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WEAK CONVERGENCE THEOREMS FOR 2-GENERALIZED HYBRID MAPPINGS AND EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we propose a new modified Ishikawa iteration for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of 2-generalized hybrid mappings in a Hilbert space. Our results generalize and improve some existing results in the literature. A numerical example is given to illustrate the usability of our results.

1. Introduction

Numerous problems in physics, optimization and economics reduce to find a solution of an equilibrium problem. Some methods have been proposed to solve the equilibrium problem; see for instance, [4, 8, 11, 17, 21].

In the recent years, many authors have interested the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem in the framework of Hilbert spaces and Banach spaces, respectively; see for instance, [3, 5, 6, 9, 15, 16, 22, 23, 24, 28] and the references therein. They have used various iterative processes to finding this common element. The most useful of these processes are Mann [19] and Ishikawa [13] iteration processes.

Ishikawa process is indeed more general than Mann process. In spite of this fact, research has been done on the latter due probably to reasons that the formulation of Mann process is simpler than that of Ishikawa process and that a convergence theorem for Mann process may lead to a convergence theorem for Ishikawa process under appropriate conditions. On the other hand, the Mann process may fail to converge while Ishikawa process can still converge for a Lipschitz pseudocontractive mapping in a Hilbert space [7]. Actually, Mann and Ishikawa iteration processes have only weak convergence, in general (see [10]).

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Recently, Alizadeh and Moradlou [2] have considered the class of m-generalized hybrid mappings in Hilbert spaces and they proved weak and strong convergence theorems for this class of nonlinear mappings.

In this paper, we modify Ishikawa iteration process for finding a common element of the set of solution of an equilibrium problem and the set of fixed points of a 2-generalized hybrid mapping.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and let E be a nonempty closed convex subset of H. The equilibrium problem for a bifunction $f : E \times E \to \mathbb{R}$ is to find $x \in E$ such that

$$f(x, y) \geq 0, \quad (\forall\, y \in E). \tag{2.1}$$

The solutions set of (2.1) is denoted by $EP(f)$, i.e.,

$$EP(f) = \{x \in E : \ f(x, y) \geq 0, \ \ \forall y \in E\}.$$

A self mapping S of E is called nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad (\forall x, y \in E).$$

We denote by $F(S)$ the set of fixed points of S.

Let $S : E \longrightarrow H$ be a mapping and let $f(x, y) = \langle Sx, y - x \rangle$ for all $x, y \in E$. Then $z \in EP(f)$ if only if $\langle Sz, y - z \rangle \geq 0$ for all $y \in E$, i.e., z is a solution of the variational inequality $\langle Sx, y - x \rangle \geq 0$ all $y \in E$. So, the formulation (2.1) includes variational inequalities as special cases.

A self mapping S of E is called: (i) *firmly nonexpansive*, if $\|Sx - Sy\|^2 \leq \langle x - y, Sx - Sy \rangle$ for all $x, y \in E$; (ii) *nonspreading* [18], if $2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|Sy - x\|^2$ for all $x, y \in E$; (iii) *hybrid* [26], if $3\|Sx - Sy\|^2 \leq \|x - y\|^2 + \|Sx - y\|^2 + \|Sy - x\|^2$ for all $x, y \in E$. Also, a self mapping S of E with $F(S) \neq \emptyset$ is called *quasi-nonexpansive* if $\|x - Sy\| \leq \|x - y\|$ for all $x \in F(S)$ and $y \in E$. It is well-known that for a *quasi-nonexpansive* mapping S, $F(S)$ is closed and convex [14].

Let E be a nonempty closed convex subset of H. A self mapping S of E is called *generalized hybrid* [3] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Sx - Sy\|^2 + (1 - \alpha)\|x - Sy\|^2 \leq \beta\|Sx - y\|^2 + (1 - \beta)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in E$. We call such a mapping an (α, β)-*generalized hybrid* mapping.

It easy to see that

- $(1, 0)$-generalized hybrid mapping is nonexpansive;
- $(2, 1)$-generalized hybrid mapping is nonspreading;
- $(\frac{3}{2}, \frac{1}{2})$-generalized hybrid mapping is hybrid.

