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TWO GENERAL ITERATION SCHEMES FOR MULTI-VALUED MAPS IN HYPERBOLIC SPACES

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Abstract. In this paper, we introduce two general iteration schemes with bounded error terms and prove some theorems related to the strong and $\triangle$-convergence of these iteration schemes for multi-valued maps in a hyperbolic space. The results which are presented here extend and improve some well-known results in the current literature.

1. Introduction and preliminaries

Most of the problems in various disciplines of science are nonlinear in nature, whereas fixed point theory proposed in the setting of normed linear space or Banach space mainly depends on the linear structure of the underlying spaces. A nonlinear framework for fixed point theory is a metric space embedded with a 'convex structure'. One such convex structure is available in a hyperbolic space. Throughout the paper, we work in the setting of hyperbolic space introduced by Kohlenbach [14], which plays a significant role in many branches of mathematics.

A hyperbolic space (or W-hyperbolic space) is a metric space (X, d) together with a map $W : X \times X \times [0, 1] \to X$ satisfying

(W1) $d(z, W(x, y, \alpha)) \leq \alpha d(z, x) + (1 - \alpha) d(z, y)$
(W2) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta| d(x, y)$
(W3) $W(x, y, \alpha) = W(y, x, (1 - \alpha))$
(W4) $d(W(x, z, \alpha), W(y, w, \alpha)) \leq (1 - \alpha) d(x, y) + \alpha d(z, w)$

for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

A subset K of a hyperbolic space X is called convex if $W(x, y, \alpha) \in K$ for all $x, y \in K$ and $\alpha \in [0, 1]$. If a metric space satisfies only (W1), then it coincides with the convex metric space introduced by Takahashi [23]. The concept of

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hyperbolic space in [14] is more restrictive than the hyperbolic type introduced by Goebel and Kirk [8] since (W1)-(W3) together are equivalent to (X, d, W) being a space of hyperbolic type in [8]. Also it is slightly more general than the hyperbolic space defined by Reich and Shafrir [20]. The class of hyperbolic spaces in [14] contains all normed linear spaces and convex subsets thereof, $\mathbb{R}$-trees, the Hilbert ball with the hyperbolic metric (see [9]), Cartesian products of Hilbert balls, Hadamard manifolds and CAT(0) spaces (see [2]).

An important example of a hyperbolic space is the open unit ball B in the complex plane $\mathbb{C}$ with respect to the Poincare metric (also called 'Poincare distance')

$$d_B(x,y) = \arg\tanh\left|\frac{x-y}{1-x\overline{y}}\right| = \arg\tanh\left(1-\sigma\left(x,y\right)\right)^{\frac{1}{2}},$$

where

$$\sigma\left(x,y\right) = \frac{\left(1-|x|^2\right)\left(1-|y|^2\right)}{|1-x\overline{y}|^2} \quad \text{for all } x, y \in B.$$

For more and detailed treatment of examples on hyperbolic space, we refer the readers to [6, 13, 14, 15, 21].

A hyperbolic space (X, d, W) is said to be uniformly convex [22] if for all $u, x, y \in X, r > 0$ and $\varepsilon \in (0, 2]$, there exists a $\delta \in (0, 1]$ such that $d\left(W\left(x, y, \frac{1}{2}\right), u\right) \leq (1-\delta)r$ whenever $d(x, u) \leq r$, $d(y, u) \leq r$ and $d(x, y) \geq \varepsilon r$.

A map $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ which provides such a $\delta = \eta(r, \varepsilon)$ for given $r > 0$ and $\varepsilon \in (0, 2]$ is called modulus of uniform convexity. We call η monotone if it decreases with r (for a fixed ε). A CAT(0) space is a uniformly convex hyperbolic space with the modulus of uniform convexity $\eta(r, \varepsilon) = \frac{\varepsilon^2}{8}$ (see [15]). Thus, the class of uniformly convex hyperbolic spaces includes both uniformly convex normed spaces and CAT(0) spaces as special cases.

Let K be a nonempty subset of a metric space (X, d). The set K is called proximinal if for each $x \in X$, there exists an element $k \in K$ such that $d(x, k) = d(x, K)$, where $d(x, K) = \inf\{d(x, y) : y \in K\}$. We denote by $2^K, CB(K)$ and $P(K)$ the family of nonempty all subsets, nonempty closed bounded all subsets and nonempty proximinal bounded all subsets of K, respectively. The Hausdorff distance on 2^K is defined by

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\} \quad \text{for all } A, B \in 2^K.$$

Let $T : K \to 2^K$ be a multi-valued map. An element $p \in K$ is a fixed point of T if $p \in Tp$. The notation $F(T)$ has been reserved for the set of all fixed points of T. The map T is said to be

(i) nonexpansive if $H(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;

(ii) quasi-nonexpansive if $F(T) \neq \emptyset$ and $H(Tx, Tp) \leq d(x, p)$ for all $x \in K$ and $p \in F(T)$;

(iii) Lipschitzian if there exists a constant $L > 0$ such that $H(Tx, Ty) \leq Ld(x, y)$ for all $x, y \in K$;

(iv) Lipschitzian quasi-nonexpansive if both (ii) and (iii) hold.

