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MINIMAXNESS AND COFINITENESS PROPERTIES OF GENERALIZED LOCAL COHOMOLOGY WITH RESPECT TO A PAIR OF IDEALS

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Abstract. Let I and J be two ideals of a commutative Noetherian ring R and M, N be two non-zero finitely generated R-modules. Let t be a non-negative integer such that $H^i_{I,J}(N)$ is (I,J)-minimax for all $i < t$. It is shown that the generalized local cohomology module $H^i_{I,J}(M,N)$ is (I,J)-Cofinite minimax for all $i < t$. Also, we prove that the R-module $Ext^j_R(R/I, H^i_{I,J}(N))$ is finitely generated for all $i \leq t$ and $j = 0, 1$.

1. Introduction

Throughout this paper, R is a commutative Noetherian local ring and I, J are ideals of R and M, N two R-modules. As a generalization of the local cohomology modules, Nam, Tri and Dong [7] (see also [2], [11]) introduced the local cohomology modules with respect to a pair of ideals (I, J). To more precise, let

$$W(I,J) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I^n \subseteq \mathfrak{p} + J \text{ for some positive integer } n\}.$$

For an R-module M, the (I,J)-torsion submodule $\Gamma_{I,J}(M)$ of M, which consists of all elements x of M with $\operatorname{Supp}(Rx) \subseteq W(I,J)$, is considered. Let i be an integer, the generalized local cohomology functor $H^i_{I,J}(M,-)$ with respect to (I,J) is defined to be the i-th right derived functor of $\Gamma_{I,J}(\operatorname{Hom}(M,-))$. This is a generalization of the local cohomology functors $H^i_{I,J}(-)$ with respect to (I,J) and is also generalization of the generalized local cohomology functors $H^i_I(M,-)$. The i-th generalized local cohomology module of M, N with respect to (I,J) is denoted by $H^i_{I,J}(M,N)$. Henceforth, in this paper, M will denote a non-zero finitely generated R-module. Then, according to [7, Proposition 2.2], $\Gamma_{I,J}(\operatorname{Hom}(M,N)) = \operatorname{Hom}(M, \Gamma_{I,J}(N))$.

In [4], Grothedieck proposed the following conjecture:

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If I is an ideal of R and N is a finitely generated R-module, then

$$\mathrm{Hom}_R(R/I, H_I^i(N))$$

is finitely generated for all i.

One year later, Hartshorne [5] provided a counterexample to Grothendieck's conjecture. He defined an R-module T to be I-cofinite if $\mathrm{Supp}(T) \subseteq V(I)$ and $\mathrm{Ext}_R^i(R/I, T)$ is finitely generated for all i and asked:

For which rings R and ideals I are the modules $H_I^i(N)$, I-cofinite for all i and all finitely generated modules N?

The aim of the present paper is to prove some results concerning cofiniteness of generalized local cohomology modules $H_{I,J}^i(M, N)$ ($i \in \mathbb{N}_0$). More precisely, we shall show that:

Theorem 1.1. *Let M and N be two finitely generated R-modules. Let I and J be two ideals of R and t be a non-negative integer such that $H_{I,J}^i(N)$ is (I, J)-minimax for all $i < t$. Then $H_{I,J}^i(M, N)$ is (I, J)-cofinite. In particular, the Goldie dimension of $H_{I,J}^i(M, N)$ is finite.*

Recall that an R-module T is said to be (I, J)-cofinite if $\mathrm{Supp}(T) \subseteq W(I, J)$ and $\mathrm{Ext}_R^i(R/I, T)$ is finitely generated for all $i \geq 0$. Also, an R-module T is said to have finite Goldie dimension if T does not contain an infinite direct sum of non-zero submodules.

We say that an R-module T is (I, J)-minimax if the Goldie dimension of (I, J)-torsion submodule $\Gamma_{I,J}(T)$ of T is finite. One of our tools for proving Theorem 1.1 is the following:

Theorem 1.2. *Let N be a finitely generated R-module. Let t be a non-negative integer such that $H_{I,J}^i(N)$ is (I, J)-minimax for all $i < t$. Then $H_{I,J}^i(N)$ is (I, J)-cofinite for all $i < t$ and $\mathrm{Ext}_R^j(R/I, H_{I,J}^i(N))$ is finitely generated for all $i \leq t$ and $j = 0, 1$.*

We refer the reader to [1] or [10] for more details about local cohomology.

2. The results

For an R-module T, the Goldie dimension of T is defined as the cardinal of the set of indecomposable submodules of $E(T)$ which appear in a decomposition of $E(T)$ into a direct sum of indecomposable submodules.

