalizations of integral transforms with hypergeometric functions have been actively investigated. Virchenko and Ovcharenko [16] presented some new integral transforms with the generalized confluent hypergeometric function due to Virchenko [15]. Here, in this paper, using the extended generalized integral transform given by Luo *et al.* [6], we introduce some new generalized integral transforms to investigate such their (potentially) useful properties as inversion formulas and Parseval-Goldstein type relations. Classical integral transforms including (for example) Laplace, Stieltjes, and Widder-Potential transforms are seen to follow as special cases of the newly-introduced integral transforms.

Recently, Luo *et al.* [6] introduced the following extended generalized hypergeometric function ${}_pF_q^{(\alpha,\beta;\kappa,\mu)}$ and investigated its various properties. The

Received December 17, 2015.
2010 *Mathematics Subject Classification.* Primary 44A10, 44A20; Secondary 33C15, 33C20.
Key words and phrases. integral transforms, Laplace transform, inversion Theorems, Parseval relations.

extended generalized hypergeometric function ${}_pF_q^{(\alpha,\beta;\kappa,\mu)}$ is defined by

$$(1.1)\qquad {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};z;\gamma\right]:=\sum_{n=0}^{\infty}\Theta\left(n/p,q\right)\frac{z^n}{n!}$$

$$(\min\{\Re(\alpha),\,\Re(\beta),\,\Re(\gamma)\}>0,\ \min\{\Re(\kappa),\,\Re(\mu)\}\geq 0),$$

whose coefficient $\Theta\left(n/p,q\right)$ is determined by

$$(1.2)\qquad \Theta\left(n/p,q\right)=\begin{cases} (a_1)_n\prod_{j=1}^{q}\frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a_{j+1}+n,b_j-a_{j+1})}{B(a_{j+1},b_j-a_{j+1})} \\ \qquad (p=q+1;\Re(b_j)>\Re(a_{j+1})>0;|z|<1), \\ \prod_{j=1}^{q}\frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a_j+n,b_j-a_j)}{B(a_j,b_j-a_j)} \\ \qquad (p=q;\Re(b_j)>\Re(a_j)>0;z\in\mathbb{C}), \\ \prod_{i=1}^{r}\frac{1}{(b_i)_n}\prod_{j=1}^{p}\frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a_j+n,b_{r+j}-a_j)}{B(a_j,b_r+j-a_j)} \\ \qquad (r=q-p,p<q;\Re(b_{r+j})>\Re(a_j)>0;z\in\mathbb{C}). \end{cases}$$

Here the generalized Beta function $B_\gamma^{(\alpha,\beta;\kappa,\mu)}(x,y)$ is defined by (see [6])

$$(1.3)\qquad B_\gamma^{(\alpha,\beta;\kappa,\mu)}(x,y):=\int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{\gamma}{t^k(1-t)^\mu}\right)dt$$

$$(\min\{\Re(\gamma),\,\Re(\kappa)\}\geq 0,\ \min\{\Re(x),\,\Re(y),\,\Re(\alpha),\Re(\beta),\,\Re(k),\,\Re(\mu)\}>0)$$

and the familiar classical Beta function $B(\alpha,\,\beta)$ may be recalled as follows (see, *e.g.*, [12, p. 8, Eq. (43)]):

$$(1.4)\qquad B(\alpha,\,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt \quad (\Re(\alpha)>0;\ \Re(\beta)>0) \\[2ex] \dfrac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad (\alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-).\end{cases}$$

The special case of the function (1.1) when $\gamma=0$ is seen to reduce to the generalized hypergeometric function ${}_pF_q$ with p numerator and q denominator parameters defined by (see, *e.g.*, [10, 12])

$$(1.5)\qquad \begin{aligned} {}_pF_q\left[\begin{array}{c} a_1,\,\ldots,\,a_p; \\ b_1,\,\ldots,\,b_q; \end{array} z\right] &= {}_pF_q(a_1,\,\ldots,\,a_p;\,b_1,\,\ldots,\,b_q;\,z) \\ &=\sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!}, \end{aligned}$$

where, in terms of the Gamma function $\Gamma(z)$ (see, *e.g.*, [12, Section 1.1]) whose Euler's integral is given by

$$(1.6)\qquad \Gamma(z)=\int_0^\infty e^{-t}t^{z-1}\,dt \quad (\Re(z)>0),$$

the widely-used Pochhammer symbol $(\lambda)_\nu$ $(\lambda,\ \nu \in \mathbb{C})$ is defined, in general, by (see, for details, [13]; see also [12])

$$(1.7)\qquad \begin{aligned}(\lambda)_\nu &:= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad \left(\lambda \in \mathbb{C}\setminus \mathbb{Z}_0^-\right)\\ &= \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}).\end{cases}\end{aligned}$$

The special case of the function (1.3) when $\gamma = 0$ would reduce immediately to the Beta function $B(x, y)$ in (1.4).