It is clear that each multi-valued nonexpansive map with $F(T) \neq \emptyset$ is a quasi-nonexpansive map. But there exist the multi-valued quasi-nonexpansive maps that are not nonexpansive (see [19]). Moreover, each multi-valued nonexpansive map is a Lipschitzian map with $L = 1$.

Yıldırım and Özdemir [25] introduced a new multi-step iteration scheme for a finite family of non-self maps. In [11], Gürsoy et al. modified this iteration scheme for a self-map in a Banach space as follows.

For an arbitrary fixed order $k \geq 2$ and $x_1 \in K$,

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n^1 + \alpha_n T y_n^1, \\ y_n^1 = (1-\beta_n^1)y_n^2 + \beta_n^1 T y_n^2, \\ y_n^2 = (1-\beta_n^2)y_n^3 + \beta_n^2 T y_n^3, \\ \quad \vdots \\ y_n^{k-2} = (1-\beta_n^{k-2})y_n^{k-1} + \beta_n^{k-2} T y_n^{k-1}, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T x_n, \quad n \geq 1, \end{cases}$$

or, in short,

$$\begin{cases} x_{n+1} = (1-\alpha_n)y_n^1 + \alpha_n T y_n^1, \\ y_n^i = (1-\beta_n^i)y_n^{i+1} + \beta_n^i T y_n^{i+1}, \ i = 1, 2, \ldots, k-2, \\ y_n^{k-1} = (1-\beta_n^{k-1})x_n + \beta_n^{k-1} T x_n, \quad n \geq 1. \end{cases} \tag{1}$$

By taking $k = 3$ and $k = 2$ in (1), we obtain the SP-iteration scheme of Phuengrattana and Suantai [18] and the two-step iteration scheme of Thianwan [24], respectively. Recently, Başarır and Şahin [1] studied the iteration scheme (1) for single-valued maps in a CAT(0) space.

Fukhar-ud-din et al. [7] introduced two iteration schemes for multi-valued maps in a hyperbolic space as follows.

(A) Let T_1 and T_2 be two multi-valued quasi-nonexpansive maps from K into $CB(K)$, where K is a convex subset of a hyperbolic space. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\alpha_n + \beta_n\}$, $\{\alpha_n^{'}\}$, $\{\beta_n^{'}\}$, $\{\alpha_n^{'} + \beta_n^{'}\}$ are real sequences in $[0, 1]$. Then for $x_1 \in K$, generate a sequence $\{x_n\}$ as

$$\begin{cases} y_n = W\left(z_n^{'}, W\left(x_n, u_n, \frac{\beta_n^{'}}{1-\alpha_n^{'}}\right), \alpha_n^{'}\right), \\ x_{n+1} = W\left(z_n, W\left(y_n, v_n, \frac{\beta_n}{1-\alpha_n}\right), \alpha_n\right), \quad n \geq 1, \end{cases} \tag{2}$$

where $z_n^{'} \in T_1 x_n, z_n \in T_2 y_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in K.

(B) Let T_1 and T_2 be two multi-valued quasi-nonexpansive maps from K into $P(K)$ and $P_{T_i}x = \{y \in T_i x : d(x, y) = d(x, T_i x)\}, i = 1, 2$. Suppose that $\{\alpha_n\}$, $\{\beta_n\}$, $\{\alpha_n + \beta_n\}$, $\{\alpha_n^{'}\}$, $\{\beta_n^{'}\}$, $\{\alpha_n^{'} + \beta_n^{'}\}$ are real sequences in $[0, 1]$. Then for $x_1 \in K$, generate a sequence $\{x_n\}$ as in (2) where $z_n^{'} \in P_{T_1} x_n$, $z_n \in P_{T_2} y_n$ and $\{u_n\}, \{v_n\}$ are bounded sequences in K.

The iteration schemes (A) and (B) coincide with the iteration schemes of Cholamjiak and Suantai [4] when $W(x, y, \alpha) = \alpha x + (1-\alpha)y$ and X is a Banach space.

Inspired and motivated by these results, we introduce the general iteration schemes in a hyperbolic space, as follows.

(C) Let $\{T_i\}_{i=1}^N$ be a finite family of multi-valued maps from K into 2^K, where K is a convex subset of a hyperbolic space. Suppose that $\{\alpha_n^i\}, \{\beta_n^i\}$ and $\{\alpha_n^i + \beta_n^i\}$ are real sequences in $[0, 1]$ for each $i = 1, 2, \ldots, N$. Then for $x_1 \in K$, define a sequence $\{x_n\}$ by

$$\begin{cases} y_n^1 = W\left(z_n^1, W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right), \alpha_n^1\right), \\ y_n^2 = W\left(z_n^2, W\left(y_n^1, u_n^2, \frac{\beta_n^2}{1-\alpha_n^2}\right), \alpha_n^2\right), \\ \quad \vdots \\ y_n^{N-2} = W\left(z_n^{N-2}, W\left(y_n^{N-3}, u_n^{N-2}, \frac{\beta_n^{N-2}}{1-\alpha_n^{N-2}}\right), \alpha_n^{N-2}\right), \\ y_n^{N-1} = W\left(z_n^{N-1}, W\left(y_n^{N-2}, u_n^{N-1}, \frac{\beta_n^{N-1}}{1-\alpha_n^{N-1}}\right), \alpha_n^{N-1}\right), \\ x_{n+1} = W\left(z_n^N, W\left(y_n^{N-1}, u_n^N, \frac{\beta_n^N}{1-\alpha_n^N}\right), \alpha_n^N\right), \quad n \geq 1, \end{cases}$$

