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COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASS FOR SPIRALLIKE FUNCTIONS DEFINED BY MEANS OF GENERALIZED ATTIYA-SRIVASTAVA OPERATOR

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Abstract. In this article, we derive a sharp estimates for the Taylor-Maclaurin coefficients of functions in some certain subclasses of spirallike functions which are defined by generalized Srivastava-Attiya operator. Several corollaries and consequences of the main result are also considered.

1. Introduction

Let $\mathbb{D}$ be the unit disk $\{z : |z| < 1\}$, $\mathcal{A}$ be the class of functions analytic in $\mathbb{D}$, satisfying the conditions

$$f(0) = 0 \text{ and } f'(0) = 1. \tag{1}$$

Then each function f in $\mathcal{A}$ has the Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{2}$$

because of the conditions (1). Let S denote class of analytic and univalent functions in $\mathbb{D}$ with the normalization conditions (1).

Definition 1.1. For $0 \leq \alpha < 1$ let $S^*(\alpha)$ and $S^c(\alpha)$ denote the class of starlike and convex univalent functions of order α, which are defined as the following, respectively.

$$S^*(\alpha) = \left\{ f(z) \in S : \operatorname{Re}\left(\frac{zf'(z)}{f(z)} \right) > \alpha,\ z \in \mathbb{D} \right\}$$

and

$$S^c(\alpha) = \left\{ f(z) \in S : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha,\ z \in \mathbb{D} \right\}.$$

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Observe that $S^*(0) = S^*$ represent standard starlike functions. A notation of α_starlikeness and α_convexity were generalized onto a complex order α by Nasr and Aouf [13]. Spaĉek [16] extend the class of starlike functions by introducing the class of spirallike functions of type β in $\mathbb{D}$ and gave the following analytical characterization of spirallikeness functions of type β in $\mathbb{D}$.

Theorem 1.2 (Spaĉek [16])**.** *Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $f(z)$ is a spirallike function of type β in $\mathbb{D}$ if and only if*

$$\text{Re}\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{D}. \tag{3}$$

We denote the class of spirallike functions of type β in $\mathbb{D}$ by $\widetilde{S}^{\beta}$. Libera [9] unified and extended the classes $S^*(\alpha)$ and $\widetilde{S}^{\beta}$ by introducing the analytic function class $\widetilde{S}_{\alpha}^{\beta}$ in $\mathbb{D}$ as follows.

Definition 1.3 (Libera [9])**.** Let the function $f(z)$ be in the normalized analytic function class $\mathcal{A}$. Also let $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\alpha \in [0, 1)$. We say that $f \in \widetilde{S}_{\alpha}^{\beta}$ if and only if

$$\text{Re}\left(e^{i\beta}\frac{zf'(z)}{f(z)}\right) > \alpha\cos\beta \quad (z \in \mathbb{D};\ 0 \le \alpha < 1). \tag{4}$$

From Definition 1.1 and Definition 1.3, we have the following inclusions:

$$\widetilde{S}_{\alpha}^{0} = S^*(\alpha) \text{ and } \widetilde{S}_{0}^{\beta} = \widetilde{S}^{\beta}.$$

Libera [9] also proved the following coefficients bounds for the functions in the class $\widetilde{S}_{\alpha}^{\beta}$.

Theorem 1.4 (Libera [9])**.** *If the function $f \in \widetilde{S}_{\alpha}^{\beta}$ is given by* (2), *then*

$$|a_n| \le \prod_{j=0}^{n-2}\left(\frac{\left|2(1-\alpha)e^{-i\beta}\cos\beta + j\right|}{j+1}\right) \quad (n \in \mathbb{N}\setminus\{1\};\ \mathbb{N} := \{1, 2, 3, \ldots\}). \tag{5}$$

The coefficient estimates in (5) *are sharp.*

In [19], Srivastava and Owa determined a representation formula and radius of starlikeness for functions in the class $S_p(\alpha, a, b)$. They also proved distortion theorem for functions in the same subclass. Several subordination properties of spirallike functions is also investigated in the article [8]. Another interesting results about functions in some certain subclasses of spirallike functions can be found in [20] and [21].