In [13], H. Zöschinger introduced the class of minimax modules, and he has in [13] and [14] given many equivalent conditions for a module to be minimax. The R-module T is said to be minimax module if there is a finitely generated submodule T' of T such that T/T' is Artinian. On the other hand, it is known that when R is a Noetherian ring, a module is minimax if and only if each of its quotients has finite Goldie dimension, [14] or [12]. Let I be an ideal of R. An R-module T is said to be minimax with respect to I or I-minimax if the I-relative Goldie dimension of any quotient module of T is finite. Also, an R-module T is said to have finite I-relative Goldie dimension if the Goldie dimension of the I-torsion submodule $\Gamma_I(T)$ of T is finite. In addition, we say that an R-module T is I-cominimax if the support of T is contained in $V(I)$ and $\mathrm{Ext}^i_R(R/I,T)$ is I-minimax.

The definition of (I,J)-cofinite modules and I-minimax modules motivate the following definition.

Definition 2.1.

(i) An R-module T is said to be (I,J)-minimax if the (I,J)-relative Goldie dimension of any quotient module of T is finite.

(ii) An R-module T is said to be (I,J)-cominimax if support of T is contained in $W(I,J)$ and $\mathrm{Ext}^i_R(R/I,T)$ is (I,J)-minimax.

Remark 2.2. It is easy to see that:

(i) For a Noetherian ring R, the class of I-minimax R-modules contains the class of (I,J)-minimax R-modules and it contains the class of minimax R-modules.

(ii) Let T be an I-torsion module. Then, by [3, Lemma 2.6] and [8, Lemma 3.1], T is minimax if and only if it is (I,J)-minimax if and only if it is I-minimax.

(iii) Let T be an (I,J)-torsion module. Then, by the definition and [8, Lemma 3.1], T is minimax if and only if it is (I,J)-minimax.

(iv) Let T be an (I,J)-cofinite. Then T is (I,J)-cominimax.

The following lemma is needed in the proof of the main theorem of this paper.

Lemma 2.3. *Let T be an R-module. Then the following holds*:

(i) *Let T be an Artinian module. If $Hom(R/I,T)$ is finitely generated, then T is (I,J)-cofinite.*

(ii) *Let T be a minimax module with support in $W(I,J)$. If $Hom(R/I,T)$ is finitely generated, then T is (I,J)-cofinite.*

(iii) *Let $0 \longrightarrow T' \longrightarrow T \longrightarrow T'' \longrightarrow 0$ be an exact sequence of R-modules. Then T is an (I,J)-cofinite minimax module if and only if T' and T'' are both (I,J)-cofinite minimax modules. In particular, any quotient of an (I,J)-cofinite minimax module, as well as any finite direct sum of (I,J)-cofinite minimax modules, is (I,J)-cofinite minimax.*

(iv) *Let T be an (I,J)-cofinite minimax and M a finitely generated R-module. Then $Ext^i_R(M,T)$ is (I,J)-cofinite minimax for all $i \geq 0$.*

Proof. (i) Since, in view of the hypothesis, $\mathrm{Hom}(R/I,T)$ has finite length and $\mathrm{Supp}(T) \subseteq V(\mathfrak{m}) \subseteq V(I) \subseteq W(I,J)$. By using [6, Proposition 4.1], we can get $\mathrm{Ext}^i_R(R/I,T)$ is finitely generated for all $i \geq 0$, as required.

(ii) Let T_1 be a finitely generated submodule of T such that $T_2 = T/T_1$ is Artinian and suppose that $\mathrm{Hom}(R/I,T)$ is finitely generated. Then exactness of $0 \to \mathrm{Hom}_R(R/I,T_1) \to \mathrm{Hom}_R(R/I,T) \to \mathrm{Hom}_R(R/I,T_2) \to \mathrm{Ext}^1_R(R/I,T_1)$

implies that $\mathrm{Hom}(R/I, T_2)$ is finitely generated. Hence we get from (i) that T_2 is Artinian and (I, J)-cofinite, therefore T is also (I, J)-cofinite.

(iii) This follows from (ii) and [6, Corollary 4.4].

(iv) Let

$$F_* : \cdots \longrightarrow F_n \longrightarrow F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

be a minimal free resolution of M. Then $\mathrm{Ext}^i_R(M, T) = H^i(\mathrm{Hom}_R(F_*, T))$ is subquotient of a direct sum of finitely many copies of T. Therefore, it follows from (iii) that $\mathrm{Ext}^i_R(M, T)$ is (I, J)-cofinite minimax for all $i \geq 0$. □

Theorem 2.4. *Let N be a finitely generated R-module. Let t be a non-negative integer, such that $H^i_{I,J}(N)$ is (I, J)-cofinite for all $i < t$. Then $Ext^j_R(R/I, H^i_{I,J}(N))$ is finitely generated for all $i \leq t$ and $j = 0, 1$.*

Proof. Consider Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_R(R/I, H^q_{I,J}(N)) \underset{p}{\Longrightarrow} \mathrm{Ext}^{p+q}_R(R/I, N).$$

