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THE MINIMAL FREE RESOLUTION OF THE UNION OF TWO LINEAR STAR-CONFIGURATIONS IN $\mathbb{P}^2$

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Abstract. In [1], the authors proved that the finite union of linear star-configurations in $\mathbb{P}^2$ has a generic Hilbert function. In this paper, we find the minimal graded free resolution of the union of two linear star-configurations in $\mathbb{P}^2$ of type $s \times t$ with $\binom{t}{2} \le s$ and $3 \le t$.

1. Introduction

Throughout the paper, let $R = \Bbbk[x_0, \ldots, x_n]$ be an $(n+1)$-variable polynomial ring over an algebraically closed field $\Bbbk$ of any characteristic, and the symbol $\mathbb{P}^n$ will denote the projective n-space over the field $\Bbbk$. Let $\mathbb{X} = \{P_1, \ldots, P_s\}$ be a set of distinct points in $\mathbb{P}^n$ $(n < s)$. Then $I = I_{\mathbb{X}}$ is the defining ideal of $\mathbb{X}$ and the ring $A = R/I$ is called the *homogeneous coordinate ring* of $\mathbb{X}$. Then the *Hilbert function* of $A = \bigoplus_{t \ge 0} R_t/I_t$ or of $\mathbb{X}$ is defined by

$$\mathbf{H}(A, t) = \mathbf{H}_{\mathbb{X}}(t) = \dim_k A_t.$$

We say that $\mathbb{X}$ has a *generic Hilbert function* if

$$\mathbf{H}_{\mathbb{X}}(t) = \min\left\{ \binom{t+n}{n}, |\ \mathbb{X}\ | \right\}$$

for all $t \ge 0$.

Since the ring $A = R/I$ has homological dimension n, R/I has a *minimal graded free resolution* $\mathbb{F}$, as an R-module, of the form:

$$\mathbb{F}: \quad 0 \to \mathbb{F}_n \to \cdots \to \mathbb{F}_i \xrightarrow{\varphi_i} \mathbb{F}_{i-1} \to \cdots \to \mathbb{F}_1 \to R \to R/I \to 0,$$

where the $\mathbb{F}_i$'s are free graded R-modules and the image of each homomorphism φ_i of free modules in the resolution lies in $(x_0, x_1, \ldots, x_n)\mathbb{F}_{i-1}$. In fact,

$$\mathbb{F}_i := \bigoplus_{t=0}^{r_i} R(-(i+1+t))^{\beta_{i,i+1+t}}.$$

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The numbers $\{\beta_{i,j}\}$ for $0 \le i \le n$ are called the i^{th} *graded Betti numbers* of the ideal I. With this resolution, note that $\mathrm{rank} F_1$ is the minimal number of generators of the ideal I. Since R/I is a Cohen-Macaulay ring, $\mathrm{rank}\, F_n$ is the *Cohen-Macaulay type* of R/I.

Recently, a star-configuration in $\mathbb{P}^n$ has been extensively studied (see [1, 2, 3, 4, 6, 8, 9, 10, 11]). In [2], the authors found the minimal graded free resolution of a star-configuration in $\mathbb{P}^n$ of codimension 2, and in [9] the authors found the minimal graded free resolution of a star-configuration in $\mathbb{P}^n$ of any codimension $2 \le r \le n$. Moreover, in [1], they proved that any finite union of linear star-configurations in $\mathbb{P}^2$ has a generic Hilbert function.

In this paper, we shall be mainly concerned with the number of minimal generators of the ideal $I_{\mathbb{X}}$ and the minimal graded free resolution of $R/I_{\mathbb{X}}$, where $\mathbb{X}$ is the union of two linear star-configurations in $\mathbb{P}^2$.

2. Star-configurations

To introduce a star-configuration, we start with a variety of some specific ideal of R.

Definition 2.1. Let $R = \Bbbk[x_0, x_1, \ldots, x_n]$ be a polynomial ring over a field $\Bbbk$. For positive integers r and s with $1 \le r \le \min\{n, s\}$, suppose $F_1, \ldots, F_s$ are general forms in R of degrees $d_1, \ldots, d_s$, respectively. We call the variety $\mathbb{X}$ defined by the ideal

$$\bigcap_{1 \le i_1 < \cdots < i_r \le s} (F_{i_1}, \ldots, F_{i_r})$$

a star-configuration in $\mathbb{P}^n$ of type (r, s). If $F_1, \ldots, F_s$ are general linear forms in R, then we call $\mathbb{X}$ *a linear star-configuration in* $\mathbb{P}^n$ *of type* (r, s).