For functions $f \in \mathcal{A}$ given by (2) and $g \in \mathcal{A}$ given by $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$, we define the Hadamard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n,\ z \in \mathbb{D}. \tag{6}$$

Let recall a general Hurwitz-Lerch zeta functions $\Phi(z,s,a)$ defined in [18] by

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{7}$$

$$\left(a \in \mathbb{C}\backslash\mathbb{Z}_0^-;\ s \in \mathbb{C},\ \text{when } |z|<1;\ \operatorname{Re}(s)>1 \text{ when } |z|=1\right),$$

where $\mathbb{Z}_0^- := \mathbb{Z}\backslash\{\mathbb{N}\}$, $(\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\})$; $\mathbb{N} := \{1, 2, 3, \ldots\}$. Choi and Srivastava [5], Ferreira and Lopez [6], Garg et al. [7], Lin and Srivastava [10], Lin et al. [11] investigate several interesting properties of this function.

Srivastava and Attiya [17] introduced and investigated the linear operator

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}.$$

This operator is defined in terms of the Hadamard product by

$$\mathcal{J}_{\mu,b}f(z) = \mathcal{G}_{b,\mu} * f(z),\ \left(z \in \mathbb{D};\ b \in \mathbb{C}\backslash\left\{\mathbb{Z}_0^-\right\};\ \mu \in \mathbb{C};\ f \in \mathcal{A}\right), \tag{8}$$

where,

$$\mathcal{G}_{b,\mu}(z) := (1+b)^{\mu}\left[\Phi(z,s,a) - b^{-\mu}\right]\ (z \in \mathbb{D}). \tag{9}$$

We recall here the following relationships (given earlier by [14], [15]) which follow easily by using (2), (8) and (9) :

$$\mathcal{J}_b^{\mu} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n. \tag{10}$$

Murugusundaramoorthy [12] generalized this operator as

$$\mathcal{J}_{\mu,b}^{m,k} f(z) = z + \sum_{n=2}^{\infty} C_n^m(b,\mu) a_n z^n, \tag{11}$$

where

$$C_n^m(b,\mu) = \left|\left(\frac{1+b}{n+b}\right)^{\mu} \frac{m!\,(n+k-2)!}{(k-2)!\,(n+m-1)!}\right|, \tag{12}$$

where $b \in \mathbb{C}\backslash\left\{\mathbb{Z}_0^-\right\}$ and $\mu \in \mathbb{C}$, $k \geq 2$ and $m > -1$. Note that $\mathcal{J}_{\mu,b}^{1,2}$ is the Srivastava-Attiya operator and $\mathcal{J}_{0,b}^{m,k}$ is the Choi-Saigo-Srivastava operator [5].

He also generalized the class of spirallike functions [12] as follows:

Definition 1.5. Let $\mathcal{G}_{b,\mu}^{m,k}(\beta,\alpha,\lambda)$ be the subclass of $\mathcal{A}$ consisting of functions of the form (2) and satisfying

$$\operatorname{Re}\left(e^{i\beta}\frac{z\left(\mathcal{J}_{\mu,b}^{m,k}f(z)\right)'}{(1-\lambda)\mathcal{J}_{\mu,b}^{m,k}f(z)+\lambda z\left(\mathcal{J}_{\mu,b}^{m,k}f(z)\right)'}\right) > \alpha\cos\beta,\ z \in \mathbb{D}, \tag{13}$$

where $0 \leq \lambda < 1$, $0 \leq \alpha < 1$, $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ and $\mathcal{J}_{\mu,b}^{m,k} f(z)$ is given by (11).

By choosing appropriate values of μ, b, m, k, we obtain the following subclasses:

Example 1.6. For $0 \leq \alpha < 1$ and if $k = 2$ and $m = 1$ with $\mu = 0$, $b = 0$, then,

$$\mathcal{G}_{0,0}^{1,2}(\beta,\alpha,\lambda) = S(\beta,\alpha,\lambda)$$
$$:= \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\beta} \frac{zf'(z)}{(1-\lambda)f(z)+\lambda zf'(z)} \right) > \alpha\cos\beta,\ |\beta| < \frac{\pi}{2},\ z \in \mathbb{D} \right\}.$$

Note that $\mathcal{G}_{0,0}^{1,2}(\beta,\alpha,0) = \widetilde{S}_{\alpha}^{\beta}$ and $\mathcal{G}_{0,0}^{1,2}(\beta,0,0) = \widetilde{S}^{\beta}$.

