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DERIVATION OF SOME INEQUALITIES USING THE (p,q)-TH LOWER ORDER AND (p,q)-TH WEAK TYPE OF ENTIRE FUNCTIONS

Tanmay Biswas, Sanjib Kumar Datta, and Jinarul Haque Shaikh

Abstract. The object of the present paper is to obtain new estimates about the (p,q)-th lower order and (p,q)-th weak type of entire functions under some interesting conditions.

1. Introduction

A single valued function of one complex variable which is analytic in the finite complex plane is called an entire (integral) function. For example $\exp z$, $\sin z$, $\cos z$ etc. are all entire functions. In the value distribution theory, one studies how an entire function assumes some values and the influence of assuming certain values in some specific manner on a function. In 1926 Rolf Nevanlinna initiated the value distribution theory of entire functions. This value distribution theory is a prominent branch of Complex Analysis and is the prime concern of this paper. Perhaps the Fundamental Theorem of Classical Algebra which states that "If f is a polynomial of degree n with real or complex coefficients, then the equation $f(z)=0$ has at least one root" is the most well known value distribution theorem. The value distribution theory deals with the various aspects of the behaviour of entire functions one of which is the study of comparative growth properties of entire functions. For any entire function f , $M_f(r)$ of $f=\sum\limits_{n=0}^{\infty}a_n z^n$ on $|z|=r$, a function of r is defined as follows:

$$M_f(r) = \max_{|z|=r}|f(z)|.$$

In this connection, we just recall the following well known inequalities for all sufficiently large r relating to the maximum moduli of any two entire functions f_i and f_j:

$$M_{f_i \pm f_j}(r) < M_{f_i}(r) + M_{f_j}(r), \tag{1.1}$$

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$$M_{f_i \pm f_j}(r) \geq M_{f_i}(r) - M_{f_j}(r) \tag{1.2}$$

and

$$M_{f_i \cdot f_j}(r) \leq M_{f_i}(r) \cdot M_{f_j}(r). \tag{1.3}$$

On the other hand, if we consider z_r to be a point on $|z| = r$, we have for all sufficiently large values of r that

$$M_{f_i \cdot f_j}(r) = \max\{|f_i \cdot f_j(z)| : |z| = r\} = \max\{|f_i(z)||f_j(z)| : |z| = r\}$$

$$i.e.,\ M_{f_i \cdot f_j}(r) \geq |f_i(z_r)||f_j(z_r)|. \tag{1.4}$$

The order and lower order of an entire function f which are generally used in computational purpose is defined in terms of the maximum modulus of f as

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

The concept of type has been introduced to determine the relative growth of two entire functions with same non zero finite order. An entire function f of order ρ_f $\left(0 < \rho_f < \infty\right)$, is said to be of type $(0 \leq \sigma_f \leq \infty)$ if

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}}.$$

Similarly, Datta and Jha [2] introduced the definition of *weak type* τ_f $(0 \leq \tau_f \leq \infty)$ of an entire function of finite positive lower order in the following way:

$$\tau_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}},\ 0 < \lambda_f < \infty.$$

The determination of the order of growth and type of entire functions are very important to study the basic properties of the value distribution theory. In this regard several researchers made close investigations on it. Accordingly, Holland [3] and Levin [7] established the addition and multiplication theorems of order and type under some different conditions.

Further in this paper we wish to prove addition and multiplication theorems of lower order and weak type in the light of lower index-pairs and (p,q)-th lower order of entire functions for any two positive integers p and q with $p \geq q$ whose definitions have been given in Section 2 headed as "Preliminary remarks" of the paper. We do not explain the standard definitions and notations in the theory of entire functions as those are available in [9].

2. Preliminary remarks

Let f be an entire function defined in the open complex plane $\mathbb{C}$ and $M_f(r) = \max\{|f(z)| : |z| = r\}$. In the sequel we use the following notation:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \ldots \text{ and}$$

$$\log^{[0]} x = x;$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \ldots \text{ and}$$
$$\exp^{[0]} x = x.$$

The following definitions are well known:

Definition 2.1. The order ρ_f and lower order λ_f of an entire function f are defined as follows:

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \quad \text{and} \quad \rho_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}.$$

Definition 2.2 ([2])**.** The weak type τ_f of an entire function f is defined as

$$\tau_f = \liminf_{r\to\infty} \frac{\log M_f(r)}{r^{\lambda_f}}, \quad 0 < \lambda_f < \infty.$$

Sato [8] gave a more generalized concept of order(lower order) which is as follows:

Definition 2.3 ([8])**.** Let l be an integer ≥ 2. The generalized order $\rho_f^{[l]}$ (respectively generalized lower order $\lambda_f^{[l]}$) of an entire function f is defined as

$$\rho_f^{[l]} = \limsup_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} \left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l]} M_f(r)}{\log r} \right).$$

When $l = 2$, Definition 2.3 coincides with Definition 2.1.

Analogously, one may define the generalized weak type $\tau_f^{[l]}$ of an entire function f in the following manner:

$$\tau_f^{[l]} = \liminf_{r\to\infty} \frac{\log^{[l-1]} M_f(r)}{r^{\lambda_f^{[l]}}}, \quad 0 < \lambda_f^{[l]} < \infty.$$

Juneja, Kapoor and Bajpai [4] defined the (p,q)-th order and (p,q)-th lower order of an entire function f which are as follows:

$$\rho_f(p,q) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \quad \text{and} \quad \lambda_f(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

where p, q are any two positive integers with $p \geq q$. If $p = l$ and $q = 1$, then we write $\rho_f(l,1) = \rho_f^{[l]}$ and $\lambda_f(l,1) = \lambda_f^{[l]}$. Also for $p = 2$ and $q = 1$, we denote $\rho_f(2,1)$ and $\lambda_f(2,1)$ by ρ_f and λ_f respectively.

In the line of Juneja, Kapoor and Bajpai [5] one can introduced the concepts of (p,q)-th weak type of entire function in order to compare the growth of entire functions having the same (p,q)-th lower order in the following way:

Definition 2.4 ([5])**.** The (p,q)-th weak type of entire function f having finite positive (p,q) th lower order $\lambda_f(p,q)$ $(b<\lambda_f(p,q)<\infty)$ is defined as:

$$\tau_f(p,q)=\liminf_{r\to\infty}\frac{\log^{[p-1]}M_f(r)}{\left(\log^{[q-1]}r\right)^{\lambda_f(p,q)}}\ ,\ \ 0\le\tau_f(p,q)\le\infty,$$

where p,q are any two positive integers, $b=1$ if $p=q$ and $b=0$ for $p>q$. If $p=2$ and $q=1$, Definition 2.4 reduces to Definition 2.2. Similarly, if we consider $p=l$ and $q=1$, then we write $\tau_f(l,1)=\tau_f^{[l]}$.

Similarly we use the growth indicator $\overline{\tau}_f(p,q)$ of entire function f having finite positive (p,q) th lower order $\lambda_f(p,q)$ $(b<\lambda_f(p,q)<\infty)$ in the following way

$$\overline{\tau}_f(p,q)=\limsup_{r\to\infty}\frac{\log^{[p-1]}M_f(r)}{\left(\log^{[q-1]}r\right)^{\lambda_f(p,q)}}\ ,\ \ 0\le\overline{\tau}_f(p,q)\le\infty,$$

where p,q are any two positive integers.