In particular, if $r = 2$, then we simply call $\mathbb{X}$ *a star-configuration in* $\mathbb{P}^n$ *of type* s.

Theorem 2.2 ([9, Theorem 3.4])**.** *Let* $\mathbb{X}^{(r,s)}$ *be a star-configuration in* $\mathbb{P}^n$ *of type* (r, s) *defined by general forms* $F_1, \ldots, F_s$ *in* $R = \Bbbk[x_0, x_1, \ldots, x_n]$ *of degrees* $d_1, d_2, \ldots, d_s$, *where* $2 \le r \le \min\{s, n\}$, *and let* $d = d_1 + \cdots + d_s$. *Then the minimal free resolution of* $I_{\mathbb{X}^{(r,s)}}$ *is*

$$0 \to \mathbb{F}_r^{(r,s)} \to \mathbb{F}_{r-1}^{(r,s)} \to \cdots \to \mathbb{F}_1^{(r,s)} \to I_{\mathbb{X}^{(r,s)}} \to 0, \tag{2.1}$$

where

$$\begin{aligned}
\mathbb{F}_r^{(r,s)} &= R^{\alpha_r^{(r,s)}}(-d), \\
\mathbb{F}_{r-1}^{(r,s)} &= \bigoplus_{1 \le i_1 \le s} R^{\alpha_{r-1}^{(r,s)}}(-(d - d_{i_1})), \\
&\ \ \vdots \\
\mathbb{F}_\ell^{(r,s)} &= \bigoplus_{1 \le i_1 < \cdots < i_{r-\ell} \le s} R^{\alpha_\ell^{(r,s)}}(-(d - (d_{i_1} + \cdots + d_{i_{r-\ell}}))),
\end{aligned}$$

$$\vdots$$

$$\mathbb{F}_2^{(r,s)} = \bigoplus_{1\le i_1<\cdots<i_{r-2}\le s} R^{\alpha_2^{(r,s)}}(-(d-(d_{i_1}+\cdots+d_{i_{r-2}}))), \quad \textit{and}$$

$$\mathbb{F}_1^{(r,s)} = \bigoplus_{1\le i_1<\cdots<i_{r-1}\le s} R^{\alpha_1^{(r,s)}}(-(d-(d_{i_1}+\cdots+d_{i_{r-1}}))),$$

with

$$\alpha_\ell^{(r,s)} = \binom{s-r+\ell-1}{\ell-1} \quad \textit{and} \quad \operatorname{rank}\mathbb{F}_\ell^{(r,s)} = \binom{s-r+\ell-1}{\ell-1}\cdot\binom{s}{r-\ell}$$

for $1\le \ell\le r$. In particular, the last free module $\mathbb{F}_r^{(r,s)}$ has only one shift d, i.e., a star-configuration $\mathbb{X}^{(r,s)}$ in $\mathbb{P}^n$ is level. Furthermore, any star-configuration $\mathbb{X}^{(r,s)}$ in $\mathbb{P}^n$ is aCM.

3. Minimal free resolutions of the union of two linear star-configurations in $\mathbb{P}^2$ of type $t\times s$

In this section, let $\mathbb{X}^{(s,t)} := \mathbb{X}\dot\cup\mathbb{Y}$ be the disjoint union of two linear star-configurations $\mathbb{X}$ and $\mathbb{Y}$ in $\mathbb{P}^2$ of type s and t (type $s\times t$ for short), otherwise specified. Assume that $\mathbb{X}$ is defined by general forms $L_1,\dots,L_s$ and $\mathbb{Y}$ is defined by general forms $M_1,\dots,M_t$ in $R=\Bbbk[x_0,x_1,x_2]$, respectively.

For the rest of this section, we define $\mathbb{X}'$ as a linear star-configuration in $\mathbb{P}^2$ defined by linear forms $L_1,\dots,L_{s-1}$, and let $\mathbb{X}^{(s-1,t)} := \mathbb{X}'\cup\mathbb{Y}$.

Let I be a homogeneous ideal of $R=\Bbbk[x_0,x_1,\dots,x_n]$. We denote the *minimal number of generators of* I by $\nu(I)$, and if $d_1,\dots,d_\ell$ are the degrees of the minimal generators of I, then we denote by $\Delta(I)$ the multi-set $\{d_1,\dots,d_\ell\}$. This is somewhat unorthodox since some of the d_j's might be equal to each other. Furthermore, we sometimes denote by $\nu_i := \nu_i(I)$ the number of minimal generators of I in degree i.