It is also noted that for $p = 2$ and $q = 1$ the definitions in (1.1) would reduce immediately to the extended hypergeometric type function defined as follows (see [6]):

$$(1.8)\qquad {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a, b \\ c \end{matrix}; z; \gamma\right] := \sum_{n=0}^{\infty} (a)_n \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}$$

$(\Re(\gamma) > 0, \Re(\kappa) \geq 0, \mu \geq 0; \min\{\Re(\alpha), \Re(\beta)\} > 0;\ \Re(c) > \Re(b) > 0,\ |z| < 1)$.

The Mellin transform of $f(x)$ is defined in the following way (see, *e.g.*, [11]):

$$(1.9)\qquad \mathcal{M}\{f(x); s\} = \int_0^\infty f(x) x^{s-1}\, dx,$$

provided the integral converges.

Our investigation needs to recall Hadamard product (see, *e.g.*, [7]) which is used to decompose a newly-emerged function into two known functions: Let $f(z) := \sum_{n=0}^\infty a_n z^n$ and $g(z) := \sum_{n=0}^\infty b_n z^n$ be two power series whose radii of convergence are denoted by R_f and R_g, respectively. Then their Hadamard product is the power series defined by

$$(1.10)\qquad (f * g)(z) = \sum_{n=0}^{\infty} a_n b_n\, z^n.$$

The radius of convergence R of the power series (1.10) is easily seen to satisfy $R_f \cdot R_g \leq R$. In particular, if one of the power series is an entire function, then the Hadamard product series defines an entire function, too. For example, the function ${}_sF_{s+r}^{(\alpha,\beta;\kappa,\mu)}$ (see [6]) is decomposed as follows: For all $z \in \mathbb{C}$,

$$(1.11)\qquad \begin{aligned}&{}_sF_{s+r}^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} x_1, \ldots, x_s \\ y_1, \ldots, y_{s+r} \end{matrix}; z; \gamma\right] \\ &= {}_1F_r\left[\begin{matrix} 1; \\ y_1, \ldots, y_r; \end{matrix}\ z\right] * {}_sF_s{}^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} x_1, \ldots, x_s \\ y_{1+r}, \ldots, y_{s+r} \end{matrix}; z; \gamma\right].\end{aligned}$$

The following Laplace transforms are required in our investigation (see, *e.g.*, [2, pp. 159–160, Entries (81) and (82)]):

$$(1.12)\qquad \mathcal{L}\{x^{-1}\sin(\alpha x)\ \sin(\beta x); u\} = \frac{1}{4}\ln\frac{u^2+(\alpha+\beta)^2}{u^2+(\alpha-\beta)^2}$$

$$(\Re(u) > |\Im(\pm\alpha\pm\beta)|)$$

and

$$\mathcal{L}\{x^{-1}\sin(\alpha x)\ \cos(\beta x); u\} = \frac{1}{2}\ \arctan\frac{2\alpha u}{u^2-\alpha^2+\beta^2} \qquad (1.13)$$
$$(\Re(u) > |\Im(\pm\alpha\pm\beta)|),$$

where the Laplace transform of a function f is defined by

$$\mathcal{L}\{f(x); u\} = \int_0^\infty e^{-xu}\ f(x)dx, \qquad (1.14)$$

provided the integral converges.

Here we define some new generalized integral transforms by using the extended generalized Gauss hypergeometric function in (1.1).

Consider the following generalization of some classical transforms:

$$\mathcal{L}^*\{f(x); y\} = F^*(y) := \int_0^\infty e^{-xy}\,{}_1F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a \\ c \end{array}; -bxy; \gamma\right] f(x)dx, \qquad (1.15)$$

where ${}_1F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a \\ c \end{array}; z; \gamma\right]$ is the extended generalized Gauss hypergeometric function in (1.1). The special case of (1.15) when $b = 0$ is seen to reduce to the classical Laplace transform (1.14).