In this connection we just recalling that for any pair of integer numbers m,n the Kroenecker function is defined by $\delta_{m,n}=1$ for $m=n$ and $\delta_{m,n}=0$ for $m\neq n$, the aforementioned properties provide the following definition:

Definition 2.5 ([4])**.** An entire function f is said to have index-pair $(1,1)$ if $0<\rho_f(1,1)<\infty$. Otherwise, f is said to have index-pair $(p,q)\neq(1,1)$, $p\ge q\ge 1$, if $\delta_{p-q,0}<\rho_f(p,q)<\infty$ and $\rho_f(p-1,q-1)\notin\mathcal{R}^+$.

Definition 2.6 ([4])**.** An entire function f is said to have lower index-pair $(1,1)$ if $0<\lambda_f(1,1)<\infty$. Otherwise, f is said to have lower index-pair $(p,q)\neq(1,1)$, $p\ge q\ge 1$, if $\delta_{p-q,0}<\lambda_f(p,q)<\infty$ and $\lambda_f(p-1,q-1)\notin\mathcal{R}^+$.

An entire function f of index-pair (p,q) is said to be of regular (p,q)-growth if it coincides with its (p,q)-th lower order, otherwise f is said to be of irregular (p,q)-growth.

Now we give the following proposition which is frequently used in the sequel:

Proposition 2.1. *Let f_i and f_j be any two entire functions with lower index-pair (p_i,q_i) and (p_j,q_j) respectively. Thus the following relations may occur:*

(i) $p_i\ge p_j, q_i=q_j$ *and* $\lambda_{f_i}(p_i,q_i)>\lambda_{f_j}(p_j,q_j)$,
(ii) $p_i\ge p_j, q_i<q_j$ *and* $\lambda_{f_i}(p_i,q_i)=\lambda_{f_j}(p_j,q_j)$,
(iii) $p_i>p_j, q_i=q_j$ *and* $\lambda_{f_i}(p_i,q_i)=\lambda_{f_j}(p_j,q_j)$,
(iv) $p_i\ge p_j, q_i<q_j$ *and* $\lambda_{f_i}(p_i,q_i)>\lambda_{f_j}(p_j,q_j)$,
(v) $p_i=p_j, q_i=q_j$ *and* $\lambda_{f_i}(p_i,q_i)=\lambda_{f_j}(p_j,q_j)$,
(vi) $p_i=p_j, q_i>q_j$ *and* $\lambda_{f_i}(p_i,q_i)>\lambda_{f_j}(p_j,q_j)$,
(vii) $p_i>p_j, q_i<q_j$ *and* $\lambda_{f_i}(p_i,q_i)<\lambda_{f_j}(p_j,q_j)$,
(viii) $p_i>p_j, q_i=q_j$ *and* $\lambda_{f_i}(p_i,q_i)<\lambda_{f_j}(p_j,q_j)$,
(ix) $p_i<p_j, q_i<q_j$ *and* $\lambda_{f_i}(p_i,q_i)>\lambda_{f_j}(p_j,q_j)$,

and

(x) $p_i > p_j, q_i > q_j$ *and* $\lambda_{f_i}(p_i, q_i) \geq \lambda_{f_j}(p_j, q_j)$.

In this connection the following definition is also relevant:

Definition 2.7 ([1]). A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds.

For examples of functions with or without the Property (A), one may see [1].

3. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 3.1 ([1]). *Suppose that f be an entire function, $\alpha > 1$, $0 < \beta < \alpha$, $s > 1$ and $0 < \mu < \lambda$. Then*
(a) $M_f(\alpha r) > \beta M_f(r)$ *and*
(b) $\lim\limits_{r\to\infty} \frac{M_f(r^s)}{M_f(r)} = \infty = \lim\limits_{r\to\infty} \frac{M_f(r^\lambda)}{M_f(r^\mu)}$.

Lemma 3.2 ([1]). *Let f be an entire function which satisfies the Property* (A). *Then for any positive integer n and for all sufficiently large r,*

$$[M_f(r)]^n \leq M_f\left(r^\delta\right)$$

holds where $\delta > 1$.

Lemma 3.3 ([6], p. 21). *Let $f(z)$ be holomorphic in the circle $|z| = 2eR$ ($R > 0$) with $f(0) = 1$ and η be an arbitrary positive number not exceeding $\frac{3e}{2}$. Then inside the circle $|z| = R$, but outside of a family of excluded circles the sum of whose radii is not greater than $4\eta R$, we have*

$$\log|f(z)| > -T(\eta)\log M_f(2eR)$$

for $T(\eta) = 2 + \log\frac{3e}{2\eta}$.

4. Main results

In this section we present the main results of the paper.

Theorem 4.1. *Let f_i and f_j be any two entire functions with lower index-pairs (p_i, q_i) and (p_j, q_j) respectively where p_i, p_j, q_i, q_j are all positive integers such that $p_i \geq q_i$ and $p_j \geq q_j$. Then*

$$\lambda_{(f_i \pm f_j)}(p, q) \leq \max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\},$$

where $p = \max\{p_i, p_j\}$ and $q = \min\{q_i, q_j\}$ and at least f_i is of regular (p_i, q_i)-growth or f_j is of regular (p_j, q_j)-growth.

The sign of equality holds when any one of the first four conditions of Proposition 2.1 holds for $i \neq j$ and f_j is of regular (p_j, q_j)-growth.

Proof. If $\lambda_{(f_i \pm f_j)}(p,q) = 0$, then the result is obvious. So we suppose that $\lambda_{(f_i \pm f_j)}(p,q) > 0$.

We can clearly assume that $\lambda_{f_k}(p_k, q_k)$ is finite for $k = i, j$.

Now for any arbitrary $\varepsilon > 0$ from the definition of (p_k, q_k)-th lower order, we have for a sequence of values of r tending to infinity that

$$M_{f_k}(r) \le \exp^{[p_k]}\left[(\lambda_{f_k}(p_k, q_k) + \varepsilon)\log^{[q_k]} r\right], \text{ where } k = i, j, \tag{4.1}$$

i.e.,

$$M_{f_k}(r) \le \exp^{[\max\{p_1,p_2\}]}\left[\left(\max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\} + \varepsilon\right)\log^{[\min\{q_1,q_2\}]} r\right],$$

where $k = i, j$, i.e.,

$$M_{f_k}(r) \le \exp^{[p]}\left[\left(\max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\} + \varepsilon\right)\log^{[q]} r\right], \tag{4.2}$$

where $k = i, j$.

Further, when f_l is of regular (p_l, q_l)-growth for $l = i, j$ and $l \ne k$, we get for all sufficiently large values of r that

$$M_{f_l}(r) \le \exp^{[p_l]}\left[(\lambda_{f_l}(p_l, q_l) + \varepsilon)\log^{[q_l]} r\right], \text{ where } l = i, j \text{ and } l \ne k. \tag{4.3}$$

Therefore in view of (4.2), we get for all sufficiently large values of r that

$$M_{f_l}(r) \le \exp^{[p]}\left[\left(\max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\} + \varepsilon\right)\log^{[q]} r\right], \tag{4.4}$$

where $l = i, j$ and $l \ne k$.