We now introduce some known results due to Geramita and Marocia (see [5]), which we shall often use in this section.

Proposition 3.1 ([5, Proposition 1.1])**.** *Let $I := I_{\mathbb{X}}$ be an ideal of a finite set $\mathbb{X}$ of points in $\mathbb{P}^n$. If $I = I_\alpha\oplus I_{\alpha+1}\oplus\cdots$, then*

$$I = (I_\alpha, I_{\alpha+1},\dots,I_\sigma),$$

where $\alpha := \alpha(I)$ is the initial degree of the ideal I and

$$\sigma := \min\{i \mid \mathbf{H}(R/I,i) = \mathbf{H}(R/I,i-1)\}.$$

Theorem 3.2 ([1])**.** *Let $\mathbb{X}^{(s_1,\dots,s_\ell)}$ be the union of linear star-configurations in $\mathbb{P}^2$ of type $s_1,\dots,s_\ell$ with $2\le\ell$ and $2\le s_i$ for every $1\le i\le\ell$. Then $\mathbb{X}^{(s_1,\dots,s_\ell)}$ has a generic Hilbert function.*

Hence, by Theorem 3.2 and Proposition 3.1, the ideal of the union of two linear star-configurations in $\mathbb{P}^2$ of type $s\times t$ has minimal generators in one or two degrees when $3\le t\le s$.

Lemma 3.3. *Let $\mathbb{X}$ and $\mathbb{Y}$ be linear star-configurations in $\mathbb{P}^2$ of type $s \times t$ with $\binom{t}{2} = s$. Then the minimal graded free resolution of $R/I_{\mathbb{X}\dot\cup\mathbb{Y}}$ is*

$$0 \to R^s(-(s+1)) \to R^{s+1}(-s) \to R \to R/I_{\mathbb{X}\dot\cup\mathbb{Y}} \to 0.$$

Proof. By Theorem 3.2, the Hilbert function of $R/I_{\mathbb{X}\dot\cup\mathbb{Y}}$ is

$$\mathbf{H}_{\mathbb{X}\dot\cup\mathbb{Y}} \ : \ 1 \quad \binom{1+2}{2} \quad \cdots \quad \binom{(s-2)+2}{2} \quad \underset{\displaystyle \binom{s}{2}+\binom{t}{2}}{\underset{\|}{\binom{(s-1)+2}{2}}} \quad \to .$$

Note that

$$\begin{aligned}\dim_{\Bbbk}(I_{\mathbb{X}\dot\cup\mathbb{Y}})_s &= \binom{s+2}{2} - \mathbf{H}_{\mathbb{X}\dot\cup\mathbb{Y}}(s)\\ &= \binom{s+2}{2} - \deg(\mathbb{X}\,\dot\cup\,\mathbb{Y})\\ &= \binom{s+2}{2} - \binom{s+1}{2} = s+1,\end{aligned}$$

and

$$\alpha(I_{\mathbb{X}\dot\cup\mathbb{Y}}) = \sigma(I_{\mathbb{X}\dot\cup\mathbb{Y}}).$$

Hence, by Proposition 3.1, $\nu(I_{\mathbb{X}\dot\cup\mathbb{Y}}) = s+1$ and $\Delta(I_{\mathbb{X}\dot\cup\mathbb{Y}}) = \{\overbrace{s,\ldots,s}^{(s+1)\text{-times}}\}$, and so by the Hilbert-Burch Theorem, we obtain the minimal graded free resolution of $R/I_{\mathbb{X}\dot\cup\mathbb{Y}}$ as above, which completes the proof. □

Lemma 3.4. *Let $\mathbb{X}$ and $\mathbb{Y}$ be as in Lemma 3.3 with $\binom{t}{2} + 1 = s$. Then the minimal graded free resolution of $R/I_{\mathbb{X}\dot\cup\mathbb{Y}}$ is*

$$0 \to R^{s-1}(-(s+1)) \to R(-(s-1)) \oplus R^{s-1}(-s) \to R \to R/I_{\mathbb{X}\dot\cup\mathbb{Y}} \to 0.$$

Proof. By Theorem 3.2, the Hilbert function of $R/I_{\mathbb{X}\dot\cup\mathbb{Y}}$ is

$$\mathbf{H}_{\mathbb{X}\dot\cup\mathbb{Y}} \ : \ 1 \quad \binom{1+2}{2} \quad \cdots \quad \binom{(s-2)+2}{2} \quad \underset{\displaystyle \binom{s}{2}+\binom{t}{2}}{\underset{\|}{\binom{(s-1)+2}{2} - 1}} \quad \to .$$