or, in short,

$$\begin{cases} y_n^1 = W\left(z_n^1, W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right), \alpha_n^1\right), \\ y_n^i = W\left(z_n^i, W\left(y_n^{i-1}, u_n^i, \frac{\beta_n^i}{1-\alpha_n^i}\right), \alpha_n^i\right), \quad i = 2, 3, \ldots, N-1, \\ x_{n+1} = W\left(z_n^N, W\left(y_n^{N-1}, u_n^N, \frac{\beta_n^N}{1-\alpha_n^N}\right), \alpha_n^N\right), \quad n \geq 1, \end{cases} \tag{3}$$

where $z_n^1 \in T_1 x_n, z_n^i \in T_i y_n^{i-1}$ for each $i = 2, 3, \ldots, N$ and $\{u_n^i\}$ is a bounded sequence in K for each $i = 1, 2, \ldots, N$.

(D) Let $\{T_i\}_{i=1}^N$ be the finite family of multi-valued maps from K into $P(K)$ and $P_{T_i} x = \{y \in T_i x : d(x, y) = d(x, T_i x)\}, i = 1, 2, \ldots, N$. Suppose that $\{\alpha_n^i\}, \{\beta_n^i\}$ and $\{\alpha_n^i + \beta_n^i\}$ are real sequences in $[0, 1]$ for each $i = 1, 2, \ldots, N$. Then for $x_1 \in K$, define a sequence $\{x_n\}$ as in (3) where $z_n^1 \in P_{T_1} x_n, z_n^i \in P_{T_i} y_n^{i-1}$ for each $i = 2, 3, \ldots, N$ and $\{u_n^i\}$ is a bounded sequence in K for each $i = 1, 2, \ldots, N$.

By taking $N = 2$ in (C) and (D), we obtain the iteration schemes (A) and (B), respectively.

In this paper, we prove the strong and $\triangle$-convergence results of the iteration schemes (C) and (D) for the finite family of multi-valued maps in a uniformly convex hyperbolic space.

The concept of $\triangle$-convergence in a metric space was introduced by Lim [16] and its analogue in a CAT(0) space has been investigated by Dhompongsa and Panyanak [5]. In order to define the concept of $\triangle$-convergence in the general setup of hyperbolic space, we first collect some basic concepts.

Let $\{x_n\}$ be a bounded sequence in a hyperbolic space X. For $x \in X$, we define a continuous functional $r(\cdot, \{x_n\}) : X \to [0, \infty)$ by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf \ \{r(x, \{x_n\}) : x \in X\}.$$

The asymptotic center of $\{x_n\}$ with respect to a subset K of X is defined as

$$A_K(\{x_n\}) = \{x \in X : r(x, \{x_n\}) \le r(y, \{x_n\}), \forall y \in K\}.$$

This is the set of minimizer of the functional $r(\cdot, \{x_n\})$. If the asymptotic center is taken with respect to X, then it is simply denoted by $A(\{x_n\})$. It is known that uniformly convex Banach spaces and even CAT(0) spaces enjoy the property that "bounded sequences have unique asymptotic center with respect to closed convex subsets". The following lemma is due to Leustean [15] and ensures that this property also holds in a complete uniformly convex hyperbolic space.

Lemma 1.1 ([15, Proposition 3.3]). *Let (X, d, W) be a complete uniformly convex hyperbolic space with a monotone modulus of uniform convexity. Then every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to any nonempty closed convex subset K of X.*

Recall that a sequence $\{x_n\}$ in X is said to $\triangle$-convergent to $x \in X$ if $\{x\}$ is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case, we write $\triangle\text{-}\lim_{n\to\infty} x_n = x$ and call x as $\triangle$-limit of $\{x_n\}$.

In the sequel, we shall need the following results.

Lemma 1.2 ([3]). *Let $\{a_n\}$ and $\{b_n\}$ be sequences of non-negative real numbers such that $a_{n+1} \le a_n + b_n$ for all $n \ge 1$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 1.3 ([13, Lemma 2.5]). *Let (X, d, W) be a uniformly convex hyperbolic space with a monotone modulus of uniform convexity. Let $x \in X$ and $\{\alpha_n\}$ be a sequence in $[a, b]$ for some $a, b \in (0, 1)$. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\limsup_{n\to\infty} d(x_n, x) \le r$, $\limsup_{n\to\infty} d(y_n, x) \le r$ and $\lim_{n\to\infty} d(W(x_n, y_n, \alpha_n), x) = r$ for some $r \ge 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$.*

Lemma 1.4 ([13, Lemma 2.6]). *Let K be a nonempty closed convex subset of a uniformly convex hyperbolic space and let $\{x_n\}$ be a bounded sequence in K such that $A(\{x_n\}) = \{y\}$ and $r(\{x_n\}) = \rho$. If $\{y_m\}$ is another sequence in K such that $\lim_{m\to\infty} r(y_m, \{x_n\}) = \rho$, then $\lim_{m\to\infty} y_m = y$.*

2. Main results

From now onward, we denote $F = \cap_{i=1}^{N} F(T_i)$ for the finite family $\{T_i\}_{i=1}^{N}$ of multi-valued maps and suppose that $F \neq \emptyset$.