A self mapping T of C is called 2-*generalized hybrid* [20] if there exist $\gamma_1, \gamma_2, \lambda_1, \lambda_2 \in \mathbb{R}$ such that

$$\begin{aligned}&\gamma_1\|T^2x - Ty\|^2 + \gamma_2\|Tx - Ty\|^2 + (1 - \gamma_1 - \gamma_2)\|x - Ty\|^2\\ \le\ &\lambda_1\|T^2x - y\|^2 + \lambda_2\|Tx - y\|^2 + (1 - \lambda_1 - \lambda_2)\|x - y\|^2\end{aligned}$$

for all $x, y \in C$. Such a mapping is called a $(\gamma_1, \gamma_2, \lambda_1, \lambda_2)$-generalized hybrid mapping. It is easy to see that a $(0, \gamma_2, 0, \lambda_2)$-generalized hybrid mapping is a (γ_2, λ_2)-generalized hybrid mapping [12]. Also, one can easily show that a 2-generalized hybrid mapping is quasi-nonexpansive if the set of it's fixed points is nonempty. In [12], Hojo et al. give two examples of 2-generalized hybrid mappings which are not generalized hybrid mappings. So, the class of 2-generalized hybrid mappings is broader than the class of generalized hybrid mappings.

Throughout this paper, we denote the weak convergence and the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively and denote $\omega_\omega(x_n)$ the weak ω-limit set of the sequence $\{x_n\}$, i.e., $\omega_\omega(x_n) := \{x \in H : \exists\{x_{n_k}\} \subset \{x_n\}; x_{n_k} \rightharpoonup x\}$.

Now, we recall some basic properties of Hilbert spaces which we will use in next section. For $x, y \in H$, we have from [25] that

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2, \quad \forall\alpha \in \mathbb{R}, \tag{2.3}$$

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle, \tag{2.4}$$

and

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle. \tag{2.5}$$

Let K be a closed convex subset of H and let P_K be metric (or nearest point) projection from H onto K (i.e., for $x \in H$, $P_K x$ is the only point in K such that $\|x - P_K x\| = \inf\{\|x - z\| : z \in K\}$). Let $x \in H$ and $z \in K$. Then $z = P_K x$ if and only if

$$\langle x - z, y - z\rangle \le 0 \tag{2.6}$$

for all $y \in K$. For more details we refer readers to [1, 25].

Lemma 2.1 ([29])**.** *Let H be a Hilbert space and $\{x_n\}$ be a sequence in H such that there exists a nonempty subset $E \subset H$ satisfying*

(i) *For every $u \in E$, $\lim_{n\to\infty}\|x_n - u\|$ exists,*
(ii) *If a subsequence $\{x_{n_j}\} \subset \{x_n\}$ converges weakly to u, then $u \in E$.*

Then there exists $x_0 \in E$ such that $x_n \rightharpoonup x_0$.

We will use the following lemmas in the proof of our main results in next section.

Lemma 2.2 ([27])**.** *Let H be a Hilbert space and E be a nonempty, closed and convex subset of H and $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - x\| \le \|x_n - x\|$ for* *all $n \in \mathbb{N}$ and $x \in E$, then $\{P_E(x_n)\}$ converges strongly to some $z \in E$, where P_E stands for the metric projection on H onto E.*

To study the equilibrium problem, we assume that $f : E \times E \longrightarrow \mathbb{R}$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in E$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \le 0$ for all $x, y \in E$;
(A3) for each $x, y, z \in E$,

$$\lim_{t \downarrow 0} f(tz + (1-t)x, y) \le f(x, y);$$

(A4) for each $x \in E$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

The following lemma can be found in [4].

Lemma 2.3. *Let E be a nonempty closed convex subset of H, let f be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and let $r > 0$ and $x \in H$. Then, there exists $z \in E$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0$$

for all $y \in E$.

The following lemma is established in [8].

Lemma 2.4. *For $r > 0$, $x \in H$, define a mapping $T_r : H \longrightarrow E$ as follows*:

$$T_r(x) = \{z \in E : f(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in E\}.$$

Then, the following statements hold:

(i) *T_r is singel-valued*;
(ii) *T_r is firmly nonexpansive, i.e., for all $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle;$$

(iii) $F(T_r) = EP(f)$;
(iv) *$EP(f)$ is closed and convex.*

3. Main results

In this section, we prove weak convergence theorems for finding a common element of the set of solution of an equilibrium problem and the set of fixed points of a 2-generalized hybrid mapping.