We define the following integral transform:

$$\tilde{S}_p\{f(x); y\} := \frac{1}{\Gamma(p)}\int_0^\infty \frac{f(x)}{(x+y)^p}\ {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} p, a \\ c \end{array}; -b\left(\frac{x}{x+y}\right); \gamma\right] dx, \qquad (1.16)$$

where ${}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a, b \\ c \end{array}; z; \gamma\right]$ is the extended generalized Gauss hypergeometric function in (1.8). The case $b = 0$ of (1.16) yields the generalized Stieltjes integral transform (see [2, 14])

$$S_p\{f(x); y\} = \frac{1}{\Gamma(p)}\int_0^\infty \frac{f(x)}{(x+y)^p}\ dx. \qquad (1.17)$$

We define the following generalized potential integral transform:

$$\tilde{P}_{m,1}\{f(x); y\} := \int_0^\infty \frac{x^{m-1}f(x)}{x^m+y^m}\ {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} 1, a \\ c \end{array}; -b\left(\frac{x^m}{x^m+y^m}\right); \gamma\right] dx, \qquad (1.18)$$

where $m \in \mathbb{N}$ and the case $b = 0$ of which gives the known potential integral transform (see [5]).

2. Inversion formulae

In this section, inversion formulae for the integral transforms (1.15), (1.16) and (1.18) are investigated.

Lemma 1. *Let* $\Re(\gamma) \geq 0$, $\min\{\Re(\alpha), \Re(\beta), \Re(k), \Re(\mu)\} > 0$, $\Re(c) > \Re(a) > 0$, *and* $|b| < 1$. *Then we have*

$$\int_0^\infty v^{s-1} e^{-v} {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{array}{c} a \\ c \end{array} ; -bv; \gamma \right] dv = \Gamma(s) \; {}_2F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{array}{c} s, a \\ c \end{array} ; -b; \gamma \right]. \tag{2.1}$$

Proof. We find from (1.1) and (1.2) that

$$\begin{aligned}
&\int_0^\infty v^{s-1} e^{-v} {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{array}{c} a \\ c \end{array} ; -bv; \gamma \right] dv \\
&= \int_0^\infty v^{s-1} e^{-v} \sum_{n=0}^\infty \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n, c-a)}{B(a, c-a)} \frac{(-bv)^n}{n!} \, dv \\
&= \sum_{n=0}^\infty \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n, c-a)}{B(a, c-a)} \frac{(-b)^n}{n!} \int_0^\infty v^{s+n-1} e^{-v} \, dv.
\end{aligned}$$

Then applying the Gamma function (1.6) to the last integral gives

$$\begin{aligned}
&\int_0^\infty v^{s-1} e^{-v} {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{array}{c} a \\ c \end{array} ; -bv; \gamma \right] dv \\
&= \sum_{n=0}^\infty \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n, c-a)}{B(a, c-a)} \frac{(-b)^n}{n!} \Gamma(s+n).
\end{aligned} \tag{2.2}$$

Finally, use of (1.7) yields the desired result (2.1). □

Theorem 2. *The inversion formula for the generalized integral transform* (1.15) *is given in the following form*:

$$\mathcal{L}^{*-1}\{F^*(y)\} = f(x) = \int_0^\infty (xy)^{-1} \, G^*(y) \, \Theta(xy) \, dy, \tag{2.3}$$

where

$$\Theta(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{x^s}{\xi(s)} \, ds, \;\; \xi(s) = \Gamma(s) \; {}_2F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{array}{c} s, a \\ c \end{array} ; -b; \gamma \right],$$

and $G^*(y) = \mathcal{L}^*\{f(x); y\}$.