Note that

$$\sigma(I_{\mathbb{X}\dot\cup\mathbb{Y}}) = \alpha(I_{\mathbb{X}\dot\cup\mathbb{Y}}) + 1.$$

Since $\dim_{\Bbbk}(I_{\mathbb{X}\dot\cup\mathbb{Y}})_{s-1} = 1$, that is, $\dim_{\Bbbk} R_1(I_{\mathbb{X}\dot\cup\mathbb{Y}})_{s-1} = 3$, and

$$\begin{aligned}\dim_{\Bbbk}(I_{\mathbb{X}\dot\cup\mathbb{Y}})_s &= \binom{s+2}{2} - \mathbf{H}_{\mathbb{X}\dot\cup\mathbb{Y}}(s)\\ &= \binom{s+2}{2} - \deg(\mathbb{X}\,\dot\cup\,\mathbb{Y})\end{aligned}$$

$$= \binom{s+2}{2} - \left[\binom{s+1}{2} - 1\right] = s+2,$$

we get that

$$\nu_s(I_{\mathbb{X}\dot{\cup}\mathbb{Y}}) = (s+2) - 3 = s-1.$$

Hence, by Proposition 3.1, $\nu(I_{\mathbb{X}\dot{\cup}\mathbb{Y}}) = s$ and $\Delta(I_{\mathbb{X}\dot{\cup}\mathbb{Y}}) = \{s-1, \overbrace{s,\dots,s}^{(s-1)\text{-times}}\}$, and so by the Hilbert-Burch Theorem, we obtain the minimal graded free resolution of $R/I_{\mathbb{X}\dot{\cup}\mathbb{Y}}$ as above, which completes the proof. □

Theorem 3.5. *Let $\mathbb{X}$ and $\mathbb{Y}$ be as in Lemma 3.3 with $\binom{t}{2} + \ell = s$ with $0 \le \ell$, and let $\mathbb{X}^{(s,t)} := \mathbb{X} \dot{\cup} \mathbb{Y}$. Then the minimal graded free resolution of $R/I_{\mathbb{X}^{(s,t)}}$ is*

$$0 \to R^{s-\ell}(-(s+1)) \to R^{\ell}(-(s-1)) \bigoplus R^{s-2\ell+1}(-s) \to R \to R/I_{\mathbb{X}^{(s,t)}} \to 0$$

for $0 \le \ell \le \binom{t}{2}$,

$$0 \to R^{\ell-\binom{t}{2}-1}(-s) \bigoplus R^{\binom{t}{2}}(-(s+1)) \to R^{\ell}(-(s-1)) \to R \to R/I_{\mathbb{X}^{(s,t)}} \to 0$$

for $\binom{t}{2} < \ell$.

Proof. We shall prove this by induction on ℓ.

Case 1. Let $0 \le \ell \le \binom{t}{2}$. If $\ell = 0$ or 1, then by Lemmas 3.3 and 3.4 it holds. Now suppose $1 < \ell \le \binom{t}{2}$. By Theorem 3.2, the Hilbert function of $R/I_{\mathbb{X}^{(s,t)}}$ is

$$\mathbf{H}_{\mathbb{X}^{(s,t)}} \quad : \quad 1 \quad \binom{1+2}{2} \quad \cdots \quad \binom{(s-2)+2}{2} \quad \underset{\binom{s}{2}+\binom{t}{2}}{\underset{\parallel}{\binom{(s-1)+2}{2} - \ell}} \quad \to .$$

Since $I_{\mathbb{X}^{(s,t)}}$ has ℓ-generators $G_1, \dots, G_\ell$ in degree $s-1$, for a general linear form $L := x_0$ in R, by induction on ℓ we may assume that $x_0 \mid G_1, \dots, G_{\ell-1}$, and $x_0 \nmid G_\ell$. Note that

$$\begin{aligned}\dim_{\Bbbk} \left[I_{\mathbb{X}^{(s-1,t)}}\right]_{s-2} &= \binom{(s-2)+2}{2} - \left[\binom{s-1}{2} + \binom{t}{2}\right] \\ &= \binom{(s-2)+2}{2} - \left[\binom{s-1}{2} + s - \ell\right] \\ &= \ell - 1.\end{aligned}$$