So in view of (4.2) and (4.4), we obtain from (1.1) for a sequence of values of r tending to infinity that

$$M_{f_i \pm f_j}(r) < 2\exp^{[p]}\left[\left(\max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\} + \varepsilon\right)\log^{[q]} r\right]. \tag{4.5}$$

Therefore in view of Lemma 3.1(a), we get from (4.5) for a sequence of values of r tending to infinity that

$$\frac{1}{2}M_{f_i \pm f_j}(r) < \exp^{[p]}\left[\left(\max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\} + \varepsilon\right)\log^{[q]} r\right]$$

$$\text{i.e., } M_{f_i \pm f_j}\left(\frac{r}{3}\right) < \exp^{[p]}\left[\left(\max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\} + \varepsilon\right)\log^{[q]} r\right]$$

$$\text{i.e., } \frac{\log^{[p]} M_{f_i \pm f_j}\left(\frac{r}{3}\right)}{\log^{[q]}\left(\frac{r}{3}\right) + O(1)} < \left(\max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\} + \varepsilon\right).$$

So

$$\lambda_{f_i \pm f_j}(p,q) = \liminf_{r\to\infty}\frac{\log^{[p]} M_{f_i \pm f_j}\left(\frac{r}{3}\right)}{\log^{[q]}\left(\frac{r}{3}\right) + O(1)} \le \left(\max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\} + \varepsilon\right).$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda_{f_i \pm f_j}(p,q) \le \max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\}. \tag{4.6}$$

Now let any one of first four conditions of Proposition 2.1 holds for $i \neq j, i = 1, 2$ and $j = 1, 2$.

As $\varepsilon (> 0)$ is arbitrary, from the definition of (p_k, q_k)-th lower order it follows for all sufficiently large values of r that

$$M_{f_k}(r) \geq \exp^{[p_k]}\left[(\lambda_{f_k}(p_k, q_k) - \varepsilon)\log^{[q_k]} r\right] \text{ for } k = i, j. \tag{4.7}$$

Therefore in view of the first four conditions of Proposition 2.1, we obtain for all sufficiently large values of r that

$$M_{f_i}(r) \geq \exp^{[p]}\left[\left(\max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\} - \varepsilon\right)\log^{[q]} r\right]. \tag{4.8}$$

Now we consider the expression

$$\frac{\exp^{[p_i]}\left[(\lambda_{f_i}(p_i, q_i) - \varepsilon)\log^{[q_i]} r\right]}{\exp^{[p_j]}\left[\left(\lambda_{f_j}(p_j, q_j) + \varepsilon\right)\log^{[q_j]} r\right]} \text{ with } i \neq j. \tag{4.9}$$

Therefore in view of the first four conditions of Proposition 2.1 and Lemma 3.1(b) we obtain from (4.9) that

$$\lim_{r\to\infty}\frac{\exp^{[p_i]}\left[(\lambda_{f_i}(p_i, q_i) - \varepsilon)\log^{[q_i]} r\right]}{\exp^{[p_j]}\left[\left(\lambda_{f_j}(p_j, q_j) + \varepsilon\right)\log^{[q_j]} r\right]} = \infty \text{ with } i \neq j. \tag{4.10}$$

Now (4.10) can also be written as

$$\lim_{r\to\infty}\frac{\exp^{[p]}\left[\left(\max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\} - \varepsilon\right)\log^{[q]} r\right]}{\exp^{[p_j]}\left[\left(\lambda_{f_j}(p_j, q_j) + \varepsilon\right)\log^{[q_j]} r\right]} = \infty, \tag{4.11}$$

where $p \geq p_j, q \leq q_j$ and $\max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\} \geq \lambda_{f_j}(p_j, q_j)$ but all the equalities do not hold simultaneously.

So from (4.11), we obtain for all sufficiently large values of r that

$$\begin{aligned} &\exp^{[p]}\left[\left(\max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\} - \varepsilon\right)\log^{[q]} r\right] \\ &> 2\exp^{[p_j]}\left[\left(\lambda_{f_j}(p_j, q_j) + \varepsilon\right)\log^{[q_j]} r\right]. \end{aligned} \tag{4.12}$$

Thus from (4.3), (4.8) and (4.12) we get for all sufficiently large values of r that

$$\begin{aligned} &M_{f_i}(r) > 2\exp^{[p_j]}\left[\left(\lambda_{f_j}(p_j, q_j) + \varepsilon\right)\log^{[q_j]} r\right] \\ &\text{i.e., } M_{f_i}(r) > 2M_{f_j}(r), \text{ where } i \neq j,\ i = 1, 2; j = 1, 2. \end{aligned} \tag{4.13}$$

So from (4.8), (4.13) and in view of Lemma 3.1(a) and (1.2), it follows for all sufficiently large values of r that

$$M_{f_i \pm f_j}(r) \geq M_{f_i}(r) - M_{f_j}(r) \text{ with } i \neq j$$
$$\text{i.e., } M_{f_i \pm f_j}(r) \geq M_{f_i}(r) - \frac{1}{2}M_{f_i}(r) \text{ with } i \neq j$$

$$\text{i.e., } M_{f_i \pm f_j}(r) \geq \frac{1}{2} M_{f_i}(r) \text{ with } i \neq j$$

$$\text{i.e., } M_{f_i \pm f_j}(r) \geq \frac{1}{2} \exp^{[p]}\left[\left(\max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\} - \varepsilon\right) \log^{[q]} r\right]$$

$$\text{i.e., } M_{f_i \pm f_j}(3r) \geq \exp^{[p]}\left[\left(\max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\} - \varepsilon\right) \log^{[q]} r\right].$$

This gives for all sufficiently large values of r that

$$\frac{\log^{[p]} M_{f_i \pm f_j}(3r)}{\log^{[q]}(3r) + O(1)} \geq \left(\max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\} + \varepsilon\right)$$

$$\text{i.e., } \liminf_{r \to \infty} \frac{\log^{[p]} M_{f_i \pm f_j}(3r)}{\log^{[q]}(3r) + O(1)} \geq \max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\}$$

$$\text{i.e., } \lambda_{f_i \pm f_j}(p, q) = \liminf_{r \to \infty} \frac{\log^{[p]} M_{f_i \pm f_j}(r)}{\log^{[q]} r} = \liminf_{r \to \infty} \frac{\log^{[p]} M_{f_i \pm f_j}(3r)}{\log^{[q]}(3r) + O(1)}$$

$$\geq \max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\}. \tag{4.14}$$

So the conclusion of the second part of the theorem follows from (4.6) and (4.14). □

Remark 4.1. The inequality sign in Theorem 4.1 cannot be removed which is evident from the following example:

Example 4.1. Given any two natural numbers l, m, the functions $f(z) = \exp^{[l]} z^m$ and $g(z) = -\exp^{[l]} z^m$ have their maximum moduli respectively as $M_f(r) = \exp^{[l]} r^m$ and $M_g(r) = \exp^{[l]} r^m$. Therefore $\frac{\log^{[k]} M_f(r)}{\log r}$ and $\frac{\log^{[k]} M_g(r)}{\log r}$ are both constants for each natural number $k \geq 2$. Thus it follows that

$$\lambda_f^{[l+1]} = \lambda_g^{[l+1]} = m$$

but $\lambda_f^{[k]} = \lambda_g^{[k]} = +\infty$ for $2 \leq k \leq l$ and $\lambda_f^{[k]} = \lambda_f^{[k]} = 0$ for $k > l + 1$.