Proof. Applying the Mellin integral transform (1.9) to (1.15) and using Lemma 1, we get

$$\begin{aligned}
\mathcal{M}\{\mathcal{L}^*\{f(x); y\}; s\} &= \mathcal{M}\{f(x); 1-s\} \; \mathcal{M} \left\{ e^{-\nu} {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{array}{c} a \\ c \end{array} ; -b\nu; \gamma \right]; 1-s \right\} \\
&= \mathcal{M}\{f(x); 1-s\} \int_0^\infty \nu^{s-1} e^{-\nu} {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{array}{c} a \\ c \end{array} ; -b\nu; \gamma \right] d\nu \\
&= \Gamma(s) \; \mathcal{M}\{f(x); 1-s\} \; {}_2F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{array}{c} s, a \\ c \end{array} ; -b; \gamma \right].
\end{aligned}$$

Using the inversion formula for Mellin integral transform (see [11]) and changing the order of integration, we obtain

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s-1}\left\{\Gamma(s)\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} s,a \\ c \end{array};-b;\gamma\right]\right\}^{-1}\mathcal{M}\{G^*(y);s\}\,ds \\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} x^{s-1}\left\{\Gamma(s)\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} s,a \\ c \end{array};-b;\gamma\right]\right\}^{-1} \\ &\quad\times\left(\int_0^\infty y^{s-1}G^*(y)\,dy\right)\,ds. \end{aligned}$$

□

Lemma 3. *Let* $\Re(\gamma)\geq 0$, $\min\{\Re(\alpha),\,\Re(\beta),\,\Re(k),\,\Re(\mu)\}>0$, $\Re(c)>\Re(a)>0$, *and* $\left|b\left(\frac{x}{x+y}\right)\right|<1$. *Then we obtain*

$$\begin{aligned} &\int_0^\infty x^{p-1}e^{-(y+z)x}\,{}_1F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a \\ c \end{array};-bxy;\gamma\right]dx \\ &= \frac{\Gamma(p)}{(y+z)^p}\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} p,a \\ c \end{array};-b\left(\frac{y}{y+z}\right);\gamma\right]. \end{aligned} \tag{2.4}$$

Proof. Let L be the left-hand side of (2.4). Using (1.1) and (1.2), we have

$$\begin{aligned} L &= \int_0^\infty x^{p-1}e^{-(y+z)x}\sum_{n=0}^\infty \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n,c-a)}{B(a,c-a)}\frac{(-bxy)^n}{n!}\,dx \\ &= \sum_{n=0}^\infty \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n,c-a)}{B(a,c-a)}\frac{(-by)^n}{n!}\int_0^\infty x^{p+n-1}e^{-(y+z)x}\,dx. \end{aligned}$$

From the definition of Laplace transform (1.14) or the Gamma function (1.6), we obtain

$$L = \sum_{n=0}^\infty \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n,c-a)}{B(a,c-a)}\frac{(-by)^n}{n!}\frac{\Gamma(p+n)}{(y+z)^{n+p}},$$

which, in view of (1.7), leads to the right-hand side of (2.4). □

Theorem 4. *The inversion formula for the generalized integral transform* (1.16) *is given in the following form*:

$$f(y) = \Gamma(p)\mathcal{L}^{*-1}\{x^{1-p}\mathcal{L}^{-1}\{h(z);x\};y\}, \tag{2.5}$$

where

$$h(z) := \tilde{S}_p\{f(y);z\} \quad \text{and} \quad \mathcal{L}^{-1}\{h(z);x\} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} h(z)e^{zx}\,dz.$$

Proof. Integrating both sides of (2.4) with respect to the variable y from 0 to ∞, we have

$$\begin{aligned}(2.6)\quad &\frac{1}{\Gamma(p)}\int_0^\infty f(y)\left(\int_0^\infty x^{p-1}e^{-(y+z)x}{}_1F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}a\\c\end{matrix};-bxy;\gamma\right]dx\right)dy\\ &=\int_0^\infty \frac{f(y)}{(y+z)^p}\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}p,a\\c\end{matrix};-b\left(\frac{y}{y+z}\right);\gamma\right]dy.\end{aligned}$$

Changing the order of integration in (2.6) yields

$$\begin{aligned}(2.7)\quad &\tilde{S}_p\{f(y);z\}\\ &=\frac{1}{\Gamma(p)}\int_0^\infty x^{p-1}e^{-zx}\left(\int_0^\infty e^{-xy}f(y){}_1F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}a\\c\end{matrix};-bxy;\gamma\right]dy\right)dx\\ &=\frac{1}{\Gamma(p)}\int_0^\infty x^{p-1}e^{-zx}\left(\mathcal{L}^*\{f(y);x\}\right)\,dx\\ &=\frac{1}{\Gamma(p)}\mathcal{L}\{x^{p-1}\mathcal{L}^*\{f(y);x\};z\}.\end{aligned}$$