Hence $I_{\mathbb{X}^{(s-1,t)}}$ has $(\ell-1)$-generators $K_1, \dots, K_{\ell-1}$ in degree $s-2$, and by induction on ℓ,

$$((s-1) - 2(\ell-1) + 1) = (s - 2\ell + 2)$$

generators $Q_1,\dots,Q_{s-2\ell+2}$ in degree $s-1$, respectively. This implies that

$$G_\ell \in (I_{\mathbb{X}^{(s,t)}})_{s-1} \subseteq (I_{\mathbb{X}^{(s-1,t)}})_{s-1} = R_1\langle K_1,\dots,K_{\ell-1}\rangle + \langle Q_1,\dots,Q_{s-2\ell+2}\rangle$$

and so for some $a_i \in \Bbbk$ and linear forms $H_j \in R_1$

$$G_\ell = \sum_{j=1}^{\ell-1} H_j K_j + \sum_{i=1}^{s-2\ell+2} a_i Q_i.$$

Since G_ℓ vanishes on $(s-1)$-points in $\mathbb{X}^{(s,t)} - \mathbb{X}^{(s-1,t)}$, which lie on the line defined by x_0, we see that G_ℓ can be chosen to satisfy the condition $G_\ell \notin R_1\langle K_1,\dots,K_{\ell-1}\rangle$. Note that, by induction ℓ,

$$R_1\langle K_1,\dots,K_{\ell-1}\rangle = \langle F_1,\dots,F_{3(\ell-1)}\rangle.$$

So

$$R_1\langle G_1,\dots,G_{\ell-1}\rangle = x_0 R_1\langle K_1,\dots,K_{\ell-1}\rangle = \langle x_0F_1,\dots,x_0F_{3(\ell-1)}\rangle,$$

i.e.,

$$\dim_k R_1\langle G_1,\dots,G_{\ell-1}\rangle = 3(\ell-1).$$

Since $G_\ell \notin \langle F_1,\dots,F_{3(\ell-1)}\rangle$, it is obvious that

$$\{x_0F_1,\dots,x_0F_{3(\ell-1)},x_0G_\ell\}$$

is linearly independent. Assume that for some α_i and β_j in $\Bbbk$

$$\alpha_1 x_0 F_1 + \cdots + \alpha_{3(\ell-1)} x_0 F_{3(\ell-1)} + \beta_0 x_0 G_\ell = \beta_1 x_1 G_\ell + \beta_2 x_2 G_\ell.$$

Then

$$x_0 \mid (\beta_1 x_1 + \beta_2 x_2) G_\ell.$$

That is,

$$x_0 \mid (\beta_1 x_1 + \beta_2 x_2) \quad \text{or} \quad x_0 \mid G_\ell,$$

which is a contradiction. This implies that

$$\{x_0F_1,\dots,x_0F_{3(\ell-1)},x_0G_\ell,x_1G_\ell,x_2G_\ell\}$$

is also linearly independent. In other words,

$$\dim_{\Bbbk}\langle x_0F_1,\dots,x_0F_{3(\ell-1)},x_0G_\ell,x_1G_\ell,x_2G_\ell\rangle = \dim_{\Bbbk} R_1\langle G_1,\dots,G_{\ell-1},G_\ell\rangle = 3\ell.$$

Moreover, since

$$\begin{aligned}\dim_{\Bbbk}(I_{\mathbb{X}^{(s,t)}})_s &= \binom{s+2}{2} - \left[\binom{s}{2} + \binom{t}{2}\right]\\ &= \binom{s+2}{2} - \left[\binom{s}{2} + s - \ell\right]\\ &= s+\ell+1,\end{aligned}$$

we get that $\nu_s(I_{\mathbb{X}^{(s,t)}}) = (s+\ell+1) - 3\ell = s-2\ell+1$. Hence, by Proposition 3.1, $\nu(I_{\mathbb{X}^{(s,t)}}) = s-\ell+1$ and $\Delta(I_{\mathbb{X}^{(s,t)}}) = \{\underbrace{s-1,\dots,s-1}_{\ell\text{-times}}, \underbrace{s,\dots,s}_{(s-2\ell+1)\text{-times}}\}$, as we wished.