Theorem 3.1. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and S be a 2-generalized hybrid self mapping of E such that $F(S) \cap EP(f) \ne \phi$ and $\|S^2 x - Sx\| \le \|Sx - x\|$ for all $x \in E$. Assume that $0 < \alpha < \alpha_n < \beta < 1$ and $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n \to \infty} r_n > 0$ and $\{\beta_n\}$ is a sequence in $[b, 1]$ for* *some* $b \in (0,1)$ *such that* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. *If* $\{x_n\}$ *and* $\{u_n\}$ *are sequences generated by* $x_1 = x \in E$ *and let*

$$\begin{cases} u_n \in E \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in E, \\ y_n = (1-\beta_n)x_n + \beta_n S u_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n, \end{cases}$$

for all $n \in \mathbb{N}$. *Then*

$$x_n \rightharpoonup v \in F(S) \cap EP(f),$$

where $v = \lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n)$.

Proof. By Lemma 2.3, $\{u_n\}, \{y_n\}$ and $\{x_n\}$ are well defined. Since S is a 2-generalized hybrid mapping such that $F(S) \neq \phi$, S is quasi-nonexpansive. So $F(S)$ is closed and convex. Also by hypothesis $EP(f) \neq \phi$. Let $q \in F(S) \cap EP(f)$.

From $u_n = T_{r_n} x_n$, we get

$$\|u_n - q\| = \|T_{r_n}x_n - T_{r_n}q\| \leq \|x_n - q\|. \tag{3.1}$$

On the other hand,

$$\begin{aligned} \|y_n - q\|^2 &= (1-\beta_n)\|x_n - q\|^2 + \beta_n\|Su_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - Su_n\|^2 \\ &\leq (1-\beta_n)\|x_n - q\|^2 + \beta_n\|x_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - Su_n\|^2 \\ &= \|x_n - q\|^2 - \beta_n(1-\beta_n)\|x_n - Su_n\|^2 \end{aligned} \tag{3.2}$$

and hence

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1-\alpha_n)x_n + \alpha_n S y_n - q\|^2 \\ &= (1-\alpha_n)\|x_n - q\|^2 + \alpha_n\|Sy_n - q\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sy_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - q\|^2 + \alpha_n\|y_n - q\|^2 - \alpha_n(1-\alpha_n)\|x_n - Sy_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - q\|^2 + \alpha_n\|x_n - q\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Su_n\|^2 \\ &- \alpha_n(1-\alpha_n)\|x_n - Sy_n\|^2 \\ &\leq \|x_n - q\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Su_n\|^2 \\ &\leq \|x_n - q\|^2. \end{aligned} \tag{3.3}$$

So, we can conclude that $\lim_{n\to\infty}\|x_n - q\|$ exists. This yields that $\{x_n\}$ and $\{y_n\}$ are bounded. It follows from (3.3) that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha_n\beta_n(1-\beta_n)\|x_n - Su_n\|^2.$$

By using $0 < \alpha < \alpha_n < \beta < 1$, it is easy to see that

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \alpha\beta_n(1-\beta_n)\|x_n - Su_n\|^2.$$

Also, we have

$$0 \leq \alpha\beta_n(1-\beta_n)\|x_n - Su_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2 \to 0$$

as $n \to \infty$, since $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. Therefore

$$\|x_n - Su_n\| \longrightarrow 0. \tag{3.4}$$

This yields that

$$\|y_n - x_n\| = \beta_n\|x_n - Su_n\| \longrightarrow 0. \tag{3.5}$$

Using (2.5) and Lemma 2.4, we get

$$\begin{aligned}\|u_n - q\|^2 &= \|T_{r_n}x_n - T_{r_n}q\|^2\\ &\le \langle T_{r_n}x_n - T_{r_n}q, x_n - q\rangle\\ &= \langle u_n - q, x_n - q\rangle\\ &= \frac{1}{2}(\|u_n - q\|^2 + \|x_n - q\|^2 - \|x_n - u_n\|^2),\end{aligned}$$

hence

$$\|u_n - q\|^2 \le \|x_n - q\|^2 - \|x_n - u_n\|^2.$$

Then, by the convexcity of $\|\cdot\|^2$, we get

$$\begin{aligned}\|y_n - q\|^2 &= \|(1-\beta_n)(x_n - q) + \beta_n(Su_n - q)\|^2\\ &\le (1-\beta_n)\|x_n - q\|^2 + \beta_n\|Su_n - q\|^2\\ &\le (1-\beta_n)\|x_n - q\|^2 + \beta_n\|u_n - q\|^2\\ &\le (1-\beta_n)\|x_n - q\|^2 + \beta_n(\|x_n - q\|^2 - \|x_n - u_n\|^2)\\ &= \|x_n - q\|^2 - \beta_n\|x_n - u_n\|^2.\end{aligned}$$