Example 1.7. For $0 \leq \alpha < 1$ and if $k = 2$ and $m = 1$ with $\mu = 1$, $b = 0$, then,

$$\mathcal{G}_{0,1}^{1,2}(\beta,\alpha,\lambda) = L(\beta,\alpha,\lambda)$$
$$:= \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\beta} \frac{z(\mathcal{L}f(z))'}{(1-\lambda)\mathcal{L}f(z)+\lambda z(\mathcal{L}f(z))'} \right) > \alpha\cos\beta,\ |\beta| < \frac{\pi}{2},\ z \in \mathbb{D} \right\},$$

where $\mathcal{L}$ is the Alexander integral operator [1] given by $\mathcal{L}f(z) = \mathcal{J}_{0,1}^{1,2}f(z)z + \sum_{n=2}^{\infty} \frac{a_n}{n} z^n$, $z \in \mathbb{D}$.

Example 1.8. For $0 \leq \alpha < 1$ and if $k = 2$ and $m = 1$ with $b = \vartheta$ $(\vartheta > -1)$ and $\mu = 1$, then

$$\mathcal{G}_{\vartheta,1}^{1,2}(\beta,\alpha,\lambda) = B_{\vartheta}(\beta,\alpha,\lambda)$$
$$:= \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\beta} \frac{z(\mathcal{F}_{\vartheta}f(z))'}{(1-\lambda)\mathcal{F}_{\vartheta}f(z)+\lambda z(\mathcal{F}_{\vartheta}f(z))'} \right) > \alpha\cos\beta,\ |\beta| < \frac{\pi}{2},\ z \in \mathbb{D} \right\},$$

where $\mathcal{F}_{\vartheta}f(z)$ is the Bernardi operator [2] given by $\mathcal{F}_{\vartheta}f(z) = \mathcal{J}_{\vartheta,1}^{1,2}f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+\vartheta}{n+\vartheta}\right) a_n z^n$, $z \in \mathbb{D}$.

Example 1.9. For $0 \leq \alpha < 1$ and if $k = 2$ and $m = 1$ with $b = 1$ and $\mu = \sigma$ $(\sigma > 0)$, then

$$\mathcal{G}_{1,\sigma}^{1,2}(\beta,\alpha,\lambda) = I^{\sigma}(\beta,\alpha,\lambda)$$
$$:= \left\{ f \in \mathcal{A} : \operatorname{Re}\left(e^{i\beta} \frac{z(I^{\sigma}f(z))'}{(1-\lambda)I^{\sigma}f(z)+\lambda z(I^{\sigma}f(z))'} \right) > \alpha\cos\beta,\ |\beta| < \frac{\pi}{2},\ z \in \mathbb{D} \right\},$$

where $I^{\sigma}f(z)$ is defined by $I^{\sigma}f(z) = \mathcal{J}_{1,\sigma}^{1,2}f(z) = z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^{\sigma} a_n z^n$, $z \in \mathbb{D}$.

Before giving our main results, we need one more definition which is given in the following.

Definition 1.10. Let $f(z)$ and $g(z)$ be analytic functions in $\mathbb{D}$. We say that $f(z)$ is subordinate to $g(z)$ in $\mathbb{D}$, and we denote

$$f(z) \prec g(z) \quad (z \in \mathbb{D}),$$

if there exists a Schwarz function $w(z)$ analytic in $\mathbb{D}$, with

$$w(0) = 0 \text{ and } |w(z)| < 1 \ (z \in \mathbb{D}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{D}).$$

In particular, if the function g is univalent in $\mathbb{D}$, the above subordination is equivalent to

$$f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

After the proof of the Bieberbach Conjecture [3] (which is also known as de Branges Theorem [4]), many authors were interested in other interesting subclasses of normalized analytic function class $\mathcal{A}$. In this paper, we obtain sharp coefficient bounds for functions in the class $\mathcal{G}_{b,\mu}^{m,k}(\beta,\alpha,\lambda)$ and we give a necessary and sufficient condition such that $f \in \mathcal{A}$ belongs to $\mathcal{G}_{b,\mu}^{m,k}(\beta,\alpha,\lambda)$.

2. Main results

In this section, we obtain coefficient conditions for functions in the class given by Definition 1.5. Also, we get sharp estimates for functions belong to $\mathcal{G}_{b,\mu}^{m,k}(\beta,\alpha,\lambda)$.