Case 2. Let $\binom{t}{2} < \ell$. If $\ell = \binom{t}{2} + 1$ and $L_s := x_0$, i.e., $\ell - 1 = \binom{t}{2}$, then

$$s - 1 = \binom{t}{2} + (\ell - 1) = \binom{t}{2} + \binom{t}{2},$$

and so by *Case 1*,

$$\Delta(I_{\mathbb{X}^{(s-1,t)}}) = \{\underbrace{s-2, \ldots, s-2}_{(\ell-1)\text{-times}}, s-1\}.$$

Note that $I_{\mathbb{X}^{(s,t)}}$ has ℓ-generators $G_1, \ldots, G_{\ell-1}, G_\ell$ in R_{s-1}, and by induction on s, we may assume that $x_0 \mid G_i$ for $1 \le i \le \ell - 1$ and $x_0 \nmid G_\ell$. Let $G_i = x_0 K_i$ for such i, where $(I_{\mathbb{X}^{(s-1,t)}})_{s-2} = \langle K_1, \ldots, K_{\ell-1}\rangle$. By induction on s, $\dim_{\Bbbk} R_1(I_{\mathbb{X}^{(s-1,t)}})_{s-2} = 3(\ell - 1)$. Moreover,

$$\begin{aligned}\dim_{\Bbbk}(I_{\mathbb{X}^{(s-1,t)}})_{s-1} &= \binom{(s-1)+2}{2} - \left[\binom{s-1}{2} + \binom{t}{2}\right]\\ &= s + \ell - 1\\ &= 3(\ell - 1) + 1,\end{aligned}$$

and hence

$$\begin{aligned}R_1(I_{\mathbb{X}^{(s-1,t)}})_{s-2} &:= \langle F_1, \ldots, F_{3(\ell-1)}\rangle \quad \text{and}\\ (I_{\mathbb{X}^{(s-1,t)}})_{s-1} &= \langle F_1, \ldots, F_{3(\ell-1)}, F_{3(\ell-1)+1}\rangle\end{aligned}$$

for some $F_i \in R_{s-1}$. Since

$$\begin{aligned}G_\ell \in (I_{\mathbb{X}^{(s,t)}})_{s-1} &\subseteq (I_{\mathbb{X}^{(s-1,t)}})_{s-1}\\ &= \langle F_1, \ldots, F_{3(\ell-1)}, F_{3(\ell-1)+1}\rangle\\ &= R_1\langle K_1, \ldots, K_{\ell-1}\rangle + \langle F_{3(\ell-1)+1}\rangle,\end{aligned}$$

we get that

$$G_\ell = \left[\sum_{i=1}^{3(\ell-1)} \alpha_i F_i\right] + \alpha_{3(\ell-1)+1} F_{3(\ell-1)+1}$$

for some $\alpha_i \in \Bbbk$. Note that

$$3(\ell - 1) - (s - 1) = \binom{t}{2} - 1 > 0$$

and G_ℓ vanishes on $(s-1)$-points in $\mathbb{X}^{(s,t)} - \mathbb{X}^{(s-1,t)}$, which lie on the line defined by x_0, and so there is a non-trivial solution $\alpha_i \in \Bbbk$ with $\alpha_{3(\ell-1)+1} = 1$. In other words,

$$x_0 G_\ell = x_0 \cdot \left[\sum\nolimits_{i=1}^{3(\ell-1)} \alpha_i F_i\right] + x_0 F_{3(\ell-1)+1},$$

and so

$$\begin{aligned}(3.1)\qquad &\langle x_0F_1,\dots,x_0F_{3(\ell-1)+1},x_0G_\ell\rangle = \langle x_0F_1,\dots,x_0F_{3(\ell-1)},x_0F_{3(\ell-1)+1}\rangle,\\ &\text{and}\\ &\langle x_0F_1,\dots,x_0F_{3(\ell-1)+1},x_0G_\ell\rangle = \langle x_0F_1,\dots,x_0F_{3(\ell-1)},x_0G_\ell\rangle.\end{aligned}$$

Thus

$$\begin{aligned}&R_1(I_{\mathbb{X}^{(s,t)}})_{s-1}\\ &= R_1\langle G_1,\dots,G_{\ell-1},G_\ell\rangle\\ &= R_1\langle G_1,\dots,G_{\ell-1}\rangle + R_1\langle G_\ell\rangle\\ &= R_1\langle x_0K_1,\dots,x_0K_{\ell-1}\rangle + R_1\langle G_\ell\rangle\\ &= x_0R_1\langle K_1,\dots,K_{\ell-1}\rangle + R_1\langle G_\ell\rangle\\ &= \langle x_0F_1,\dots,x_0F_{3(\ell-1)},x_0G_\ell,x_1G_\ell,x_2G_\ell\rangle\\ &= \langle x_0F_1,\dots,x_0F_{3(\ell-1)},x_0F_{3(\ell-1)+1},x_1G_\ell,x_2G_\ell\rangle \qquad \text{(by equation (3.1))}.\end{aligned}$$