Therefore

$$\beta_n\|x_n - u_n\|^2 \le \|x_n - q\|^2 - \|y_n - q\|^2. \tag{3.6}$$

Since $\{\beta_n\} \subset [b, 1]$, it follows from (3.6) that

$$\begin{aligned}b\|x_n - u_n\|^2 &\le \beta_n\|x_n - u_n\|^2\\ &\le \|x_n - q\|^2 - \|y_n - q\|^2\\ &= (\|x_n - q\| - \|y_n - q\|)(\|x_n - q\| + \|y_n - q\|)\\ &\le \|y_n - x_n\|(\|x_n - q\| + \|y_n - q\|).\end{aligned}$$

By using the boundedness of $\{x_n\}$ and $\{y_n\}$, it follows from (3.5) and the above inequality that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.7}$$

Since $\liminf_{n\to\infty} r_n > 0$, we get

$$\lim_{n\to\infty} \left\|\frac{x_n - u_n}{r_n}\right\| = \lim_{n\to\infty} \frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.8}$$

As $\beta_n Su_n = y_n - (1-\beta_n)x_n$, we get

$$\begin{aligned} b\|u_n - Su_n\| &\le \beta_n\|Su_n - u_n\| = \|y_n - (1-\beta_n)x_n - \beta_n u_n\| \\ &\le \|y_n - u_n\| + (1-\beta_n)\|x_n - u_n\| \\ &\le \|y_n - x_n\| + \|x_n - u_n\| + \|x_n - u_n\| \\ &= \|y_n - x_n\| + 2\|x_n - u_n\|. \end{aligned}$$

From (3.5) and (3.7), we obtain

$$\lim_{n\to\infty} \|u_n - Su_n\| = 0. \tag{3.9}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup u$. By (3.7) we obtain $u_{n_i} \rightharpoonup u$. We know that E is closed and convex and $\{u_{n_i}\} \subset E$, therefore $u \in E$.

Now, we show that $u \in F(S) \cap EP(f)$. Since $u_n = T_{r_n} x_n$, we get

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0$$

for all $y \in E$. From the condition $(A2)$, we obtain

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge f(y, u_n)$$

for all $y \in E$, therefore

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \ge f(y, u_{n_i}) \tag{3.10}$$

for all $y \in E$. It follows from (3.8), (3.10) and condition $(A4)$ that

$$0 \ge f(y, u)$$

for all $y \in E$. Suppose that $t \in (0, 1]$, $y \in E$ and $y_t = ty + (1-t)u$. Therefore, $y_t \in E$ and so $f(y_t, u) \ge 0$. Hence

$$0 = f(y_t, y_t) \le tf(y_t, y) + (1-t)f(y_t, u) \le tf(y_t, y),$$

and dividing by t, we have $f(y_t, y) \ge 0$ for all $y \in E$. By taking the limit as $t \downarrow 0$ and using $(A3)$, we get $u \in EP(f)$.

Next we show that $u \in F(S)$. Since S is a 2-generalized hybrid mapping, then

$$\begin{aligned} &\alpha_1\|S^2x - Sy\|^2 + \alpha_2\|Sx - Sy\|^2 + (1-\alpha_1-\alpha_2)\|x - Sy\|^2 \\ &\le \beta_1\|T^2x - y\|^2 + \beta_2\|Sx - y\|^2 + (1-\beta_1-\beta_2)\|x - y\|^2 \end{aligned}$$

hence

$$\begin{aligned} 0 \le\ & \beta_1\|S^2x - y\|^2 + \beta_2\|Sx - y\|^2 + (1-\beta_1-\beta_2)\|x - y\|^2 - \alpha_1\|S^2x - Sy\|^2 \\ & - \alpha_2\|Sx - Sy\|^2 - (1-\alpha_1-\alpha_2)\|x - Sy\|^2, \end{aligned}$$

replacing x and y by u_n and u in above inequality, respectively, we get

$$0 \le \beta_1(\|S^2u_n\|^2 - 2\langle S^2u_n, u\rangle + \|u\|^2) + \beta_2(\|Su_n\|^2 - 2\langle Su_n, u\rangle + \|u\|^2)$$