Theorem 2.1. *Let $\alpha \in [0,1)$ and $\beta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ and let $f(z)$ is in the form (2) such that $\mathcal{J}_{\mu,b}^{m,k} f(z) \neq 0$ for $z \in \mathbb{D}\backslash\{0\}$. Then, $f(z)$ belongs to the class $\mathcal{G}_{b,\mu}^{m,k}(\beta,\alpha,\lambda)$ if and only if*

(14)
$$\sum_{n=1}^{\infty}\left\{(n-1)(1-\lambda(\alpha+i\tan\beta))+2e^{2i\beta}-\lambda(1-\alpha)\left(1-e^{2i\beta}\right)(n-1)\right\}A_n z^n \neq 0$$
$$(z \in z \in \mathbb{D}\backslash\{0\}),$$

where

$$A_n = C_n^m(b,\mu)a_n \text{ and } C_n^m(b,\mu) = \left|\left(\frac{1+b}{n+b}\right)^{\mu}\frac{m!\,(n+k-2)!}{(k-2)!\,(n+m-1)!}\right|.$$

Proof. Let the function $f \in S$ be defined by (2). Define a function

$$h(z) = \mathcal{J}_{\mu,b}^{m,k} f(z) = z + \sum_{n=2}^{\infty} A_n z^n,\ z \in \mathbb{D}. \tag{15}$$

Consider the function

$$p(z) = \frac{e^{i\beta}\sec\beta\left(\frac{zh(z)}{(1-\lambda)h(z)+\lambda zh'(z)}\right) - i\tan\beta - \alpha}{1-\alpha}$$

is an analytic function which satisfies $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$, then $f \in \mathcal{G}_{b,\mu}^{m,k}(\beta,\alpha,\lambda)$ if and only if

$$p(z) \neq \frac{1-e^{2i\beta}}{1+e^{2i\beta}}$$

or,

$$\frac{e^{i\beta}\sec\beta zh'(z) - (\alpha+i\tan\beta)((1-\lambda)h(z)+\lambda zh'(z))}{(1-\alpha)((1-\lambda)h(z)+\lambda zh'(z))} \neq \frac{1-e^{2i\beta}}{1+e^{2i\beta}}.$$

By using the series expansion of $h(z)$ which is given by (15), we get the following

$$\left(1+e^{2i\beta}\right)\sum_{n=1}^{\infty}\left[(n-1)\left(1-\alpha\lambda-i\lambda\tan\beta\right)+(1-\alpha)\right]A_nz^n$$
$$\neq\ (1-\alpha)\left(1-e^{2i\beta}\right)\sum_{k=1}^{\infty}\left(1+(n-1)\lambda\right)A_nz^n$$

for $z\neq 0$. It is equivalent to

$$\sum_{n=1}^{\infty}\left\{(n-1)\left(1-\lambda\left(\alpha+i\tan\beta\right)\right)+2e^{2i\beta}-(1-\alpha)\left(1-e^{2i\beta}\right)(n-1)\lambda\right\}A_nz^n\neq 0,$$

which completes the proof. □

Now, we prove our coefficient estimates for functions which belong to the class $\mathcal{G}_{b,\mu}^{m,k}(\beta,\alpha,\lambda)$.

Theorem 2.2. *Let* $\alpha\in[0,1)$ *and* $\beta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ *and let* $f(z)$ *be in the form* (2) *such that* $\mathcal{J}_{\mu,b}^{m,k}f(z)\neq 0$ *for* $z\in\mathbb{D}\setminus\{0\}$. *If* $f(z)$ *belongs to the class* $\mathcal{G}_{b,\mu}^{m,k}(\beta,\alpha,\lambda)$, *then*

$$(16)\qquad |a_n|\leq\frac{(k-2)!}{m!\,(n+k-2)!\,(1-\lambda)^{n-1}}\binom{n+m-1}{n-1}\left|\left(\frac{1+b}{n+b}\right)^{\mu}\right|$$
$$\times\prod_{j=0}^{n-2}\left|j(1-\lambda)+2(1-\alpha)e^{i\beta}\cos\beta\,(1+\lambda j)\right|$$
$$\left(n\in\mathbb{N}\setminus\{1\};\ \mathbb{N}:=\{1,2,3,\ldots\}\right).$$

Proof. Since $f\in\mathcal{G}_{b,\mu}^{m,k}(\beta,\alpha,\lambda)$ there exists a Schwarz function $w(z)$, which is already introduced in Definition 1.10, such that

$$e^{i\beta}\sec\beta\left(\frac{z\left(\mathcal{J}_{\mu,b}^{m,k}f(z)\right)'}{(1-\lambda)\,\mathcal{J}_{\mu,b}^{m,k}f(z)+\lambda z\left(\mathcal{J}_{\mu,b}^{m,k}f(z)\right)'}\right)-i\tan\beta=\frac{1+(1-2\alpha)\,w(z)}{1-w(z)}.$$