We start with the following key lemmas.

Lemma 2.1. *Let K be a nonempty, closed and convex subset of a hyperbolic space X and let $\{T_i\}_{i=1}^N$ be the family of multi-valued quasi-nonexpansive maps from K into 2^K such that $T_ip = \{p\}$ for all $p \in F$ and $i = 1, 2, \ldots, N$. Let $\{x_n\}$ be the sequence defined by (3) with $\sum_{n=1}^{\infty}(1 - \alpha_n^i - \beta_n^i) < \infty$ for all $i = 1, 2, \ldots, N$. Then $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$.*

Proof. Since $\{u_n^i\}_{i=1}^N$ are bounded sequences, we set

$$\max_{i\in\{1,2,\ldots,N\}} \left\{\sup_{n\in\mathbb{N}} d(u_n^i, p)\right\} < M \text{ for some } M > 0.$$

For any $p \in F$, it follows from (3) that

$$\begin{aligned}
d(y_n^1, p) &= d\left(W\left(z_n^1, W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right), \alpha_n^1\right), p\right)\\
&\le \alpha_n^1 d(z_n^1, p) + (1-\alpha_n^1) d\left(W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right), p\right)\\
&\le \alpha_n^1 d(z_n^1, p) + \beta_n^1 d(x_n, p) + (1-\alpha_n^1-\beta_n^1) d(u_n^1, p)\\
&\le \alpha_n^1 H(T_1x_n, T_1p) + \beta_n^1 d(x_n, p) + (1-\alpha_n^1-\beta_n^1)M\\
&\le \alpha_n^1 d(x_n, p) + \beta_n^1 d(x_n, p) + (1-\alpha_n^1-\beta_n^1)M\\
&\le d(x_n, p) + (1-\alpha_n^1-\beta_n^1)M \qquad (4)
\end{aligned}$$

and

$$\begin{aligned}
d(y_n^2, p) &\le \alpha_n^2 d(z_n^2, p) + (1-\alpha_n^2) d\left(W\left(y_n^1, u_n^2, \frac{\beta_n^2}{1-\alpha_n^2}\right), p\right)\\
&\le \alpha_n^2 d(z_n^2, p) + \beta_n^2 d(y_n^1, p) + (1-\alpha_n^2-\beta_n^2) d(u_n^2, p)\\
&\le \alpha_n^2 H(T_2y_n^1, T_2p) + \beta_n^2 d(y_n^1, p) + (1-\alpha_n^2-\beta_n^2)M\\
&\le d(y_n^1, p) + (1-\alpha_n^2-\beta_n^2)M. \qquad (5)
\end{aligned}$$

Similarly, we have

$$d(y_n^3, p) \le d(y_n^2, p) + (1-\alpha_n^3-\beta_n^3)M.$$

Therefore

$$d(y_n^3, p) \le d(x_n, p) + M\sum_{i=1}^{3}(1-\alpha_n^i-\beta_n^i).$$

Continuing the above process, we obtain

$$d(y_n^{N-1}, p) \le d(x_n, p) + M\sum_{i=1}^{N-1}(1-\alpha_n^i-\beta_n^i). \qquad (6)$$

In addition, we have

$$\begin{aligned}
d(x_{n+1}, p) &\le \alpha_n^N d(z_n^N, p) + (1-\alpha_n^N) d\left(W\left(y_n^{N-1}, u_n^N, \frac{\beta_n^N}{1-\alpha_n^N}\right), p\right)\\
&\le \alpha_n^N d(z_n^N, p) + \beta_n^N d(y_n^{N-1}, p) + (1-\alpha_n^N-\beta_n^N) d(u_n^N, p)
\end{aligned}$$

$$\leq \alpha_n^N H(T_N y_n^{N-1}, T_N p) + \beta_n^N d(y_n^{N-1}, p) + (1 - \alpha_n^N - \beta_n^N)M$$

$$\leq d(y_n^{N-1}, p) + (1 - \alpha_n^N - \beta_n^N)M. \tag{7}$$

Combining (6) and (7), we get

$$d(x_{n+1}, p) \leq d(x_n, p) + M \sum_{i=1}^{N} (1 - \alpha_n^i - \beta_n^i).$$

Since $\sum_{n=1}^{\infty}(1 - \alpha_n^i - \beta_n^i) < \infty$ for all $i = 1, 2, \ldots, N$, then, by Lemma 1.2, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. □

Lemma 2.2. *Let K be a nonempty, closed and convex subset of a uniformly convex hyperbolic space X with a monotone modulus of uniform convexity and let $\{T_i\}_{i=1}^N$ be the family of multi-valued Lipschitzian quasi-nonexpansive maps from K into $CB(K)$ such that $T_i p = \{p\}$ for all $p \in F$ and $i = 1, 2, \ldots, N$. Let $\{x_n\}$ be the sequence defined by* (3) *with $0 < l \leq \alpha_n^i \leq m < 1$ and $\sum_{n=1}^{\infty}(1 - \alpha_n^i - \beta_n^i) < \infty$ for each $i = 1, 2, \ldots, N$. Then $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for each $i = 1, 2, \ldots, N$.*