From (2.7), we obtain

$$(2.8)\qquad \mathcal{L}^*\{f(y);x\}=\Gamma(p)\,x^{1-p}\,\mathcal{L}^{-1}\{h(z);x\},$$

which, upon taking inverse of the transform $\mathcal{L}^*$ given in Theorem 2, leads to the desired result (2.5). $\square$

Lemma 5. *Let* $\Re(\gamma)\geq 0$, $\min\{\Re(\alpha),\ \Re(\beta),\ \Re(k),\ \Re(\mu)\}>0$, $\Re(c)>\Re(a)>0$, $|b|<1$ *and* $m\in\mathbb{N}$. *Then we have*

$$\begin{aligned}(2.9)\quad &\int_0^\infty \frac{v^{s-1}}{1+v^m}\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}1,a\\c\end{matrix};\frac{-b}{1+v^m};\gamma\right]dv\\ &=\frac{\Gamma(1-\frac{s}{m})\Gamma(\frac{s}{m})}{m}\left\{{}_1F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}a\\c\end{matrix};-b;\gamma\right]*{}_1F_0\left[\begin{matrix}1-\frac{s}{m}\\-\end{matrix};-b\right]\right\}.\end{aligned}$$

Proof. Let L be the left-hand side of (2.9). Using (1.8), we have

$$(2.10)\qquad L=\int_0^\infty \frac{v^{s-1}}{1+v^m}\sum_{n=0}^\infty (1)_n\frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n,c-a)}{B(a,c-a)}\frac{(\frac{-b}{1+v^m})^n}{n!}\,dv.$$

Changing the order of integration and summation in (2.10), which is guaranteed under the given conditions, we get

$$(2.11)\qquad L=\sum_{n=0}^\infty (1)_n\frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n,c-a)}{B(a,c-a)}\frac{(-b)^n}{n!}\int_0^\infty \frac{v^{s-1}}{1+v^m}\left(\frac{1}{1+v^m}\right)^n dv.$$

Setting $1/(1+v^m)=t$ in the integral in (2.11) and using (1.4), we obtain

$$L=\frac{\Gamma(1-\frac{s}{m})\Gamma(\frac{s}{m})}{m}\sum_{n=0}^\infty (1)_n\frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n,c-a)}{B(a,c-a)}\frac{(-b)^n}{n!}\frac{\Gamma(1-\frac{s}{m}+n)}{\Gamma(1-\frac{s}{m})\Gamma(n+1)},$$

which, in view of (1.11), leads to the right-hand side of (2.9). □

Theorem 6. *The inversion formula for the generalized potential integral transform* (1.18) *is given in the following form*:

$$f(x) = m \int_0^\infty y^{-1}\, F(y)\, R\left(\frac{y}{x}\right)\, dx, \tag{2.12}$$

where

$$F(y) := \tilde{P}_{m,1}\{f(x); y\}, \quad R(x) := \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{x^s}{\xi(s)}\, ds,$$

$$\xi(s) := \frac{\Gamma(1-\frac{s}{m})\Gamma(\frac{s}{m})}{m}\, \left\{ {}_1F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a \\ c \end{array}; -b; \gamma\right] * {}_1F_0\left[\begin{array}{c} 1-\frac{s}{m} \\ - \end{array}; -b\right]\right\}.$$

Proof. Applying the Mellin integral transform (1.9) to (1.18) and using (2.9), we obtain

$$\begin{aligned} &\mathcal{M}\{\tilde{P}_{m,1}\{f(x); y\}; s\} \\ &= \mathcal{M}\{f(u); s\}\, \mathcal{M}\left\{\frac{v^{s-1}}{1+v^m}\, {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} 1, a \\ c \end{array}; \frac{-b}{1+v^m}; \gamma\right]; s\right\} \\ &= \mathcal{M}\{f(u); s\}\, \frac{\Gamma(1-\frac{s}{m})\Gamma(\frac{s}{m})}{m}\, \left\{ {}_1F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a \\ c \end{array}; -b; \gamma\right] * {}_1F_0\left[\begin{array}{c} 1-\frac{s}{m} \\ - \end{array}; -b\right]\right\}, \end{aligned}$$

which gives

$$\mathcal{M}\{f(x); s\} = m\, \xi(s)^{-1}\, \mathcal{M}\{\tilde{P}_{m,1}\{f(x); y\}; s\}. \tag{2.13}$$

Applying the inversion formula for the Mellin integral transform (see [11]) to (2.13) yields the desired result (2.12). □

3. Parseval-type relations

In this section, some Parseval-type relations (see [3]) for the generalized integral transforms are considered.