Therefore

$$\lambda_{f+g}^{[l+1]} = 0 < \lambda_f^{[l+1]} + \lambda_g^{[l+1]} = 2m.$$

Theorem 4.2. *Let f_i and f_j be any two entire functions with lower index-pairs (p_i, q_i) and (p_j, q_j) respectively where p_i, p_j, q_i, q_j are all positive integers such that $p_i \geq q_i$ and $p_j \geq q_j$. Further suppose that $\lambda_{f_i}(p_i, q_i)$ and $\lambda_{f_j}(p_j, q_j)$ are both non zero and finite. Then for $p = \max\{p_i, p_j\}$ and $q = \min\{q_i, q_j\}$,*

$$\tau_{f_i \pm f_j}(p, q) = \tau_{f_i}(p_i, q_i),$$

when any one of the first four conditions of Proposition 2.1 holds for $i \neq j$ and f_j is of regular (p_j, q_j)-growth.

Proof. Suppose that any one of the first four conditions of Proposition 2.1 holds for $i \neq j$. Also let $\varepsilon\,(>0)$ and $\varepsilon_1\,(>0)$ be arbitrary.

Now from the definition of (p_k, q_k)-weak type, we have for a sequence of values of r tending to infinity that
(4.15)
$$M_{f_k}(r) \leq \exp^{[p_k-1]}\left\{(\tau_{f_k}(p_k,q_k)+\varepsilon)\left(\log^{[q_k-1]} r\right)^{\lambda_{f_k}(p_k,q_k)}\right\} \text{ for } k=i,j,$$

and for all sufficiently large values of r we obtain that
(4.16)
$$M_{f_k}(r) \geq \exp^{[p_k-1]}\left\{(\tau_{f_k}(p_k,q_k)-\varepsilon)\left(\log^{[q_k-1]} r\right)^{\lambda_{f_k}(p_k,q_k)}\right\} \text{ for } k=i,j.$$

Similarly from the definition of $\overline{\tau}_{f_k}(p_k,q_k)$, we get for all sufficiently large values of r that
(4.17)
$$M_{f_k}(r) \leq \exp^{[p_k-1]}\left\{(\overline{\tau}_{f_k}(p_k,q_k)+\varepsilon)\left(\log^{[q_k-1]} r\right)^{\lambda_{f_k}(p_k,q_k)}\right\} \text{ for } k=i,j.$$

Therefore from (1.1), (4.15) and (4.17) we get for a sequence of values of r tending to infinity that

$$M_{f_i\pm f_j}(r) \leq \exp^{[p_i-1]}\left\{(\tau_{f_i}(p_i,q_i)+\varepsilon)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}$$
$$\times\left[1+\frac{\exp^{[p_j-1]}\left\{(\overline{\tau}_{f_j}(p_j,q_j)+\varepsilon)\left(\log^{[q_j-1]} r\right)^{\lambda_{f_j}(p_j,q_j)}\right\}}{\exp^{[p_i-1]}\left\{(\tau_{f_i}(p_i,q_i)+\varepsilon)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}}\right], \tag{4.18}$$

where $i \neq j$.

Now in view of any one of the first four conditions of Proposition 2.1 for $i \neq j$ and for all sufficiently large values of r, we can make the term

$$\left[1+\frac{\exp^{[p_j-1]}\left\{(\overline{\tau}_{f_j}(p_j,q_j)+\varepsilon)\left(\log^{[q_j-1]} r\right)^{\lambda_{f_j}(p_j,q_j)}\right\}}{\exp^{[p_i-1]}\left\{(\tau_{f_i}(p_i,q_i)+\varepsilon)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}}\right]$$

sufficiently small.

Hence for any $\alpha > 1+\varepsilon_1$, it follows from Lemma 3.1(a) and (4.18) for a sequence of values of r tending to infinity that

$$M_{f_i\pm f_j}(r) \leq \exp^{[p_i-1]}\left\{(\tau_{f_i}(p_i,q_i)+\varepsilon)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}(1+\varepsilon_1)$$

$$\text{i.e., } \frac{1}{(1+\varepsilon_1)}M_{f_i\pm f_j}(r) \leq \exp^{[p_i-1]}\left\{(\tau_{f_i}(p_i,q_i)+\varepsilon)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}$$

$$\text{(4.19)}\quad \text{i.e., } M_{f_i \pm f_j}(r) \leq \exp^{[p_i-1]}\left\{\alpha\left(\tau_{f_i}(p_i,q_i)+\varepsilon\right)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}.$$

Thus from (4.19), it follows for a sequence of values of r tending to infinity that

(4.20)

$$M_{f_i \pm f_j}(r) \leq \exp^{[p-1]}\left\{\alpha\left(\tau_{f_i}(p_i,q_i)+\varepsilon\right)\left(\log^{[q-1]} r\right)^{\max\left\{\lambda_{f_1}(p_1,q_1),\lambda_{f_2}(p_2,q_2)\right\}}\right\}.$$

Therefore in view of Theorem 4.1, we have from (4.20) and for a sequence of values of r tending to infinity that

$$\log^{[p-1]} M_{f_i \pm f_j}(r) \leq \alpha\left(\tau_{f_i}(p_i,q_i)+\varepsilon\right)\left(\log^{[q-1]} r\right)^{\max\left\{\lambda_{f_1}(p_1,q_1),\lambda_{f_2}(p_2,q_2)\right\}}$$

$$\text{i.e.,}\quad \frac{\log^{[p-1]} M_{f_i \pm f_j}(r)}{\left[\log^{[q-1]}(r)\right]^{\lambda_{(f_1 \pm f_2)}(p,q)}}$$

$$\text{(4.21)}\quad \leq \frac{\alpha\left(\tau_{f_i}(p_i,q_i)+\varepsilon\right)\left(\log^{[q-1]} r\right)^{\max\left\{\lambda_{f_1}(p_1,q_1),\lambda_{f_2}(p_2,q_2)\right\}}}{\left[\log^{[q-1]}(r)\right]^{\max\left\{\lambda_{f_1}(p_1,q_1),\lambda_{f_2}(p_2,q_2)\right\}}}.$$

Hence making $\alpha \to 1+$, we obtain from (4.21) for a sequence of values of r tending to infinity that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M_{f_i \pm f_j}(r)}{\left[\log(r)\right]^{\lambda_{(f_1 \pm f_2)}(p,q)}} \leq \tau_{f_i}(p_i,q_i)$$

$$\text{(4.22)}\quad \text{i.e., } \tau_{f_i \pm f_j}(p,q) \leq \tau_{f_i}(p_i,q_i).$$

Again from (1.2), (4.16) and (4.17) we get for all sufficiently large values of r that

$$M_{f_i \pm f_j}(r) \geq \exp^{[p_i-1]}\left\{\left(\tau_{f_i}(p_i,q_i)-\varepsilon\right)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}$$

$$\text{(4.23)}\quad \times\left[1-\frac{\exp^{[p_j-1]}\left\{\left(\overline{\tau}_{f_j}(p_j,q_j)+\varepsilon\right)\left(\log^{[q_j-1]} r\right)^{\lambda_{f_j}(p_j,q_j)}\right\}}{\exp^{[p_i-1]}\left\{\left(\tau_{f_i}(p_i,q_i)-\varepsilon\right)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}}\right],$$

where $i \neq j$.