Proof. In fact, it follows from Lemma 2.1 that $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F$. We may assume that $\lim_{n\to\infty} d(x_n, p) = r$. The case $r = 0$ is trivial. Next, we deal with the case $r > 0$. As $\{x_n\}, \{y_n^i\}, \{u_n^i\}$ are bounded sequences, so

$$\max_{i\in\{1,2,\ldots,N-1\}} \left\{ \sup_{n\in\mathbb{N}} d(u_n^1, x_n), \sup_{n\in\mathbb{N}} d(u_n^{i+1}, y_n^i) \right\} < \infty.$$

Taking limsup on both sides in the inequality (4), we obtain $\limsup_{n\to\infty} d(y_n^1, p) \leq r$. On the other hand, from (7), we have

$$\begin{aligned}
d(x_{n+1}, p) &\leq d(y_n^{N-1}, p) + (1 - \alpha_n^N - \beta_n^N)M \\
&\leq d(y_n^{N-2}, p) + M \sum_{i=N-1}^{N} (1 - \alpha_n^i - \beta_n^i) \\
&\ \ \vdots \\
&\leq d(y_n^2, p) + M \sum_{i=3}^{N} (1 - \alpha_n^i - \beta_n^i) \qquad (8) \\
&\leq d(y_n^1, p) + M \sum_{i=2}^{N} (1 - \alpha_n^i - \beta_n^i).
\end{aligned}$$

This implies that $\liminf_{n\to\infty} d(y_n^1, p) \geq r$. Then we get

$$\lim_{n\to\infty} d\left(W\left(z_n^1, W\left(x_n, u_n^1, \frac{\beta_n^1}{1 - \alpha_n^1} \right), \alpha_n^1 \right), p \right) = \lim_{n\to\infty} d(y_n^1, p) = r. \tag{9}$$

Since T_1 is a quasi-nonexpansive map, then we have $d(z_n^1, p) \leq H(T_1 x_n, T_1 p) \leq d(x_n, p)$. Hence

$$\limsup_{n\to\infty} d(z_n^1, p) \leq r. \tag{10}$$

In addition, since

$$\begin{aligned} d\left(W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right), p\right) &\leq \frac{\beta_n^1}{1-\alpha_n^1} d(x_n, p) + \left(1 - \frac{\beta_n^1}{1-\alpha_n^1}\right) d(u_n^1, p) \\ &\leq \frac{\beta_n^1}{1-\alpha_n^1} d(x_n, p) \\ &\quad + \left(1 - \frac{\beta_n^1}{1-\alpha_n^1}\right)\left[d(u_n^1, x_n) + d(x_n, p)\right] \\ &\leq d(x_n, p) + \left(\frac{1-\alpha_n^1-\beta_n^1}{1-m}\right) d(u_n^1, x_n), \end{aligned}$$

then we have

$$\limsup_{n\to\infty} d\left(W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right), p\right) \leq r. \tag{11}$$

With the help of (9)-(11) and Lemma 1.3, we have

$$\lim_{n\to\infty} d\left(z_n^1, W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right)\right) = 0. \tag{12}$$

From

$$\begin{aligned} d(z_n^1, y_n^1) &= d\left(z_n^1, W\left(z_n^1, W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right), \alpha_n^1\right)\right) \\ &\leq (1-\alpha_n^1) d\left(z_n^1, W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right)\right) \end{aligned}$$

and

$$\begin{aligned} d(y_n^1, x_n) &= d\left(W\left(z_n^1, W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right), \alpha_n^1\right), x_n\right) \\ &\leq \alpha_n^1 d(z_n^1, x_n) + (1-\alpha_n^1) d\left(W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right), x_n\right) \\ &\leq \alpha_n^1 d(z_n^1, x_n) + (1-\alpha_n^1-\beta_n^1) d\left(u_n^1, x_n\right), \end{aligned}$$

we obtain

$$\begin{aligned} d(z_n^1, x_n) &\leq d(z_n^1, y_n^1) + d(y_n^1, x_n) \\ &\leq (1-\alpha_n^1) d\left(z_n^1, W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1}\right)\right) \\ &\quad + \alpha_n^1 d(z_n^1, x_n) + (1-\alpha_n^1-\beta_n^1) d\left(u_n^1, x_n\right). \end{aligned}$$

Rearranging the terms in the above inequality and using $0 < l \le \alpha_n^1 \le m < 1$, we have

$$d(z_n^1, x_n) \le \left(\frac{1-\alpha_n^1-\beta_n^1}{1-m} \right) d\left(u_n^1, x_n\right) + d\left(z_n^1, W\left(x_n, u_n^1, \frac{\beta_n^1}{1-\alpha_n^1} \right) \right).$$