For each $r \geq 2$ and $i = 0, 1$, we consider the exact sequence

$$(1) \qquad 0 \longrightarrow \ker d_r^{i,t} \longrightarrow E_r^{i,t} \xrightarrow{d_r^{i,t}} E_r^{i+r,t-r+1}.$$

It follows from hypothesis that the R module $E_r^{i+r,t-r+1}$ is finitely generated. Note that $E_r^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $p, q \in \mathbb{N}_0$. There is an integer $\ell \geq 2$ such that $E_\infty^{i,t} = E_r^{i,t}$ for all $r \geq \ell$. Also, there is a bounded filtration

$$0 = \varphi^{t+1}H^t \subseteq \varphi^t H^t \subseteq \cdots \subseteq \varphi^1 H^t \subseteq \varphi^0 H^t = \mathrm{Ext}^t_R(R/I, N)$$

such that $E_\infty^{p,t-p} \cong \varphi^p H^t/\varphi^{p+1}H^t$ for all $p = 0, 1, \ldots, t$. Thus $E_\infty^{p,q}$ is finitely generated for all p, q. Since $E_\ell^{i,t} = (\ker d_{\ell-1}^{i,t}/\mathrm{im} d_{\ell-1}^{i-\ell+1,t+\ell-2})$ and $\mathrm{im} d_{\ell-1}^{i-\ell+1,t+\ell-2} = 0$ (for all $\ell > 2$ and $i = 0, 1$), it follows that $\ker d_{\ell-1}^{i,t}$ is finitely generated. Hence by using (1) for $r = \ell - 1$, we deduce that $E_{\ell-1}^{i,t}$ is a finitely generated R-module. By continuing this argument repeatedly for integer $\ell-1, \ell-2, \ldots, 3$ in stead of ℓ, we obtain that $E_2^{i,t}$ is a finitely generated R-module for $i = 0, 1$. This completes the proof. □

Theorem 2.5. *Let N be a finitely generated R-module. Let t be a non-negative integer, such that $H^i_{I,J}(N)$ is (I, J)-minimax for all $i < t$. Then $H^i_{I,J}(N)$ is (I, J)-cofinite for all $i < t$.*

Proof. Remark 2.2 allows us to assume that $H^i_{I,J}(N)$ is minimax for all $i < t$. Now, we prove the result by induction on i. It is clear $H^0_{I,J}(N)$ is (I, J)-cofinite. Assume that $i > 0$ and the result holds true for smaller values than i. Thus we obtain that $H^j_{I,J}(N)$ is (I, J)-cofinite minimax for all $j \leq i - 1$ by the inductive hypothesis. Also, in view of Theorem 2.4, $\mathrm{Hom}(R/I, H^i_{I,J}(N))$ is finitely generated. Therefore by Lemma 2.3(ii), $H^i_{I,J}(N)$ is (I, J)-cofinite minimax. This completes proof. □

Theorem 2.6. *Let M, N be two finitely generated R-modules. Let t be a non-negative integer such that $H^i_{I,J}(N)$ is (I,J)-minimax for all $i < t$. Then $H^i_{I,J}(M,N)$ is (I,J)-cofinite minimax for all $i < t$. In particular, the Goldie dimension of $H^i_{I,J}(M,N)$ is finite for all $i < t$.*

Proof. Let $G(-) = \Gamma_{I,J}(-)$ and $F(-) = \mathrm{Hom}(M,-)$ be functors from category of R-modules to itself. Then $FG(-) = \Gamma_{I,J}(M,-)$ and F is left exact. For any injective module E

$$R^iF(G(E)) = R^i\mathrm{Hom}_R(M, \Gamma_{I,J}(E)) = 0$$

for all $i > 0$, as $\Gamma_{I,J}(E)$ is an injective R-module. By [9, Theorem 10.47] there is a Grothendieck spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_R(M, H^q_{I,J}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{I,J}(M,N).$$

Using Theorem 2.5, $H^i_{I,J}(N)$ is (I,J)-cofinite minimax for all $i < t$, and M is a finitely generated R-module hence $E_2^{p,q}$ is (I,J)-cofinite minimax for all $p \geq 0$ and $q < t$, see Lemma 2.3(iv). Since $E_i^{p,q}$ is a subquotient of $E_2^{p,q}$ for $i \geq 2$, by Lemma 2.3(iii), we deduce that $E_i^{p,q}$ is (I,J)-cofinite minimax for all $i \geq 2$, $p \geq 0$, and $q < t$. There is a finite filtration

$$0 = \varphi^{j+1}H^j \subseteq \varphi^j H^j \subseteq \cdots \subseteq \varphi^1 H^j \subseteq \varphi^0 H^j = H^j_{I,J}(M,N)$$