Now in view of any one of the first four conditions of Proposition 2.1 for $i \neq j$ and for all sufficiently large values of r, we can make the term

$$\left[1-\frac{\exp^{[p_j-1]}\left\{\left(\overline{\tau}_{f_j}(p_j,q_j)+\varepsilon\right)\left(\log^{[q_j-1]} r\right)^{\lambda_{f_j}(p_j,q_j)}\right\}}{\exp^{[p_i-1]}\left\{\left(\tau_{f_i}(p_i,q_i)-\varepsilon\right)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}}\right]$$

sufficiently small.

Hence for any $\beta > \frac{1}{1-\varepsilon_1}$, it follows from Lemma 3.1(a) and (4.23) for all sufficiently large values of r that

$$M_{f_i \pm f_j}(r) \geq \exp^{[p_i-1]}\left\{(\tau_{f_i}(p_i,q_i)-\varepsilon)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}(1-\varepsilon_1)$$

$$\text{i.e., } \frac{1}{(1-\varepsilon_1)} M_{f_i \pm f_j}(r) \geq \exp^{[p_i-1]}\left\{(\tau_{f_i}(p_i,q_i)-\varepsilon)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}$$

$$\text{i.e., } M_{f_i \pm f_j}(\beta r) \geq \exp^{[p_i-1]}\left\{(\tau_{f_i}(p_i,q_i)-\varepsilon)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right\}. \tag{4.24}$$

Therefore in view of the first four conditions of Proposition 2.1 for $i \neq j$, it follows from (4.24) for all sufficiently large values of r that

$$M_{f_i \pm f_j}(\beta r) \geq \exp^{[p-1]}\left\{(\tau_{f_i}(p_i,q_i)-\varepsilon)\left(\log^{[q-1]} r\right)^{\max\left\{\lambda_{f_1}(p_1,q_1),\lambda_{f_2}(p_2,q_2)\right\}}\right\}.$$

Hence making $\beta \to 1+$, we get from above that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M_{f_i \pm f_j}(r)}{\left[\log^{[q-1]}(r)\right]^{\max\left\{\lambda_{f_1}(p_1,q_1),\lambda_{f_2}(p_2,q_2)\right\}}} \geq \tau_{f_i}(p_i,q_i). \tag{4.25}$$

Thus in view of Theorem 4.1, we obtain from (4.25) that

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} M_{f_i \pm f_j}(r)}{\left[\log^{[q-1]}(r)\right]^{\lambda_{(f_1\pm f_2)}(p,q)}} \geq \tau_{f_i}(p_i,q_i)$$

$$\text{i.e., } \tau_{f_i \pm f_j}(p,q) \geq \tau_{f_i}(p_i,q_i). \tag{4.26}$$

Thus the theorem follows from (4.22) and (4.26). □

In the next theorem we wish to find out the condition for which the equality sign of Theorem 4.1 holds in the case of Proposition 2.1(v).

Theorem 4.3. *Let f_1 and f_2 be any two entire functions such that $\lambda_{f_1}(p,q) = \lambda_{f_2}(p,q)$ $(0 < \lambda_{f_1}(p,q) = \lambda_{f_2}(p,q) < \infty)$ and $\tau_{f_1}(p,q) \neq \tau_{f_2}(p,q)$. Then*

$$\lambda_{f_1 \pm f_2}(p,q) = \lambda_{f_1}(p,q) = \lambda_{f_2}(p,q),$$

where p,q are any two positive integers with $p \geq q$, $(f_1 \pm f_2)$ is of regular (p,q)-th growth and at least f_1 or f_2 is also of regular (p,q)-th growth.

Proof. Suppose that $\lambda_{f_1}(p,q) = \lambda_{f_2}(p,q)$ $(0 < \lambda_{f_1}(p,q) = \lambda_{f_2}(p,q) < \infty)$ and $\tau_{f_1}(p,q) \neq \tau_{f_2}(p,q)$.

Now in view of Theorem 4.1, it is easy to see that

$$\lambda_{f_1 \pm f_2}(p,q) \leq \lambda_{f_1}(p,q) = \lambda_{f_2}(p,q).$$

If we consider

$$\lambda_{f_1 \pm f_2}(p,q) < \lambda_{f_1}(p,q) = \lambda_{f_2}(p,q),$$

then in view of Theorem 4.2, we obtain that

$$\tau_{f_1}(p,q) = \tau_{f_1 \pm f_2 \mp f_2}(p,q) = \tau_{f_2}(p,q)$$

which is a contradiction.

Hence

$$\lambda_{f_1 \pm f_2}(p,q) = \lambda_{f_1}(p,q) = \lambda_{f_2}(p,q).$$

Thus the theorem follows. □

Theorem 4.4. *Let f_i and f_j be any two entire functions with lower index-pairs (p_i, q_i) and (p_j, q_j) respectively where p_i, p_j, q_i, q_j are all positive integers such that $p_i \geq q_i$ and $p_j \geq q_j$. Then*

$$\lambda_{f_i \cdot f_j}(p,q) \leq \max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\},$$

where $p = \max\{p_i, p_j\}$ and $q = \min\{q_i, q_j\}$ and at least f_i is of regular (p_i, q_i)-growth or f_j is of regular (p_j, q_j)-growth.

The sign of equality holds when any one of the first four conditions of Proposition 2.1 holds for $i \neq j$ and f_j is of regular (p_j, q_j)-growth.

Proof. Suppose that $\lambda_{(f_i \cdot f_j)}(p,q) > 0$. Otherwise if $\lambda_{(f_i \cdot f_j)}(p,q) = 0$, then the result is obvious.

Also suppose that $\max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\} = \lambda$.

We can clearly assume that $\lambda_{f_k}(p_k, q_k)$ is finite for $k = i, j$.

Now for any arbitrary $\frac{\varepsilon}{2} > 0$, we obtain in view of (4.2) and for a sequence of values of r tending to infinity that

$$M_{f_k}(r) \leq \exp^{[p]}\left[\left(\lambda + \frac{\varepsilon}{2}\right)\log^{[q]} r\right], \; k = i, j. \tag{4.27}$$

Also for any arbitrary $\frac{\varepsilon}{2} > 0$, we obtain in view of (4.3) and for all sufficiently large values of r that

$$M_{f_l}(r) \leq \exp^{[p]}\left[\left(\lambda + \frac{\varepsilon}{2}\right)\log^{[q]} r\right], \text{ where } l = i, j \text{ and } l \neq k. \tag{4.28}$$

Further we consider the expression $\frac{\exp^{[p-1]}\left[(\lambda+\varepsilon)\log^{[q]} r\right]}{\exp^{[p-1]}\left[\left(\lambda+\frac{\varepsilon}{2}\right)\log^{[q]} r\right]}$ for all sufficiently large values of r.