Theorem 7. *The following equality holds true*:

$$\int_0^\infty g(u)\, \mathcal{L}^*\{f(t); u\}\, du = \int_0^\infty f(t)\, \mathcal{L}^*\{g(u); t\}\, dt, \tag{3.1}$$

provided that the involved integrals exist and converge absolutely.

Proof. We find from (1.15) that

$$\begin{aligned} &\int_0^\infty g(u)\, \mathcal{L}^*\{f(t); u\}\, du \\ &= \int_0^\infty g(u)\, \left(\int_0^\infty e^{-ut}\, {}_1F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a \\ c \end{array}; -but; \gamma\right] f(t) dt\right)\, du \\ &= \int_0^\infty f(t)\, \left(\int_0^\infty e^{-ut}\, {}_1F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a \\ c \end{array}; -but; \gamma\right] g(u) du\right)\, dt \end{aligned}$$

$$= \int_0^\infty f(t)\, \mathcal{L}^*\{g(u);t\}\, dt. \qquad \square$$

Let

$$\Phi(u) := u\mathcal{L}^*\{f(t);u\} \quad \text{and} \quad \Psi(t) := t\mathcal{L}^*\{g(u);t\}.$$

Then, for $b = 0$ in (1.15), we obtain the following well known result (see [2]):

$$\int_0^\infty \frac{\Phi(u)g(u)}{u}\, du = \int_0^\infty \frac{\Psi(t)f(t)}{t}\, dt. \tag{3.2}$$

Lemma 8. *The following equality holds true*:

$$\mathcal{L}\{\mathcal{L}^*\{g(u);x\};y\} = \tilde{P}_{1,1}\{g(u);y\}, \tag{3.3}$$

provided that the involved integral transforms converge.

Proof. Using (1.15) and (1.18), we obtain

$$\begin{aligned}
&\mathcal{L}\{\mathcal{L}^*\{g(u);x\};y\} \\
&= \int_0^\infty e^{-xy}\, \mathcal{L}^*\{g(u);x\}\, dx \\
&= \int_0^\infty e^{-xy} \left(\int_0^\infty e^{-ux}\, {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} a \\ c \end{matrix} ; -bux; \gamma \right] g(u) du \right) dx \\
&= \int_0^\infty \left(\int_0^\infty e^{-(u+y)x}\, {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} a \\ c \end{matrix} ; -bux; \gamma \right] g(u) dx \right) du.
\end{aligned}$$

Considering (2.4), we get

$$\mathcal{L}\{\mathcal{L}^*\{g(u);x\};y\} = \int_0^\infty \frac{g(u)}{y+u}\, {}_2F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} 1, a \\ c \end{matrix} ; -b\left(\frac{u}{y+u}\right); \gamma \right] du, \tag{3.4}$$

which, upon using (1.18), yields the desired result (3.3). $\square$

Theorem 9. *The following Parseval-Goldstein type identity holds true*:

$$\int_0^\infty \mathcal{L}\{h(y);x\}\{\mathcal{L}^*\{g(u);x\}\, dx = \int_0^\infty h(y)\, \tilde{P}_{1,1}\{g(u);y\}\, dy. \tag{3.5}$$