Hence, by $\sum_{n=1}^{\infty}(1-\alpha_n^1-\beta_n^1) < \infty$, the boundedness of $\left\{ d\left(u_n^1, x_n\right) \right\}$ and using (12), we obtain

$$\lim_{n\to\infty} d(z_n^1, x_n) = 0. \tag{13}$$

As $z_n^1 \in T_1x_n$, so $d(x_n, T_1x_n) \le d(x_n, z_n^1)$ which implies that

$$\lim_{n\to\infty} d(x_n, T_1x_n) = 0.$$

Combining (4) and (5), we have $d(y_n^2, p) \le d(x_n, p) + M\sum_{i=1}^{2}(1-\alpha_n^i-\beta_n^i)$. This implies that $\limsup_{n\to\infty} d(y_n^2, p) \le r$. On the other hand, taking liminf on both sides in (8), we obtain $\liminf_{n\to\infty} d(y_n^2, p) \ge r$. Then we get

$$\lim_{n\to\infty} d\left(W\left(z_n^2, W\left(y_n^1, u_n^2, \frac{\beta_n^2}{1-\alpha_n^2} \right), \alpha_n^2 \right), p \right) = \lim_{n\to\infty} d(y_n^2, p) = r. \tag{14}$$

Since $d(z_n^2, p) \le H(T_2y_n^1, T_2p) \le d(y_n^1, p)$, then we have

$$\limsup_{n\to\infty} d(z_n^2, p) \le r. \tag{15}$$

In addition, since

$$d\left(W\left(y_n^1, u_n^2, \frac{\beta_n^2}{1-\alpha_n^2} \right), p \right) \le d(y_n^1, p) + \left(\frac{1-\alpha_n^2-\beta_n^2}{1-m} \right) d(u_n^2, y_n^1),$$

then we have

$$\limsup_{n\to\infty} d\left(W\left(y_n^1, u_n^2, \frac{\beta_n^2}{1-\alpha_n^2} \right), p \right) \le r. \tag{16}$$

Appealing to Lemma 1.3 and utilizing estimates (14)-(16), we get

$$\lim_{n\to\infty} d\left(z_n^2, W\left(y_n^1, u_n^2, \frac{\beta_n^2}{1-\alpha_n^2} \right) \right) = 0. \tag{17}$$

From

$$d(z_n^2, y_n^2) \le (1-\alpha_n^2) d\left(z_n^2, W\left(y_n^1, u_n^2, \frac{\beta_n^2}{1-\alpha_n^2} \right) \right)$$

and

$$d(y_n^2, y_n^1) \le \alpha_n^2 d(z_n^2, y_n^1) + (1-\alpha_n^2-\beta_n^2) d\left(u_n^2, y_n^1\right),$$

we get

$$\begin{aligned} d(z_n^2, y_n^1) &\le d(z_n^2, y_n^2) + d(y_n^2, y_n^1) \\ &\le (1-\alpha_n^2) d\left(z_n^2, W\left(y_n^1, u_n^2, \frac{\beta_n^2}{1-\alpha_n^2} \right) \right) \\ &\quad + \alpha_n^2 d(z_n^2, y_n^1) + (1-\alpha_n^2-\beta_n^2) d\left(u_n^2, y_n^1\right). \end{aligned}$$

Then, we have

$$d(z_n^2, y_n^1) \leq \left(\frac{1-\alpha_n^2-\beta_n^2}{1-m}\right) d\left(u_n^2, y_n^1\right) + d\left(z_n^2, W\left(y_n^1, u_n^2, \frac{\beta_n^2}{1-\alpha_n^2}\right)\right).$$

Hence, we obtain

$$\lim_{n\to\infty} d(z_n^2, y_n^1) = 0. \tag{18}$$

It follows from (13) that

$$d(x_n, y_n^1) \leq \alpha_n^1 d(z_n^1, x_n) + (1-\alpha_n^1-\beta_n^1) d\left(u_n^1, x_n\right) \to 0 \text{ as } n \to \infty. \tag{19}$$

Using (18), (19) and the fact that $z_n^2 \in T_2 y_n^1$, we get

$$\begin{aligned} d(x_n, T_2 x_n) &\leq d(x_n, y_n^1) + d(y_n^1, z_n^2) + d(z_n^2, T_2 x_n) \\ &\leq d(x_n, y_n^1) + d(y_n^1, z_n^2) + H(T_2 y_n^1, T_2 x_n) \\ &\leq (1+L) d(x_n, y_n^1) + d(y_n^1, z_n^2) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Continuing the above process, we can prove that

$$\lim_{n\to\infty} d(y_n^i, p) = r, \ \lim_{n\to\infty} d(z_n^i, y_n^{i-1}) = 0 \text{ and } \lim_{n\to\infty} d(x_n, T_i x_n) = 0$$

for all $i = 3, \ldots, N-1$. Using $\lim_{n\to\infty} d(x_{n+1}, p) = r$, we can also get that

$$\lim_{n\to\infty} d(z_n^N, y_n^{N-1}) = 0 \text{ and } \lim_{n\to\infty} d(x_n, T_N x_n) = 0.$$

Consequently, $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for each $i = 1, 2, \ldots, N$. □

Our next result deals with the $\triangle$-convergence of the iteration schemes (C) in a hyperbolic space.