Thus for any $\delta > 1$, it follows from the above expression for all sufficiently large values of r, say $r \geq r_1 \geq r_0$ that

$$\frac{\exp^{[p-1]}\left[(\lambda+\varepsilon)\log^{[q]} r_0\right]}{\exp^{[p-1]}\left[\left(\lambda+\frac{\varepsilon}{2}\right)\log^{[q]} r_0\right]} = \delta. \tag{4.29}$$

Now from (4.27), (4.28) and in view of (1.3), we have for a sequence of values of r tending to infinity that

$$M_{f_i \cdot f_j}(r) < \left[\exp^{[p]}\left[\left(\lambda + \frac{\varepsilon}{2}\right)\log^{[q]} r\right]\right]^2. \tag{4.30}$$

Also in view of Lemma 3.2, we obtain from (4.29) and (4.30) for a sequence of values of r tending to infinity that

$$M_{f_i \cdot f_j}(r) < \exp^{[p]}\left[\left(\lambda + \frac{\varepsilon}{2}\right)\log^{[q]} r\right]^{\delta}$$

$$\text{i.e., } M_{f_i \cdot f_j}(r) < \exp^{[p]}\left[(\lambda + \varepsilon)\log^{[q]} r\right].$$

Therefore from above, we get that

$$\frac{\log^{[p]} M_{f_i \cdot f_j}(r)}{\log^{[q]} r} \leq (\lambda + \varepsilon).$$

So

$$\lambda_{f_i \cdot f_j}(p, q) = \liminf_{r \to \infty} \frac{\log^{[p]} M_{f_i \cdot f_j}(r)}{\log^{[q]} r} \leq (\lambda + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary,

$$\lambda_{f_i \cdot f_j}(p, q) \leq \lambda = \max\left\{\lambda_{f_i}(p_i, q_i), \lambda_{f_j}(p_j, q_j)\right\}. \tag{4.31}$$

Now let any one of first four conditions of Proposition 2.1 hold for $i \neq j$ and f_j is of regular (p_j, q_j)-growth.

Without loss of any generality, we may assume that $f_k(0) = 1$ where $k = i, j$.

Also we may suppose that $r > R$.

Now from (4.7) and in view of the first four conditions of Proposition 2.1, we obtain for all sufficiently large values of R that

$$M_{f_i}(R) \geq \exp^{[p]}\left[(\lambda - \varepsilon)\log^{[q]} R\right]. \tag{4.32}$$

Also from (4.3), we get for all sufficiently large values of r that

$$M_{f_j}(r) \leq \exp^{[p]}\left[(\lambda + \varepsilon)\log^{[q]} r\right]. \tag{4.33}$$

Thus in view of Lemma 3.3, if we take $f_j(z)$ for $f(z)$, $\eta = \frac{1}{16}$ and $2R$ for R, it follows that

$$\log|f_j(z)| > -T(\eta)\log M_{f_j}(2e \cdot 2R),$$

where

$$T(\eta) = 2 + \log\left(\frac{3e}{2.\frac{1}{16}}\right) = 2 + \log(24e).$$

Therefore

$$\log|f_j(z)| > -(2 + \log(24e))\log M_{f_j}(4e \cdot R)$$

holds within and on $|z| = 2R$ but outside a family of excluded circles the sum of whose radii is not greater than

$$4 \cdot \frac{1}{16} \cdot 2R = \frac{R}{2}.$$

If $r \in (R, 2R)$, then on $|z| = r$

$$\log|f_j(z)| > -7\log M_{f_j}(4e \cdot R). \tag{4.34}$$

Since $r > R$, we have from (4.32) for all sufficiently large values of r that

$$M_{f_i}(r) > M_{f_i}(R) > \exp^{[p]}\left[(\lambda-\varepsilon)\log^{[q]} R\right]$$

$$> \exp^{[p]}\left[(\lambda-\varepsilon)\log^{[q]} \frac{r}{2}\right]. \tag{4.35}$$

Now let z_r be a point on $|z| = r$ such that $M_{f_i}(r) = |f_i(z_r)|$.

Therefore as $r > R$, from $(1.3), (4.33), (4.34)$ and (4.35) it follows for all sufficiently large values of r that

$$M_{f_i \cdot f_j}(r) \geq |f_j(z_r)| M_{f_i}(r)$$

$$\text{i.e., } M_{f_i \cdot f_j}(r) \geq \left[M_{f_j}(4eR)\right]^{-7} M_{f_i}(r) \tag{4.36}$$

i.e.,

$$M_{f_i \cdot f_j}(r) \geq \left[\exp^{[p]}\left[(\lambda+\varepsilon)\log^{[q]}(4eR)\right]\right]^{-7} \times \exp^{[p]}\left[(\lambda-\varepsilon)\log^{[q]}\left(\frac{r}{2}\right)\right]$$

i.e.,

$$M_{f_i \cdot f_j}(r) \geq \left[\exp^{[p]}\left[(\lambda+\varepsilon)\log^{[q]}(4er)\right]\right]^{-7} \times \exp^{[p]}\left[(\lambda-\varepsilon)\log^{[q]}\left(\frac{4er}{8e}\right)\right]. \tag{4.37}$$

As we have

$$\frac{\exp^{[p-1]}\left[(\lambda-\varepsilon)\log^{[q]}\left(\frac{4er}{8e}\right)\right]}{\exp^{[p-1]}\left[(\lambda+\varepsilon)\log^{[q]}(4er)\right]} \to \infty \text{ as } r \to \infty,$$

we may write for all sufficiently large values of r with $r_n > r_1 > r_0$,

$$\frac{\log\left[(\lambda-\varepsilon)\log^{[q]}\left(\frac{4er_n}{8e}\right)\right]}{\log\left[(\lambda+\varepsilon)\log^{[q]}(4er_n)\right]} > \frac{\log\left[(\lambda-\varepsilon)\log^{[q]}\left(\frac{4er_0}{8e}\right)\right]}{\log\left[(\lambda+\varepsilon)\log^{[q]}(4er_0)\right]} = \delta \text{ (say)}.$$

Therefore, clearly

$$\delta > 1.$$

Hence for the above value of δ, one can easily verify that

$$\exp^{[p]}\left[(\lambda-\varepsilon)\log^{[q]}\left(\frac{4er}{8e}\right)\right] \geq \exp^{[p]}\left[\left[(\lambda+\varepsilon)\log^{[q]}(4er)\right]^{\delta}\right]. \tag{4.38}$$

Also from Lemma 3.2, we get for all sufficiently large values of r that

$$\exp^{[p]}\left[\left[(\lambda+\varepsilon)\log^{[q]}(4er)\right]^{\delta}\right] \geq \left[\exp^{[p]}\left[(\lambda+\varepsilon)\log^{[q]}(4er)\right]\right]^{8}. \tag{4.39}$$

Now from $(4.37), (4.38)$ and (4.39), it follows for all sufficiently large values of r that

$$M_{f_i \cdot f_j}(r) \geq \exp^{[p]}\left[(\lambda+\varepsilon)\log^{[q]}(4er)\right]$$

$$i.e., \ \frac{\log^{[p]} M_{f_i \cdot f_j}(r)}{\log^{[q]} r + O(1)} \geq \lambda + \varepsilon$$

$$\text{i.e., } \lambda_{f_i \cdot f_j}(p,q) = \liminf_{r\to\infty} \frac{\log^{[p]} M_{f_i \cdot f_j}(r)}{\log^{[q]} r}$$

$$\leq \lambda = \max\left\{\lambda_{f_i}(p_i,q_i), \lambda_{f_j}(p_j,q_j)\right\}. \tag{4.40}$$

Consequently the second part of the theorem follows from (4.31) and (4.40). □

Remark 4.2. The following example shows that the inequality sign in Theorem 4.4 cannot be removed:

Example 4.2. Given any two natural numbers k, n, the functions $f(z) = \exp^{[k]} z^n$ and $g(z) = \exp^{[k]}(-z^n)$ have their maximum moduli respectively as $M_f(r) = \exp^{[k]} r^n$ and $M_g(r) = \exp^{[k]}(-r^n)$. Therefore $\frac{\log^{[l]} M_f(r)}{\log r}$ and $\frac{\log^{[ll]} M_g(r)}{\log r}$ are both constants for each natural $l \geq 2$. Thus it follows that

$$\lambda_f^{[k+1]} = \lambda_g^{[k+1]} = n$$

but $\lambda_f^{[l]} = \lambda_g^{[l]} = +\infty$ for $2 \leq l \leq k$ and $\rho_f^{[l]} = \rho_f^{[l]} = 0$ for $l > k+1$. Hence

$$\lambda_{f\cdot g}^{[k+1]} = 0 < \lambda_f^{[k+1]} + \lambda_g^{[k+1]} = 2n.$$

Theorem 4.5. *Let f_i and f_j be any two entire functions with lower index-pairs (p_i, q_i) and (p_j, q_j) respectively where p_i, p_j, q_i, q_j are all positive integers such that $p_i \geq q_i$ and $p_j \geq q_j$. Further suppose that $\lambda_{f_i}(p_i, q_i)$ and $\lambda_{f_j}(p_j, q_j)$ are both non zero and finite. Then for $p = \max\{p_i, p_j\}$ and $q = \min\{q_i, q_j\}$*

$$\tau_{f_i \cdot f_j}(p,q) = \tau_{f_i}(p_i, q_i),$$

when any one of the first four conditions of Proposition 2.1 holds for $i \neq j$, $q > 1$ and f_j is of regular (p_j, q_j)-growth.

Proof. Suppose that $\tau_{f_i \cdot f_j}(p,q) > 0$. Otherwise if $\tau_{f_i \cdot f_j}(p,q) = 0$ then the result is obvious.

We can clearly assume that $\tau_{f_k}(p_k, q_k)$ for $k = i, j$ is finite. Also suppose any one of the first four conditions of Proposition 2.1 holds for $i \neq j$. Also suppose that $\max\left\{\lambda_{f_i}(p_i,q_i), \rho\lambda_{f_j}(p_j,q_j)\right\} = \lambda_{f_i}(p_i,q_i) = \lambda$ and $\tau_{f_i}(p_i,q_i) = \tau$.

Further let $\varepsilon\,(>0)$ and $\varepsilon_1\,(>0)$ are arbitrary.

We now consider the expression $\frac{\exp^{[p-2]}(\tau+\varepsilon)\left(\log^{[q-1]} r\right)^{\lambda}}{\exp^{[p-2]}\left(\tau+\frac{\varepsilon}{2}\right)\left(\log^{[q-1]} r\right)^{\lambda}}$ for all sufficiently large values of r.

Thus for any $\delta > 1$, it follows from the above expression for all sufficiently large values of r, say $r \geq r_1 \geq r_0$ that

$$\frac{\exp^{[p-2]}(\tau+\varepsilon)\left(\log^{[q-1]} r_0\right)^{\lambda}}{\exp^{[p-2]}\left(\tau+\frac{\varepsilon}{2}\right)\left(\log^{[q-1]} r_0\right)^{\lambda}} = \delta. \tag{4.41}$$

Now in view of (1.3), we have from (4.15) and (4.17) for a sequence of values of r tending to infinity that

$$M_{f_i \cdot f_j}(r) \leq \exp^{[p_i-1]}\left[\left(\tau_{f_i}(p_i,q_i)+\frac{\varepsilon}{2}\right)\left(\log^{[q_i-1]} r\right)^{\lambda_{f_i}(p_i,q_i)}\right] \cdot \exp^{[p_j-1]}\left[\left(\overline{\tau}_{f_j}(p_j,q_j)+\frac{\varepsilon}{2}\right)\left(\log^{[q_j-1]} r\right)^{\lambda_{f_j}(p_j,q_j)}\right]$$

$$\text{i.e., } M_{f_i \cdot f_j}(r) \leq \exp^{[p-1]}\left\{\left(\tau+\frac{\varepsilon}{2}\right)\left(\log^{[q-1]} r\right)^{\lambda}\right\} \cdot \exp^{[p_j-1]}\left[\left(\overline{\tau}_{f_j}(p_j,q_j)+\frac{\varepsilon}{2}\right)\left(\log^{[q_j-1]} r\right)^{\lambda_{f_j}(p_j,q_j)}\right].$$

Now in view of any one of the first four conditions of Proposition 2.1 for $i \neq j$ and for all sufficiently large values of r, we get that

$$\exp^{[p-1]}\left\{\left(\tau+\frac{\varepsilon}{2}\right)\left(\log^{[q-1]} r\right)^{\lambda}\right\} > \exp^{[p_j-1]}\left[\left(\overline{\tau}_{f_j}(p_j,q_j)+\frac{\varepsilon}{2}\right)\left(\log^{[q_j-1]} r\right)^{\lambda_{f_j}(p_j,q_j)}\right]$$

and therefore from above and (4.41) it follows for a sequence of values of r tending to infinity that

$$M_{f_i \cdot f_j}(r) \leq \exp^{[p-1]}\left[\left(\tau+\frac{\varepsilon}{2}\right)\left(\log^{[q-1]} r\right)^{\lambda}\right]^2$$

$$\text{i.e., } M_{f_i \cdot f_j}(r) \leq \exp^{[p-1]}\left[(\tau+\varepsilon)\left(\log^{[q-1]} r\right)^{\lambda}\right].$$

Now in view of Theorem 4.4, we get from above for a sequence of values of r tending to infinity that

$$\frac{\log^{[p-1]} M_{f_i \cdot f_j}(r)}{\left(\log^{[q-1]} r\right)^{\lambda}} < (\tau+\varepsilon)$$

$$\text{i.e., } \liminf_{r\to\infty} \frac{\log^{[p-1]} M_{f_i \cdot f_j}(r)}{\left(\log^{[q-1]} r\right)^{\lambda_{f_i \cdot f_j}(p,q)}} \leq \tau+\varepsilon$$

Since $\varepsilon > 0$ is arbitrary,

$$\tau_{f_i \cdot f_j}(p,q) \leq \tau_{f_i}(p_i,q_i). \tag{4.42}$$

Now without loss of any generality, we may assume $f_k(0) = 1$ where $k = i, j$. Also let $r > R$.

Now from (4.16), we obtain for all sufficiently large values of R that

$$M_{f_i}(R) \geq \exp^{[p-1]}\left\{(\tau_{f_i}(p_i,q_i)-\varepsilon)\left(\log^{[q-1]} R\right)^{\lambda}\right\}. \tag{4.43}$$

Also from (4.17) we have for all sufficiently large values of r that

$$M_{f_j}(r) \leq \exp^{[p_j-1]}\left\{\left(\overline{\tau}_{f_j}(p_j,q_j)+\varepsilon\right)\left(\log^{[q_j-1]} r\right)^{\lambda_{f_j}(p_j,q_j)}\right\}.$$

Since in view of any one of the first four conditions of Proposition 2.1,

$$\begin{aligned}&\exp^{[p_j-1]}\left\{\left(\overline{\tau}_{f_j}(p_j,q_j)+\varepsilon\right)\left(\log^{[q_j-1]} r\right)^{\lambda_{f_j}(p_j,q_j)}\right\}\\ <\ &\exp^{[p-1]}\left\{(\tau+\varepsilon)\left(\log^{[q-1]} r\right)^{\lambda}\right\},\end{aligned}$$

we get from above for all sufficiently large values of r that

$$M_{f_j}(r) < \exp^{[p-1]}\left\{(\tau+\varepsilon)\left(\log^{[q-1]} r\right)^{\lambda}\right\}. \tag{4.44}$$

Since $r > R$, we have from (4.43) for all sufficiently large values of r that

$$\begin{aligned}M_{f_i}(r) > M_{f_i}(R) &> \exp^{[p-1]}\left\{(\tau-\varepsilon)\left(\log^{[q-1]} R\right)^{\lambda}\right\}\\ &> \exp^{[p-1]}\left\{(\tau-\varepsilon)\left(\log^{[q-1]} \frac{r}{2}\right)^{\lambda}\right\}.\end{aligned} \tag{4.45}$$

Further let z_r be a point on $|z| = r$ such that $M_{f_i}(r) = |f_i(z_r)|$.