such that $E_\infty^{i,j-i} \cong \varphi^i H^j/\varphi^{i+1}H^j$ for all $0 \leq i \leq j$. Since $E_i^{p,q} \cong E_\infty^{p,q}$ for i sufficiently large, we have that $E_\infty^{p,q}$ is (I,J)-cofinite minimax for all $q < t$. Hence, using the exact sequence

$$0 \longrightarrow \varphi^{i+1}H^j \longrightarrow \varphi^i H^j \longrightarrow E_\infty^{i,j-i} \longrightarrow 0 \qquad (0 \leq i \leq j)$$

we get that $H^j_{I,J}(M,N)$ is (I,J)-cofinite minimax for all $j < t$, and so

$$\mathrm{Gdim} H^i_{I,J}(M,N)$$

is finite for all $j < t$. □

The following corollary immediately follows by Theorem 2.6.

Corollary 2.7. *Let M, N be two finitely generated R-modules. Let t be a non-negative integer such that $H^i_{I,J}(N)$ is Artinian for all $i < t$. Then $H^i_{I,J}(M,N)$ is (I,J)-cofinite for all $i < t$.*

Proof. Apply Theorem 2.6 and the fact that the class of minimax modules includes all Artinian modules. □

Corollary 2.8. *Let M, N be two finitely generated R-modules. Then*

$$\inf\{i \mid H^i_{I,J}(N) \text{ is not Artinian}\} \leq \inf\{i \mid H^i_{I,J}(M,N) \text{ is not Artinian}\}.$$

Proof. In view of the spectral sequence

$$E_2^{p,q} = \mathrm{Ext}^p_R(M, H^q_{I,J}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{I,J}(M,N),$$

the result follows by similar argument as used in Theorem 2.6. □

Theorem 2.9. *Let T be an R-module such that $\mathrm{Ext}^j_R(R/I,T)$ is (I,J)-minimax for all j. If t is a non-negative integer such that $H^i_{I,J}(T)$ is (I,J)-cominimax for all $i \neq t$, then $H^t_{I,J}(T)$ is (I,J)-cominimax.*

Proof. In view of Remark 2.2 and Definition 2.1 we may assume that

$$\mathrm{Ext}^j_R(R/I,T)$$

is minimax for all j and $\mathrm{Ext}^j_R(R/I,H^i_{I,J}(T))$ is minimax for all j and $i \neq t$. It is enough for us to show that the R-module $\mathrm{Ext}^j_R(R/I,H^t_{I,J}(T))$ is minimax for all j. Using [9, Theorem 10.47] there exists a Grothendieck' spectral sequence

$$E^{p,q}_2 = \mathrm{Ext}^p_R(R/I,H^q_{I,J}(T)) \underset{p}{\Longrightarrow} \mathrm{Ext}^{p+q}_R(R/I,T).$$

Also, there is a bounded filtration

$$0 = \varphi^{n+1}H^n \subseteq \varphi^n H^n \subseteq \cdots \subseteq \varphi^1 H^n \subseteq \varphi^0 H^n = \mathrm{Ext}^n_R(R/I,T)$$

such that $E^{i,n-i}_\infty \cong \varphi^i H^n/\varphi^{i+1}H^n$ for all $0 \leq i \leq n$, and hence $E^{p,q}_\infty$ is minimax. Note that $E^{p,q}_\infty = E^{p,q}_r$ for large r and each p and q. It follows that there is an integer $\ell \geq 2$ such that $E^{p,q}_r$ is minimax for all $r \geq \ell$. We now argue by descending induction on ℓ. Now, assume that $2 < \ell < r$ and that the claim holds for ℓ. Since $E^{p,q}_r$ is a subquotient of $E^{p,q}_2$ for all $p,q \in \mathbb{N}_0$, the hypotheses give that $E^{p+r,t-r+1}_r$ is minimax for all $r \geq 2$. In addition, $E^{p,t}_\ell = (\mathrm{ker} d^{p,t}_{\ell-1}/\mathrm{im} d^{p-\ell+1,t+\ell-2}_{\ell-1})$ and $\mathrm{im} d^{p-\ell+1,t+\ell-2}_{\ell-1}$ are minimax for all $p \geq 0$. It follows that $\mathrm{ker} d^{p,t}_{\ell-1}$ is minimax for all $\ell > 2$ and $p \geq 0$. Let $r \geq 2$ and $p \geq 0$, we consider the exact sequence

$$0 \longrightarrow \mathrm{ker} d^{p,t}_r \longrightarrow E^{p,t}_r \longrightarrow E^{p+r,t-r+1}_r.$$

Since $\mathrm{ker} d^{p,t}_{\ell-1}$ and $E^{p+\ell-1,t-\ell+2}_{\ell-1}$ are minimax, it follows that $E^{p,t}_{\ell-1}$ is minimax for $p \geq 0$. This completes the inductive step. □

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