$$
\begin{aligned}
&+(1-\beta_1-\beta_2)(\|u_n\|^2-2\langle u_n,u\rangle+\|u\|^2)\\
&-\alpha_1(\|S^2u_n\|^2-2\langle S^2u_n,Su\rangle+\|Su\|^2)\\
&-\alpha_2(\|Su_n\|^2-2\langle Su_n,Su\rangle+\|Su\|^2)\\
&-(1-\alpha_1-\alpha_2)(\|u_n\|^2-2\langle u_n,Su\rangle+\|Su\|^2)\\
=\ &\|u\|^2-\|Su\|^2+\sum_{k=1}^{2}(\beta_k-\alpha_k)(\|S^{3-k}u_n\|^2-\|u_n\|^2)\\
&+2\sum_{k=1}^{2}\alpha_k\langle S^{3-k}u_n-u_n,Su\rangle-2\sum_{k=1}^{2}\beta_k\langle S^{3-k}u_n-u_n,u\rangle\\
&+2\langle u_n,Su-u\rangle\\
\le\ &\|u\|^2-\|Su\|^2+\sum_{k=1}^{2}(\beta_k-\alpha_k)(\|S^{3-k}u_n\|+\|u_n\|)(\|S^{3-k}u_n-u_n\|)\\
&+2\sum_{k=1}^{2}\alpha_k\langle S^{3-k}u_n-u_n,Su\rangle-2\sum_{k=1}^{2}\beta_k\langle S^{3-k}u_n-u_n,u\rangle\\
&+2\langle u_n,Su-u\rangle.
\end{aligned}
$$

Now, substituting n by n_i, we have
(3.11)

$$
\begin{aligned}
0\le\ &\|u\|^2-\|Su\|^2+\sum_{k=1}^{2}(\beta_k-\alpha_k)(\|S^{3-k}u_{n_i}\|+\|u_{n_i}\|)(\|S^{3-k}u_{n_i}-u_{n_i}\|)\\
&+2\sum_{k=1}^{2}\alpha_k\langle S^{3-k}u_{n_i}-u_{n_i},Su\rangle-2\sum_{k=1}^{2}\beta_k\langle S^{3-k}u_{n_i}-u_{n_i},u\rangle\\
&+2\langle u_{n_i},Su-u\rangle
\end{aligned}
$$

for all $i\in\mathbb{N}$. Since $u_{n_i}\rightharpoonup u$ as $i\to\infty$, it follows from (3.4) and (3.11) that

$$
\begin{aligned}
0&\le\|u\|^2-\|Su\|^2+2\langle u,Su-u\rangle\\
&=2\langle u,Su\rangle-\|u\|^2-\|Su\|^2\\
&=-\|u-Su\|^2.
\end{aligned}
$$

So, we have $Su=u$, i.e., $u\in F(S)$. Therefore the condition (ii) of Lemma 2.1, satisfies for $E=F(S)\cap EP(f)$. On the other hand, we see that $\lim_{n\to\infty}\|x_n-q\|$ exists for $q\in F(S)\cap EP(f)$. Hence, it follows from Lemma 2.1 that there exists $v\in F(S)\cap EP(f)$ such that $x_n\rightharpoonup v$. In addition, for all $q\in F(S)\cap EP(f)$, we have

$$\|x_{n+1}-q\|\le\|x_n-q\|,\qquad\forall\ n\in\mathbb{N},$$

so, Lemma 2.2 implies that there exists some $w \in F(S) \cap EP(f)$ such that $P_{F(T)\cap EP(f)}(x_n) \to w$. Then

$$\Big\langle v - P_{F(T)\cap EP(f)}(x_n), x_n - P_{F(T)\cap EP(f)}(x_n) \Big\rangle \le 0.$$

Hence, we get

$$\langle v - w, v - w \rangle = \|v - w\|^2 \le 0.$$

Therefore $v = w$, i.e., $x_n \rightharpoonup v = \lim_{n\to\infty} P_{F(T)\cap EP(f)}(u_n)$. □

Corollary 3.2. *Let E be a nonempty closed convex subset of a real Hilbert space H and S be a 2-generalized hybrid self mapping of E with $F(S) \neq \phi$ and $\|S^2x - Sx\| \le \|Sx - x\|$ for all $x \in E$. Assume that $0 < \alpha \le \alpha_n \le 1$ and $\{\beta_n\}$ is a sequence in $[b, 1]$ for some $b \in (0, 1)$ such that $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. If $\{x_n\}$ and $\{u_n\}$ are sequences generated by $x_1 = x \in E$ and*

$$\begin{cases} u_n \in E \text{ such that } \quad \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall\, y \in E, \\ y_n = (1 - \beta_n)x_n + \beta_n S u_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n, \end{cases}$$

for all $n \in \mathbb{N}$. Then $x_n \rightharpoonup v \in F(S)$, where $v = \lim_{n\to\infty} P_{F(S)}(x_n)$.