Therefore as $r > R$, from (4.36), (4.44) and (4.45) it follows for all sufficiently large values of R that

$$\begin{aligned}&M_{f_i\cdot f_j}(r)\\ \geq\ &\left[\exp^{[p-1]}\left\{(\tau+\varepsilon)\left(\log^{[q-1]} 4eR\right)^{\lambda}\right\}\right]^{-7}\cdot\exp^{[p-1]}\left\{(\tau-\varepsilon)\left(\log^{[q-1]} \frac{r}{2}\right)^{\lambda}\right\}\end{aligned}$$

$$\begin{aligned}\text{i.e., } M_{f_i\cdot f_j}(r) \geq\ &\left[\exp^{[p-1]}\left\{(\tau+\varepsilon)\left(\log^{[q-1]} 4er\right)^{\lambda}\right\}\right]^{-7}\\ &\times\exp^{[p-1]}\left\{(\tau-\varepsilon)\left(\log^{[q-1]} \frac{4er}{8e}\right)^{\tau}\right\}.\end{aligned} \tag{4.46}$$

Now we have

$$\frac{\exp^{[p-2]}\left\{(\tau-\varepsilon)\left(\log^{[q-1]} \frac{4er}{8e}\right)^{\lambda}\right\}}{\exp^{[p-2]}\left\{(\tau+\varepsilon)\left(\log^{[q-1]} 4er\right)^{\lambda}\right\}} \to \infty \text{ as } r \to \infty.$$

So we may write for all sufficiently large values of r with $r_n > r_1 > r_0$,

$$\frac{\log\left\{(\tau-\varepsilon)\left(\log^{[q-1]} \frac{4er_n}{8e}\right)^{\lambda}\right\}}{\log\left\{(\tau+\varepsilon)\left(\log^{[q-1]} 4er_n\right)^{\lambda}\right\}} > \frac{\log\left\{(\tau-\varepsilon)\left(\log^{[q-1]} \frac{4er_0}{8e}\right)^{\lambda}\right\}}{\log\left\{(\tau+\varepsilon)\left(\log^{[q-1]} 4er_0\right)^{\lambda}\right\}}$$

$$= \delta \text{ (say)}.$$

Therefore clearly

$$\delta > 1.$$

So for the above value of δ, one can easily verify that

$$\exp^{[p-1]}\left\{(\tau-\varepsilon)\left(\log^{[q-1]}\frac{4er}{8e}\right)^{\lambda}\right\}$$

$$\geq \exp^{[p-1]}\left[\left\{(\tau+\varepsilon)\left(\log^{[q-1]}4er\right)^{\lambda}\right\}^{\delta}\right]. \tag{4.47}$$

Also from Lemma 3.2, we get for all sufficiently large values of r that

$$\exp^{[p-1]}\left[\left\{(\tau+\varepsilon)\left(\log^{[q-1]}4er\right)^{\lambda}\right\}^{\delta}\right]$$

$$\geq \left[\exp^{[p-1]}\left\{(\tau+\varepsilon)\left(\log^{[q-1]}4er\right)^{\lambda}\right\}\right]^{8}. \tag{4.48}$$

Now in view of in view of Theorem 4.4, it follows from (4.46) , (4.47) and (4.48) for all sufficiently large values of r that

$$M_{f_i\cdot f_j}(r) \geq \exp^{[p-1]}\left\{(\tau+\varepsilon)\left(\log^{[q-1]}4er\right)^{\lambda}\right\}$$

$$\text{i.e., } \frac{\log^{[p-1]}M_{f_i\cdot f_j}(r)}{\left(\log^{[q-1]}4er\right)^{\rho}} \geq (\tau+\varepsilon)$$

$$\text{i.e., } \limsup_{r\to\infty}\frac{\log^{[p-1]}M_{f_i\cdot f_j}(r)}{\left(\log^{[q-1]}r+O(1)\right)^{\lambda_{f_i\cdot f_j}(p,q)}} \geq \tau+\varepsilon, \text{ when } q>1.$$

$$\text{i.e., } \tau_{f_i\cdot f_j}(p,q) \geq \max\left\{\tau_{f_i}(p_i,q_i), \tau_{f_j}(p_j,q_j)\right\} \quad \text{when } q>1. \tag{4.49}$$

So the theorem follows from (4.42) and (4.49). □

In the next theorem we wish to find out the condition for which the equality sign of Theorem 4.4 hold in case of Proposition 2.1(v).

Theorem 4.6. *Let f_1 and f_2 be any two entire functions such that $\lambda_{f_1}(p,q) = \lambda_{f_2}(p,q)$ $(0 < \lambda_{f_1}(p,q) = \lambda_{f_2}(p,q) < \infty)$ and $\tau_{f_1}(p,q) \neq \tau_{f_2}(p,q)$. Then*

$$\lambda_{f_1\cdot f_2}(p,q) = \lambda_{f_1}(p,q) = \lambda_{f_2}(p,q),$$

where p, q are any two positive integers with $p \geq q > 1$, $(f_1 \cdot f_2)$ is of regular (p,q)-th growth and at least f_1 or f_2 is also of regular (p,q)-th growth.

The proof of Theorem 4.6 is omitted as it can be carried out in the line of Theorem 4.3.

5. Conclusion

In Theorem 4.1, Theorem 4.2, Theorem 4.4 and Theorem 4.5 of the present paper, the authors have discussed about the limiting value of the lower bound under any one of the first four conditions of Proposition 2.1. On the other hand, in Theorem 4.3 and Theorem 4.6, the present authors have also find out the limiting value of the lower bound in case of Proposition 2.1 (v) under some different conditions. Now question may arise about the limiting value of the lower bound when any one of the last five cases of Proposition 2.1 is considered and this may be a further scope of study for the future researchers in this branch.

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Tanmay Biswas
Rajbari, Rabindrapalli, R. N. Tagore Road
P.O.-Krishnagar, Dist-Nadia
PIN-741101 West Bengal, India
E-mail address: tanmaybiswas_math@rediffmail.com

Sanjib Kumar Datta
Department of Mathematics
University of Kalyani
P.O.-Kalyani, Dist-Nadia
PIN-741235, West Bengal, India
E-mail address: sanjib_kr_datta@yahoo.co.in

Jinarul Haque Shaikh
Department of Mathematics
University of Kalyani
P.O.-Kalyani, Dist-Nadia
PIN-741235, West Bengal, India
E-mail address: jnrlhqshkh188@gmail.com