Proof. Let L be the left-hand side of (3.5). Then we have

$$\begin{aligned}
L &= \int_0^\infty \left(\int_0^\infty e^{-xy} h(y)\, dy \right) \left(\int_0^\infty e^{-ux}\, {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} a \\ c \end{matrix} ; -bux; \gamma \right] g(u)\, du \right) dx \\
&= \int_0^\infty h(y) \left(\int_0^\infty e^{-xy} \left(\int_0^\infty e^{-ux}\, {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} a \\ c \end{matrix} ; -bux; \gamma \right] g(u)\, du \right) dx \right) dy \\
&= \int_0^\infty h(y) \left(\int_0^\infty e^{-xy} (\{\mathcal{L}^*\{g(u);x\})\, dx \right) dy \\
&= \int_0^\infty h(y)\, (\mathcal{L}\{\mathcal{L}^*\{g(u);x\})\, dy,
\end{aligned}$$

which, in view of (3.3), leads to the right-hand side of (3.5). $\square$

Theorem 10. *The following Parseval-Goldstein type equality holds true*:

$$\int_0^\infty \mathcal{L}\{f(y); x+z\}\{\mathcal{L}^*\{g(u); x\} \, dx = \mathcal{L}\{f(y)\tilde{P}_{1,1}\{g(u); y\}; z\}, \tag{3.6}$$

provided the integrals involved exist and converge absolutely.

Proof. Setting $h(y) = e^{-yz} f(y)$ in (3.5) and using the identity

$$\mathcal{L}\{e^{-yz} f(y); x\} = \mathcal{L}\{f(y); x+z\}, \tag{3.7}$$

one can easily obtain (3.6). □

4. Extended integral transforms

Here three $\mathcal{L}^*$ transforms and an integral of $\mathcal{L}^*$ transform are considered as in the following theorem.

Theorem 11. *The following formulas hold true*:

(i) *Let* α, β, γ, κ, μ, $\rho \in \mathbb{C}$ *with* $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\rho)\} > 0$ *and* $\min\{\Re(\kappa), \Re(\mu)\} \geq 0$. *Then the following formula holds true*:

$$\mathcal{L}^*\{x^{\rho-1}; y\} = \frac{\Gamma(\rho)}{y^\rho} \, {}_2F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} \rho, a \\ c \end{matrix} ; -b; \gamma \right]. \tag{4.1}$$

(ii) *The following formula holds true*:

$$\int_0^\infty \mathcal{L}^*\{e^{-xu} g(x); y\} \, dy = {}_2F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} 1, a \\ c \end{matrix} ; -b; \gamma \right] \mathcal{L}\left\{ \frac{g(x)}{x}; u \right\}. \tag{4.2}$$

(iii) *The following formulas hold true*:

$$\begin{aligned} &\int_0^\infty L^*\{e^{-xu} \sin \alpha x \ \cos \beta x; y\} \, dy \\ &= \frac{1}{2} \, {}_2F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} 1, a \\ c \end{matrix} ; -b; \gamma \right] \arctan \frac{2\alpha u}{u^2 - \alpha^2 + \beta^2} \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} &\int_0^\infty L^*\{e^{-xu} \sin \alpha x \ \sin \beta x; y\} \, dy \\ &= \frac{1}{4} \, {}_2F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} 1, a \\ c \end{matrix} ; -b; \gamma \right] \ln \frac{u^2 + (\alpha+\beta)^2}{u^2 + (\alpha-\beta)^2}. \end{aligned} \tag{4.4}$$

Proof. We begin by proving (4.1). Using (1.1) and (1.8), we have

$$\mathcal{L}^*\{x^{\rho-1}; y\} = \int_0^\infty e^{-xy} \, {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} a \\ c \end{matrix} ; -bxy; \gamma \right] x^{\rho-1} \, dx. \tag{4.5}$$

By changing the order of integration and summation, we obtain

$$\mathcal{L}^*\{x^{\rho-1}; y\} = \sum_{n=0}^\infty \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n, c-a)}{B(a, c-a)} \frac{(-by)^n}{n!} \int_0^\infty e^{-xy} x^{\rho+n-1} dx. \tag{4.6}$$

Using (1.6) or (1.14), we get

$$\mathcal{L}^*\{x^{\rho-1};y\} = \sum_{n=0}^{\infty} \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(a+n,c-a)}{B(a,c-a)} \frac{(-by)^n}{n!} \frac{\Gamma(\rho+n)}{y^{\rho+n}}, \tag{4.7}$$

which, in view of (1.7), yields the right-hand side of (4.1).