Proof. Letting $f(x, y) = 0$ for all $x, y \in E$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.1, we get the desired result. □

Corollary 3.3. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and S be a 2-generalized hybrid self mapping of E with $F(S) \cap EP(f) \neq \phi$ and $\|S^2x - Sx\| \le \|Sx - x\|$ for all $x \in E$. Assume that $\{r_n\} \subset (0, \infty)$ satisfies* $\liminf_{n\to\infty} r_n > 0$ *and $\{\beta_n\}$ is a sequence in $[b, 1]$ for some $b \in (0, 1)$ such that* $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. *If $\{x_n\}$ and $\{u_n\}$ are sequences generated by $x_1 = x \in E$ and*

$$\begin{cases} u_n \in E \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall\, y \in E, \\ x_{n+1} = S((1 - \beta_n)x_n + \beta_n S u_n), \end{cases}$$

for all $n \in \mathbb{N}$. Then

$$x_n \rightharpoonup v \in F(S) \cap EP(f),$$

where $v = \lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n)$.

Proof. Letting $\alpha_n = 1$ for all $n \in \mathbb{N}$, in Theorem 3.1, we get the desired result. □

Theorem 3.4. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let f be a bifunction from $E \times E$ to $\mathbb{R}$ satisfying* (A1)-(A4) *and S be a (γ, λ)-generalized hybrid self mapping of E with $F(S) \cap EP(f) \neq \phi$. Assume that $0 < \alpha \le \alpha_n \le 1$ and $\{r_n\} \subset (0, \infty)$ satisfies* $\liminf_{n\to\infty} r_n > 0$ *and $\{\beta_n\}$*

is a sequence in $[b,1]$ *for some* $b \in (0,1)$ *such that* $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. *If* $\{x_n\}$ *and* $\{u_n\}$ *are sequences generated by* $x_1 = x \in E$ *and*

$$\begin{cases} u_n \in E \text{ such that } \quad f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall\, y \in E, \\ y_n = (1-\beta_n)x_n + \beta_n S u_n, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S y_n, \end{cases}$$

for all $n \in \mathbb{N}$. *Then*

$$x_n \rightharpoonup v \in F(S) \cap EP(f),$$

where $v = \lim_{n\to\infty} P_{F(S)\cap EP(f)}(x_n)$.

Proof. Since S is a (γ, λ)-generalized hybrid mapping, hence S is a $(0, \gamma, 0, \lambda)$-generalized hybrid mapping. Therefore by Theorem 3.1, we get the desired result. □

4. Numerical example

Now, we demonstrate Theorem 3.1 with an example.

Example 4.1. Let $H = \mathbb{R}$ and $E = [-2,2]$. Define $f(u,y) := 5y^2 + uy - 6u^2$. We see that f satisfies the conditions (A1)-(A4) as follows:

(A1) $f(u,u) = 5u^2 + u^2 - 6u^2 = 0$ for all $u \in [-2,2]$,

(A2) $f(u,y) + f(y,u) = -(y-u)^2 \leq 0$ for all $u, y \in [-2,2]$, i.e., f is monotone,

(A3) for each $u, y, z \in [-2,2]$,

$$\begin{aligned} \lim_{t\downarrow 0} f(tz + (1-t)u, y) &= \lim_{t\downarrow 0}(5y^2 + (tz + (1-t)u)y - 6(tz + (1-t)u)^2) \\ &= 5y^2 + uy - 6u^2 \\ &= f(u,y), \end{aligned}$$

(A4) it is easily seen that for each $u \in [-2,2]$, $y \to (5y^2 + uy - 6u^2)$ is convex and lower semicontinuous.