Consider the function $h(z)$ defined by (15). Then, we get

$$(17)\qquad \sum_{n=2}^{\infty}\left[ne^{i\beta}\sec\beta-(1+i\tan\beta)\left(1-(n-1)\lambda\right)\right]A_nz^n$$
$$=\left(\sum_{n=1}^{\infty}\left[ne^{i\beta}\sec\beta+(1-2\alpha-i\tan\beta)\left(1+(n-1)\lambda\right)\right]A_nz^n\right)w(z).$$

The last equation (17) may be written (for $n\in\mathbb{N}$) as follows:

$$(18)\qquad \sum_{n=2}^{m}\left[ne^{i\beta}\sec\beta-(1+i\tan\beta)\left(1-(n-1)\lambda\right)\right]A_nz^n+\sum_{n=m+1}^{\infty}b_nz^n$$

$$= \left(\sum_{n=1}^{m-1} \left[n e^{i\beta} \sec\beta + (1 - 2\alpha - i\tan\beta)(1 + (n-1)\lambda) \right] A_n z^n \right) w(z).$$

The last sum on the left-hand side of (18) is convergent in $\mathbb{D}$ for $m = 2, 3, \ldots$.

Since, by hypothesis, $|w(z)| < 1$ $(z \in \mathbb{D})$, it is not difficult to find by appealing to Parseval's Theorem that

$$\sum_{n=1}^{m-1} \left| n e^{i\beta} \sec\beta (1 - 2\alpha - i\tan\beta)(1 + (n-1)\lambda) \right|^2 |A_n|^2$$
$$\geq \sum_{n=2}^{m} \left| n e^{i\beta} \sec\beta - (1 + i\tan\beta)(1 - (n-1)\lambda) \right|^2 |A_n|^2$$

or

$$\sum_{n=1}^{m-1} 4(1-\alpha)(n - \alpha(1 + (n-1)\lambda)) |A_n|^2 \geq \frac{(m-1)^2 (1-\lambda)^2}{\cos^2\beta} |A_m|^2, \tag{19}$$

where $A_1 = 1$.

We claim that

$$|A_n| \leq \frac{1}{(n-1)!(1-\lambda)^{n-1}} \prod_{j=0}^{n-2} \left| j(1-\lambda) + 2(1-\alpha)\cos\beta e^{i\beta}(1 + j\lambda) \right|. \tag{20}$$

For $n = 2$, we get from (19)

$$|A_2| \leq \frac{2(1-\alpha)\cos\beta}{1-\lambda},$$

which is equivalent to (20). (20) is obtained for larger n from inequality (19) by the principle of the mathematical induction.

Fix n, $n \geq 3$, and suppose that (16) holds for $k = 2, 3, \ldots, n-1$. Then from (19) we get the following inequality

$$|A_n|^2 \leq \frac{4(1-\alpha)\cos^2\beta}{(n-1)^2(1-\lambda)^2} \left\{ 1 - \alpha + \sum_{k=2}^{n-1} B(k, j, \alpha) \right\}, \tag{21}$$

where

$$B(k, j, \alpha) = \frac{(1+(k-1)\lambda)(k-\alpha(k-1)\lambda)}{\left((k-1)!(1-\lambda)^{k-1}\right)^2} \prod_{j=0}^{k-2} \left| j(1-\lambda) + 2(1-\alpha)\cos\beta e^{i\beta}(1 + j\lambda) \right|^2.$$

We must show that the square of the right side of (20) is equal to the right side of (21); that is

$$\frac{\prod_{j=0}^{n-2} \left| j(1-\lambda) + 2(1-\alpha)\cos\beta e^{i\beta}(1 + j\lambda) \right|^2}{\left[(n-1)!(1-\lambda)^{n-1}\right]^2} \tag{22}$$

$$= \frac{4(1-\alpha)\cos^2\beta}{(n-1)^2(1-\lambda)^2}\left\{1-\alpha+\sum_{k=2}^{n-1}B(k,j,\alpha)\right\}$$