To prove (ii), let L be the left-hand side of (4.2). We find from (1.15) that

$$\begin{aligned} L &= \int_0^\infty \left(\int_0^\infty e^{-xy} {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} a \\ c \end{matrix} ; -bxy; \gamma \right] e^{-xu} g(x)\, dx \right) dy \\ &= \int_0^\infty \left(\int_0^\infty e^{-xy} {}_1F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} a \\ c \end{matrix} ; -bxy; \gamma \right] dy \right) g(x) e^{-xu} dx. \end{aligned}$$

Setting $xy = v$ and using (2.1), we obtain

$$L = {}_2F_1^{(\alpha,\beta;\kappa,\mu)} \left[\begin{matrix} 1, a \\ c \end{matrix} ; -b; \gamma \right] \int_0^\infty e^{-xu} \frac{g(x)}{x}\, dx, \tag{4.8}$$

which, in view of (1.14), leads to the right-hand side of (4.2).

Lastly we prove (iii). Replacing $g(x)$ in (4.2) by $\sin(\alpha x)\cos(\beta x)$ and $\sin(\alpha x)\sin(\beta x)$, and using (1.13) and (1.12), respectively, we can obtain the desired results (4.3) and (4.4). □

Acknowledgements. The authors should express their deep gratitude for the reviewer's detailed and helpful comments.

References

[1] R. Agarwal, S. Jain, and R. P. Agarwal, *Solution of fractional volterra integral equation and non-homogeneous time fractional heat equation using integral transform of pathway type*, Progr. Fract. Differ. Appl. **1** (2015), no. 3, 145–155.

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms. Vol. I*, McGraw-Hill Book Company, New York, Toronto and London, 1954.

[3] S. Goldstein, *Operational representations of Whittaker's confluent hypergeometric function and Weber's parabolic cylinder function*, Proc. London Math. Soc. **34** (1932), no. 1, 103–125.

[4] K. B. Kachhia and J. C. Prajapati, *Solution of fractional partial differential equation aries in study of heat transfer through diathermanous materials*, J. Interdisciplinary Math. **18** (2015), no. 1-2, 125–132.

[5] A. A. Kilbas and M. Saiogo, *H-Transforms*, Chapman and Hall/CRC. Boca Raton, London, New York, 2004.

[6] M. J. Luo, G. V. Milovanovic, and P. Agarwal, *Some results on the extended beta and extended hypergeometric functions*, Appl. Math. Comut. **248** (2014), 631–651.

[7] T. Pohlen, *The Hadamard product and universal power series*, Dissertation, Universität Trier, 2009.

[8] J. C. Prajapati and K. B. Kachhia, *Fractional modeling of temperature distribution and heat flux in the semi infinite solid*, J. Fract. Calc. Appl. **5** (2014), no. 2, 38–43.

[9] J. C. Prajapati, A. D. Patel, K. N. Pathak, and A. K. Shukla, *Fractional calculus approach in the study of instability phenomenon in fluid dynamics*, Palest. J. Math. **1** (2012), no. 2, 95–103.

[10] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.
[11] I. N. Sneddon, *The Use of Integral Transform*, Tata McGraw-Hill, New Delhi, India, 1979.
[12] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam-London-New York, 2012.
[13] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.
[14] H. M. Srivastava and O. Yurelki, *A theorem on a Stieltjes-type integral transform and its applications*, Complex Variables Theory Appl. **28** (1995), no. 2, 159–168.
[15] N. Virchenko, *On the generalized confluent hypergeometric function and its applications*, Fract. Calc. Appl. Anal. **9** (2006), no. 2, 101–108.
[16] N. O. Virchenko and O. V. Ovcharenko, *Some new integral transforms with the generalized hypergeometric function*, Integral Transforms Spec. Funct. **22** (2011), no. 9, 647–653.

Junesang Choi
Department of Mathematics
Dongguk University
Gyeongju 38066, Korea
E-mail address: junesang@mail.dongguk.ac.kr

Krunal B. Kachhia
Department of Mathematical Sciences
Faculty of Applied Sciences
Charotar University of Science and Technology (CHARUSAT)
Changa, Anand 388421, Gujarat, India
E-mail address: krunalmaths@hotmail.com

Jyotindra C. Prajapati
Department of Mathematics
Faculty of Technology and Engineering
Marwadi Education Foundation Group of Institutions (MEFGI)
Rajkot 360003, Gujarat, India
E-mail address: jyotindra18@rediffmail.com

Sunil Dutt Purohit
Department of HEAS (Mathematics)
Rajasthan Technical University
Kota 342010, Rajasthan, India
E-mail address: sunil_a_purohit@yahoo.com