On the other hand,

$$\frac{1}{r}\langle y-u, u-x\rangle = \frac{1}{r}(y-u)(u-x) = \frac{1}{r}(uy - u^2 + ux - xy).$$

From condition (i) of Lemma 2.4, T_r is s ingle-valued. Let $u = T_r x$, for any $y \in [-2,2]$ and $r > 0$, we have

$$f(u,y) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0.$$

Thus

$$5ry^2 + ruy - 6ru^2 + uy - u^2 + ux - xy = 5ry^2 + (ru + u - x)y - 6ru^2 - u^2 + ux \geq 0.$$

Now, let $a = 5r$, $b = ru + u - x$ and $c = -6ru^2 - u^2 + ux$. Hence, we should have $\Delta = b^2 - 4ac \leq 0$, i.e.,

$$\Delta = ((r+1)u - x)^2 + 20ru((6r+1)u - x)$$

$$
\begin{aligned}
&= 121r^2u^2 + 22ru^2 + u^2 + x^2 - 22rux - 2ux \\
&= ((11r+1)u - x)^2 \\
&\leq 0.
\end{aligned}
$$

So, it follows that $u = \frac{x}{11r+1}$. Therefore, $T_r x = \frac{x}{11r+1}$.

This implies that in Theorem 3.1, $u_n = T_{r_n} x_n = \frac{x_n}{11r_n+1}$. Since $F(T_{r_n}) = 0$, from condition (iii) of Lemma 2.4, $EP(f) = \{0\}$.

Define $S : E \to E$ by $Sx = \frac{1}{3}x$ for all $x \in E$, thus $F(S) = \{0\}$.

It is easy to see that S is a $(\frac{1}{2}, \frac{1}{2}, \frac{16}{18}, \frac{1}{18})$-generalized hybrid mapping.

Assume that $\alpha_n = \frac{1}{2} + \frac{1}{3n}$, $\beta_n = \frac{1}{3} - \frac{1}{6n}$ and $r_n = \frac{1}{11}$, so $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy in Theorem 3.1. Since $u_n = \frac{1}{2}x_n$, we get

$$
\begin{aligned}
y_n &:= (\frac{2}{3} + \frac{1}{6n})x_n + \frac{1}{6}(\frac{1}{3} - \frac{1}{6n})x_n \\
&= (\frac{13}{18} + \frac{5}{36n})x_n,
\end{aligned}
$$

also

$$
\begin{aligned}
x_{n+1} &:= (1 - \alpha_n)x_n + \frac{1}{3}\alpha_n y_n \\
&= (\frac{1}{2} - \frac{1}{3n})x_n + \frac{1}{3}(\frac{1}{2} + \frac{1}{3n})(\frac{13}{18} + \frac{5}{36n})x_n \qquad (4.1) \\
&= (\frac{67}{108} - \frac{149}{648n} + \frac{5}{324n^2})x_n.
\end{aligned}
$$

Numerical results for $x_1 = 0.5$ and $x_1 = -2$			
n	x_n	n	x_n
1	0.5	1	-2
2	0.2029	2	-0.8117
3	0.1033	3	-0.4134
	$\vdots$		$\vdots$
45	$6.85e-11$	45	$-2.74e-10$
46	$4.22e-11$	46	$-1.69e-10$
47	$2.59e-11$	47	$-1.049e-10$
	$\vdots$		$\vdots$
98	$3.24e-22$	98	$-1.30e-21$
99	$2.00e-22$	99	$-8.02e-22$
100	$1.24e-22$	100	$-4.95e-22$

Since $F(S) \cap EP(f) = \{0\}$, we get $P_{F(S)\cap EP(f)}(x_n) = 0$ for all $n \geq 1$. Taking the limit as $n \to \infty$ in (4.1), we obtain $\lim_{n\to\infty} x_n = 0$. See Figure 1 and Figure 2 for the values $x_1 = 0.5$ and $x_1 = -2$. The computations associated with example were performed using MATLAB software.

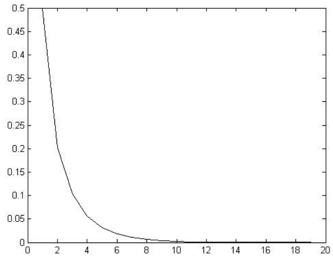

FIGURE 1

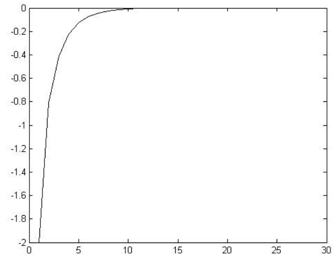

FIGURE 2

References

[1] R. P. Agarwal, D. O'Regan, and D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer 2009.