By the same argument as in the previous case, one can show that

$$\{x_0F_1,\dots,x_0F_{3(\ell-1)},x_0F_{3(\ell-1)+1},x_1G_\ell,x_2G_\ell\}$$

is linearly independent. Hence

$$(3.2)\qquad \dim_{\Bbbk} R_1(I_{\mathbb{X}^{(s,t)}})_{s-1} = 3\ell.$$

Note that

$$\begin{aligned}\binom{s+2}{2} - 3\ell &= \binom{s}{2} + (2s+1) - 3\ell\\ &= \binom{s}{2} + (2s+1) - 3\left(\binom{t}{2}+1\right)\\ &= \binom{s}{2} + \binom{t}{2} \qquad \left(\text{since } s = 2\cdot\binom{t}{2}+1\right)\\ &= \mathbf{H}_{\mathbb{X}^{(s,t)}}(s).\end{aligned}$$

In other words,

$$\begin{aligned}&\dim_{\Bbbk}(I_{\mathbb{X}^{(s,t)}})_s = \binom{s+2}{2} - \mathbf{H}_{\mathbb{X}^{(s,t)}}(s) = 3\ell, \text{ and so, by equation (3.2),}\\ &\dim_{\Bbbk} R_1(I_{\mathbb{X}^{(s,t)}})_{s-1} = \dim_{\Bbbk}(I_{\mathbb{X}^{(s,t)}})_s = 3\ell.\end{aligned}$$

I.e., $I_{\mathbb{X}^{(s,t)}}$ has no generators in degree s. Hence by Proposition 3.1, $\nu(I_{\mathbb{X}^{(s,t)}}) = \nu_{s-1}(I_{\mathbb{X}^{(s,t)}}) = \ell$ and $\Delta(I_{\mathbb{X}^{(s,t)}})) = \{\underbrace{s-1,\dots,s-1}_{\ell\text{-times}}\}$.

Now suppose $2\cdot\binom{t}{2}+1 < s$ and $L_s := x_0$. Let $\ell := s - \binom{t}{2}$. Recall that $I_{\mathbb{X}^{(s,t)}}$ has ℓ-generators $G_1,\dots,G_\ell$ in degree $(s-1)$. Note that, by induction on s,

$$\begin{aligned}\dim_{\Bbbk}(I_{\mathbb{X}^{(s-1,t)}})_{s-1} &= \dim_{\Bbbk} R_1(I_{\mathbb{X}^{(s-1,t)}})_{s-2}\\ &= \binom{(s-1)+2}{2} - \deg(\mathbb{X}^{(s-1,t)})\\ &= \binom{(s-1)+2}{2} - \left[\binom{s-1}{2} + \binom{t}{2}\right]\\ &= 2s-1-\binom{t}{2} := \alpha\end{aligned}$$

and let $(I_{\mathbb{X}^{(s-1,t)}})_{s-1} = \langle F_1, \dots, F_\alpha\rangle$. Since

$$x_0(I_{\mathbb{X}^{(s-1,t)}})_{s-1} = x_0R_1(I_{\mathbb{X}^{(s-1,t)}})_{s-2} = R_1x_0(I_{\mathbb{X}^{(s-1,t)}})_{s-2} \subseteq R_1(I_{\mathbb{X}^{(s,t)}})_{s-1},$$

one can see that

$$2s-1-\binom{t}{2} \le \dim_{\Bbbk} R_1(I_{\mathbb{X}^{(s,t)}})_{s-1}.$$

Notice that

$$\dim_{\Bbbk} R_1(I_{\mathbb{X}^{(s,t)}})_{s-1} \le \dim_{\Bbbk}(I_{\mathbb{X}^{(s,t)}})_s = \binom{s+2}{2} - \mathbf{H}_{\mathbb{X}^{(s,t)}} = 2s+1-\binom{t}{2}.$$

Since $(I_{\mathbb{X}^{(s-1,t)}})_{s-2} = \langle K_1, \dots, K_{\ell-1}\rangle$ for some $K_i \in R_{s-2}$ and $\langle x_0K_1, \dots, x_0K_{\ell-1}\rangle \subseteq (I_{\mathbb{X}^{(s,t)}})_{s-1}$, we may assume that $x_0 \mid G_i$ for $1 \le i \le \ell-1$ and $x_0 \nmid G_\ell$. Moreover, since

$$\begin{aligned}&G_\ell \in (I_{\mathbb{X}^{(s,t)}})_{s-1} \subseteq I_{\mathbb{X}^{(s-1,t)}})_{s-1} = \langle F_1, \dots, F_\alpha\rangle, \quad \text{and}\\ &\alpha-(s-1) = s-\binom{t}{2} > 0,\end{aligned}$$