Theorem 2.3. *Let X, K and $\{x_n\}$ satisfy the hypotheses of Lemma 2.2, X be complete and let $\{T_i\}_{i=1}^N$ be the family of multi-valued nonexpansive maps from K into $CB(K)$ such that $T_i p = \{p\}$ for all $p \in F$ and $i = 1, 2, \ldots, N$. Then the sequence $\{x_n\}$ is $\triangle$-convergent to a point in F.*

Proof. It follows from Lemma 2.1 that $\{x_n\}$ is bounded. Therefore by Lemma 1.1, $\{x_n\}$ has a unique asymptotic center, that is, $A_K(\{x_n\}) = \{x\}$. Let $\{u_n\}$ be any subsequence of $\{x_n\}$ such that $A_K(\{u_n\}) = \{u\}$. It follows from Lemma 2.2 that

$$\lim_{n\to\infty} d(u_n, T_i u_n) = 0 \text{ for each } i = 1, 2, \ldots, N. \tag{20}$$

Now, we claim that u is a common fixed point of $\{T_i\}_{i=1}^N$. For each $i = 1, 2, \ldots, N$, we define a sequence $\{z_m\}$ in K by $z_m \in T_i u$. So, we calculate

$$\begin{aligned} d(z_m, u_n) &\leq d(z_m, T_i u_n) + d(T_i u_n, u_n) \\ &\leq H(T_i u, T_i u_n) + d(T_i u_n, u_n) \\ &\leq d(u, u_n) + d(T_i u_n, u_n). \end{aligned}$$

Taking limsup on both sides of the above inequality and using (20), we have

$$r(z_m, \{u_n\}) = \limsup_{n\to\infty} d(z_m, u_n) \leq \limsup_{n\to\infty} d(u, u_n) = r(u, \{u_n\}).$$

This implies that $|r(z_m, \{u_n\}) - r(u, \{u_n\})| \to 0$ as $m \to \infty$. It follows from Lemma 1.4 that $\lim_{m\to\infty} z_m = u$. Since $\{T_i u\}_{i=1}^N$ is the finite family of closed set, therefore $u \in T_i u$ for each $i = 1, 2, \ldots, N$. Hence $u \in F$. Next, we claim that for the common fixed point $u, \{u\}$ is the unique asymptotic center for each subsequence $\{u_n\}$ of $\{x_n\}$. Assume contrarily, that is, $x \neq u$. Since $\lim_{n\to\infty} d(x_n, u)$ exists (by Lemma 2.1), therefore by the uniqueness of asymptotic center, we have

$$\begin{aligned}
\limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x)\\
&\leq \limsup_{n\to\infty} d(x_n, x)\\
&< \limsup_{n\to\infty} d(x_n, u)\\
&= \limsup_{n\to\infty} d(u_n, u)
\end{aligned}$$

a contradiction. Hence, we get that $x = u$. Since $\{u_n\}$ is an arbitrary subsequence of $\{x_n\}$, therefore $A(\{u_n\}) = \{u\}$ for each subsequence $\{u_n\}$ of $\{x_n\}$. This proves that the sequence $\{x_n\}$ is $\triangle$-convergent to a common fixed point of $\{T_i\}_{i=1}^N$. □

The following result gives a necessary and sufficient condition for the strong convergence of the iteration scheme (C) in a hyperbolic space.

Theorem 2.4. *Let $X, K, \{T_i\}_{i=1}^N$ and $\{x_n\}$ satisfy the hypotheses of Lemma 2.2 and X be complete. Then the sequence $\{x_n\}$ is convergent strongly to some $p \in F$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. If $\{x_n\}$ converges to $p \in F$, then $\lim_{n\to\infty} d(x_n, p) = 0$. Since $0 \leq d(x_n, F) \leq d(x_n, p)$, we have $\liminf_{n\to\infty} d(x_n, F) = 0$. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. It follows from Lemma 2.1 that $\lim_{n\to\infty} d(x_n, F)$ exists. Thus by hypothesis, we get $\lim_{n\to\infty} d(x_n, F) = 0$. The proof of the remaining part follows the proof of Theorem 2.5 in [7] with $h_n^N = M \sum_{i=1}^N (1 - \alpha_n^i - \beta_n^i)$ for some $M > 0$ as in Lemma 2.1. □

Remark 2.5. In Theorem 2.4, the condition $\liminf_{n\to\infty} d(x_n, F) = 0$ may be replaced with $\limsup_{n\to\infty} d(x_n, F) = 0$.

Now, we give an example to illustrate Theorem 2.3 and Theorem 2.4.

Example 2.6. Let $K = [0, 1]$ be endowed with the Euclidean metric and $T_i : K \to CB(K)$ be defined by $T_i(x) = \left[0, \frac{x}{i+1}\right]$ for each $i = 1, 2, \ldots, N$. It is proved in [12, Example 2] that both T_1 and T_3 are multi-valued nonexpansive mappings. Similarly, $\{T_i\}_{i=1}^N$ is the family of multi-valued nonexpansive mappings. Therefore, it is the family of multi-valued Lipschitzian quasi-nonexpansive maps. Obviously, $F = \{0\}$. For each $i = 1, 2, \ldots, N$, we set $\alpha_n^i = \frac{1}{4}, \beta_n^i = \frac{3n^2+6n-1}{4(n+1)^2}$ for all $n \geq 1$. It is easy to see that

$$\sum_{n=1}^{\infty}(1-\alpha_n^i-\beta_n^i) = \sum_{n=1}^{\infty}\left(1-\frac{1}{4}-\frac{3n^2+6n-1}{4(n+1)^2}\right) = \sum_{n=1}^{\infty}\frac{1}{(n+1)^2} < \infty$$

for each $i = 1, 2, \ldots, N$. Hence the results of Theorem 2.3 and Theorem 2.4 can be easily seen.