[2] S. Alizadeh and F. Moradlou, *Weak and strong convergence theorems for m-generalized hybrid mappings in Hilbert spaces*, Topol. Methods Nonlinear Anal. **46** (2015), no. 1, 315–328.

[3] ______, *A strong convergence theorem for equilibrium problems and generalized hybrid mappings*, Mediterr. J. Math. **13** (2016), no. 1, 379–390.

[4] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Student **63** (1994), no. 1-4, 123–145.

[5] L. C. Ceng, S. Al-Homidan, Q. H. Ansari, and J. C. Yao, *An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings*, J. Comput. Appl. Math. **223** (2009), no. 2, 967–974.

[6] L. C. Ceng and J. C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems*, J. Comput. Appl. Math. **214** (2008), no. 1, 186–201.

[7] C. E. Chidume and S. A. Mutangadura, *An example of the Mann iteration method for Lipschitz pseudocontractions*, Proc. Amer. Math. Soc. **129** (2001), no. 8, 2359–2363.

[8] P. L. Combettes and S. A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), no. 1, 117–136.

[9] S. D. Flam and A. S. Antipin, *Equilibrium programming using proximal-like algorithms*, Math. Programming **78** (1997), no. 1, Ser. A, 29–41.

[10] A. Genel and J. Lindenstrass, *An example concerning fixed points*, Israel J. Math. **22** (1975), no. 1, 81–86.

[11] F. Giannessi, A. Maugeri, and P. M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models*, Kluwer Academics Publishers, Dordrecht, Holland, 2001.

[12] M. Hojo, W. Takahashi, and I. Termwuttipong, *Strong convergence theorem for* 2*-generalized hybrid mapping in Hilbert spaces*, Nonlinear Anal. **75** (2012), no. 4, 2166–2176.

[13] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc. **40** (1974), 147–150.

[14] S. Itoh and W. Takahashi, *The common fixed point theory of singlevalued mappings and multivalued mappings*, Pacific J. Math. **79** (1978), no. 2, 493–508.

[15] C. Jaiboon and P. Kumam, *A hybrid extragradient viscosity approximation method for solving equilibrium problems and fixed point problems of infinitely many nonexpansive mappings*, Fixed Point Theory Appl. **2009** (2009), Article ID 374815, 32 pages.

[16] C. Jaiboon and P. Kumam, and H. W. Humphries, *Weak convergence theorem by an extragradient method for variational inequality, equilibrium and fixed point problems*, Bull. Malays. Math. Sci. Soc. (2) **32** (2009), no. 2, 173–185.
[17] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications*, Academic Press, New York, 1980.
[18] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), no. 2, 166–177.
[19] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc. **4** (1953), 506–510.
[20] T. Maruyama, W. Takahashi, and M. Yao, *Fixed point and mean ergodic theorems for new nonlinear mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), no. 1, 185–197.
[21] A. Moudafi and M. Thera, *Proximal and dynamical approaches to equilibrium problems*, Ill-posed variational problems and regularization techniques (Trier, 1998), 187–201, Lecture Notes in Econom. and Math. Systems, 477, Springer, Berlin, 1999.
[22] S. Plubtieng and P. Kumam, *Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings*, J. Comput. Appl. Math. **224** (2009), no. 2, 614–621.
[23] X. Qin, Y. J. Cho, and S. M. Kang, *Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces*, J. Comput. Appl. Math. **225** (2009), no. 1, 20–30.
[24] S. Takahashi and W. Takahashi, *Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces*, J. Math. Anal. Appl. **331** (2007), no. 1, 506–515.
[25] W. Takahashi, *Introduction to Nonlinear and Convex Analysis*, Yokohoma Publishers, Yokohoma, 2009.
[26] ———, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), no. 1, 79–88.
[27] W. Takahashi and M. Toyoda, *Weak convergence theorems for nonexpansive mappings and monotone mappings*, J. Optim. Theory Appl. **118** (2003), no. 2, 417–428.
[28] W. Takahashi and K. Zembayashi, *Strong convergence theorems by hybrid methods for equilibrium problems and relatively nonexpansive mapping*, Fixed Point Theory Appl. **2008** (2008), Article ID 528476, 11 pages.
[29] F. Yan, Y. Su, and Q. Feng, *Convergence of Ishikawa method for generalized hybrid mappings*, Commun. Korean Math. Soc. **28** (2013), no. 1, 135–141.

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