by the same argument as above, one can show that

$$\begin{aligned}&R_1(I_{\mathbb{X}^{(s,t)}})_{s-1} = \langle x_0F_1, \dots, x_0F_\alpha, x_1G_\ell, x_2G_\ell\rangle = (I_{\mathbb{X}^{(s,t)}})_s, \quad \text{and so}\\ &\dim_{\Bbbk}\left[I_{\mathbb{X}^{(s,t)}}\right]_s = \dim_{\Bbbk} R_1(I_{\mathbb{X}^{(s,t)}})_{s-1} = \alpha+2 = 2s+1-\binom{t}{2},\end{aligned}$$

i.e., $I_{\mathbb{X}^{(s,t)}}$ does not have any generators of degree s. Hence, by Proposition 3.1, $\nu(I_{\mathbb{X}^{(s,t)}}) = \nu_{s-1}(I_{\mathbb{X}^{(s,t)}}) = \ell$ and $\Delta(I_{\mathbb{X}^{(s,t)}}) = \{\underbrace{s-1, \dots, s-1}_{\ell\text{-times}}\}$, as we wished.

Therefore, by Hilbert-Burch theorem, we obtain the minimal graded free resolution of $R/I_{\mathbb{X}^{(s,t)}}$ as above. This completes the proof. □

The following corollary is immediate from Theorem 3.5 with $t = 3$.

Corollary 3.6. *Let $\mathbb{X}^{(s,3)}$ be the union of two linear star-configurations in $\mathbb{P}^2$ of type $s \times 3$ for $s = 3+\ell$ with $0 \le \ell$. Then the minimal graded free resolution of $R/I_{\mathbb{X}^{(s,3)}}$ is*

$$0 \to R^3(-(s+1)) \to R^\ell(-(s-1)) \bigoplus R^{s-2\ell+1}(-s) \to R \to R/I_{\mathbb{X}^{(s,t)}} \to 0$$

$$\begin{aligned}&\textit{for} \quad 0 \le \ell \le 3,\\ &0 \to R^{\ell-4}(-s) \bigoplus R^3(-(s+1)) \to R^{\ell}(-(s-1)) \to R \to R/I_{\mathbb{X}^{(s,t)}} \to 0\\ &\textit{for} \quad 3 < \ell.\end{aligned}$$

Example 3.7 (CoCoA)**.** Using results from CoCoA calculations, one can obtain the minimal graded resolution of $R/I_{\mathbb{X}^{(s,4)}}$ for $s = 4, 5$ as follows:

$$0 \to R(-6)^2 \to R(-4)^3 \to R \to R/I_{\mathbb{X}^{(4,4)}} \to 0, \quad \text{and}$$
$$0 \to R(-6)^3 \bigoplus R(-7) \to R(-5)^5 \to R \to R/I_{\mathbb{X}^{(5,4)}} \to 0,$$

respectively. Hence with Theorem 3.5, we obtain the following corollary as well.

Corollary 3.8. *Let $\mathbb{X}^{(s,4)}$ be the union of two linear star-configurations in $\mathbb{P}^2$ of type $s \times 4$ for $s = 4, 5$, and $s = 6 + \ell$ with $0 \le \ell$. Then the minimal graded free resolution of $R/I_{\mathbb{X}^{(s,4)}}$ is*

$$0 \to R(-6)^2 \to R(-4)^3 \to R \to R/I_{\mathbb{X}^{(4,4)}} \to 0,$$
$$0 \to R(-6)^3 \oplus R(-7) \to R(-5)^5 \to R \to R/I_{\mathbb{X}^{(5,4)}} \to 0,$$
$$0 \to R^6(-(s+1)) \to \begin{bmatrix} R^{\ell}(-(s-1)) \\ \oplus \\ R^{s-2\ell+1}(-s) \end{bmatrix} \to R \to R/I_{\mathbb{X}^{(s,4)}} \to 0 \quad \textit{for } 0 \le \ell \le 6,$$
$$0 \to \begin{bmatrix} R^{\ell-7}(-s) \\ \oplus \\ R^6(-(s+1)) \end{bmatrix} \to R^{\ell}(-(s-1)) \to R \to R/I_{\mathbb{X}^{(s,t)}} \to 0 \quad \textit{for } 6 < \ell.$$

These results then prompt the following natural question:

Question 3.9. What is the minimal graded free resolution of $R/I_{\mathbb{X}^{(s,t)}}$ for $5 \le t \le s < \binom{t}{2}$?

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