Recall that a multi-valued map $T : K \to 2^K$ is semi-compact if every bounded sequence $\{x_n\} \subset K$ satisfying $d(x_n, Tx_n) \to 0$ as $n \to \infty$ has a strongly convergent subsequence.

Gu and He [10] defined the concept of condition $(A^{'})$ as follows.

N multi-valued maps $\{T_i\}_{i=1}^N$ are said to satisfy the condition $(A^{'})$ if there exists a non-decreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that

$$f(d(x, F)) \ \leq \frac{1}{N}\sum_{i=1}^{N} d(x, T_i x) \ \text{ for all } \ x \in K.$$

By using the above definitions, we prove the strong convergence results of the iteration scheme (C) in a hyperbolic space.

Theorem 2.7. *Let $X, K, \{T_i\}_{i=1}^N$ and $\{x_n\}$ be the same as in Theorem 2.4.*

(i) *If one of the maps $\{T_i\}_{i=1}^N$ is semi-compact, then the sequence $\{x_n\}$ is convergent strongly to a point in F.*

(ii) *If $\{T_i\}_{i=1}^N$ satisfy condition $(A^{'})$, then the sequence $\{x_n\}$ is convergent strongly to a point in F.*

Proof. (i) It follows from Lemma 2.1 that $\{x_n\}$ is bounded. Moreover, Lemma 2.2 implies that $\lim_{n\to\infty} d(x_n, T_i x_n) = 0$ for each $i = 1, 2, \ldots, N$. Then, by the semi-compactness of T_1 (say), there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\{x_{n_k}\}$ converges strongly to some point $p \in K$. Moreover, by the continuity of T_i, for each $i = 1, 2, \ldots, N$

$$d(p, T_i p) = \lim_{k\to\infty} d(x_{n_k}, T_i x_{n_k}) = 0.$$

This implies that $p \in F$. Again, by Lemma 2.1, $\lim_{n\to\infty} d(x_n, p)$ exists. Hence p is the strong limit of the sequence $\{x_n\}$. As a result, the sequence $\{x_n\}$ is convergent strongly to a point p in F.

(ii) By virtue of Lemma 2.1, $\lim_{n\to\infty} d(x_n, F)$ exists. Further, by condition $(A^{'})$ and Lemma 2.2, we have

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \frac{1}{N}\sum_{i=1}^{N} d(x_n, T_i x_n) = 0.$$

That is, $\lim_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a non-decreasing function satisfying $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, it follows that $\lim_{n\to\infty}$

$d(x_n, F) = 0$. Then Theorem 2.4 implies that the sequence $\{x_n\}$ is convergent strongly to a point in F. □

Remark 2.8. Since each multi-valued nonexpansive map is Lipschitzian quasi-nonexpansive, Theorem 2.4 and Theorem 2.7 hold for the finite family of multi-valued nonexpansive maps.

For further development, we need the following technical result.

Lemma 2.9 ([17, Lemma 2.2]). *Let K be a nonempty subset of a metric space (X, d) and let $T : K \to P(K)$ be a multi-valued mapping. Then*
(i) $d(x, T(x)) = d(x, P_T(x))$ *for all* $x \in K$;
(ii) $x \in F(T) \iff x \in F(P_T) \iff P_T(x) = \{x\}$;
(iii) $F(T) = F(P_T)$.

To avoid the restriction of $\{T_i\}_{i=1}^N$, that is, $T_i p = \{p\}$ for all $p \in F$ and $i = 1, 2, \ldots, N$, we use the iteration scheme defined by (D). Since the calculations in the following theorem are similar to those in the above results with the help of Lemma 2.9, we omit its proof.

Theorem 2.10. *Let X and K be the same as in Theorem 2.3. Suppose that $\{T_i\}_{i=1}^N$ is the family of multi-valued maps from K into $P(K)$ such that $\{P_{T_i}\}_{i=1}^N$ are nonexpansive. If $\{x_n\}$ is the sequence defined by* (D) *with $0 < l \leq \alpha_n^i \leq m < 1$ and $\sum_{n=1}^{\infty}(1 - \alpha_n^i - \beta_n^i) < \infty$ for all $i = 1, 2, \ldots, N$, then the followings hold*:
(i) *The sequence $\{x_n\}$ is $\triangle$-convergent to a point in F.*
(ii) *The sequence $\{x_n\}$ is convergent strongly to some $p \in F$ if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0 \quad \text{or} \limsup_{n\to\infty} d(x_n, F) = 0.$$

(iii) *If one of the maps $\{T_i\}_{i=1}^N$ is semi-compact or $\{T_i\}_{i=1}^N$ satisfy condition (A'), then the sequence $\{x_n\}$ is convergent strongly to a point in F.*

Remark 2.11. Our results generalize the results of Fukhar-ud-din et al. [7] since the iteration schemes (C) and (D) is reduced to (A) and (B), respectively, when $N = 2$.

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