for $n = 3, 4, \ldots$. After necessary calculations we can show that (22) is true for $n = 3$ and proves our claim for $n = 3$. Assume that (22) is valid for all k, $3 < k \le n-1$; then from (19) and (21) we obtain

$$|A_n|^2 \le \frac{4(1-\alpha)\cos^2\beta}{(n-1)^2(1-\lambda)^2}\left\{1-\alpha+\sum_{k=2}^{n-2}B(k,j,\alpha)+B(n-1,j,\alpha)\right\}$$

$$|A_n|^2 \le \frac{4(1-\alpha)\cos^2\beta}{(n-1)^2(1-\lambda)^2}\{1-\alpha$$
$$+\sum_{k=2}^{n-2}\frac{(1+(k-1)\lambda)(k-\alpha(k-1)\lambda)}{\left((k-1)!(1-\lambda)^{k-1}\right)^2}\prod_{j=0}^{k-2}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2$$
$$\left.+\frac{(1+(n-2)\lambda)(n-1-\alpha(n-2)\lambda)}{\left((n-2)!(1-\lambda)^{n-2}\right)^2}\prod_{j=0}^{n-3}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2\right\}$$

$$= \frac{\prod_{j=0}^{n-3}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2}{\left((n-2)!(1-\lambda)^{n-2}\right)^2}\left\{\frac{(n-2)^2}{(n-1)^2}\right.$$
$$\left.+4(1-\alpha)\cos^2\beta\frac{(1+(n-2)\lambda)(n-1-\alpha(n-2)\lambda)}{(n-1)^2(1-\lambda)^2}\right\}$$

$$= \frac{\prod_{j=0}^{n-3}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2}{\left((n-1)!(1-\lambda)^{n-1}\right)^2}\left\{(n-2)^2(1-\lambda)^2\right.$$
$$\left.+4(1-\alpha)\cos^2\beta(1+(n-2)\lambda)(n-1-\alpha(n-2)\lambda)\right\}$$
$$= \frac{1}{\left((n-1)!(1-\lambda)^{n-1}\right)^2}\prod_{j=0}^{n-2}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|^2.$$

From equality (11) and (12), we get the desired result. □

3. Corollaries and consequences

By choosing appropriate values of μ, b, m, k in Theorem 4 above, we obtain the corresponding results for several subclasses of S.

Corollary 3.1. *Let the function $f(z) \in \mathcal{A}$ be given by (2). If $f \in S(\beta, \alpha, \lambda)$, then*

$$|a_k| \le \frac{1}{(n-1)!(1-\lambda)^{n-1}}\prod_{j=0}^{n-2}\left|j(1-\lambda)+2(1-\alpha)\cos\beta e^{i\beta}(1+j\lambda)\right|.$$

Corollary 3.2. *Let the function $f(z) \in \mathcal{A}$ be given by* (2). *If $f \in L(\beta, \alpha, \lambda)$, then*

$$|a_k| \leq \frac{1}{n!\,(1-\lambda)^{n-1}} \prod_{j=0}^{n-2} \left| j\,(1-\lambda) + 2\,(1-\alpha)\cos\beta e^{i\beta}\,(1+j\lambda) \right|.$$

Corollary 3.3. *Let the function $f(z) \in \mathcal{A}$ be given by* (2). *If $f \in B_\vartheta(\beta, \alpha, \lambda)$, then*

$$|a_k| \leq \frac{1}{n!\,(1-\lambda)^{n-1}} \left(\frac{1+\vartheta}{n+\vartheta}\right) \prod_{j=0}^{n-2} \left| j\,(1-\lambda) + 2\,(1-\alpha)\cos\beta e^{i\beta}\,(1+j\lambda) \right|.$$

Corollary 3.4. *Let the function $f(z) \in \mathcal{A}$ be given by* (2). *If $f \in I^\sigma(\beta, \alpha, \lambda)$, then*

$$|a_k| \leq \frac{1}{n!\,(1-\lambda)^{n-1}} \left(\frac{2}{n+1}\right)^\sigma \prod_{j=0}^{n-2} \left| j\,(1-\lambda) + 2\,(1-\alpha)\cos\beta e^{i\beta}\,(1+j\lambda) \right|.$$

In addition, if $f(z) \in \mathcal{G}_{0,0}^{1,2}(\beta, \alpha, 0)$ or $f(z) \in \mathcal{G}_{0,0}^{1,2}(\beta, 0, 0)$, then we get the results in Theorem 1.2 and Theorem 1.4, respectively.

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