Time-Domain Analysis of Wireless Power Transfer System Behavior Based on Coupled-Mode Theory

Hyunjin Shim[1,*] · Sangwook Nam[1] · Bomson Lee[2]

Abstract

In this paper, coupled-mode theory (CMT) is used to obtain a transient solution analytically for a wireless power transfer system (WPTS) when unit energy is applied to one of two resonators. The solutions are compared with those obtained using equivalent circuit-based analysis. The time-domain CMT is accurate only when resonant coils are weakly coupled and have large quality factors, and the reason for this inaccuracy is outlined. Even though the time-domain CMT solution does not describe the WPTS behavior precisely, it is accurate enough to allow for an understanding of the mechanism of energy exchange between two resonators qualitatively. Based on the time-domain CMT solution, the critical coupling coefficient is derived and a criterion is suggested for distinguishing inductive coupling and magnetic resonance coupling of the WPTS.

Key Words: Coupling Coefficient, Coupled-Mode Theory (CMT), Power Transfer Efficiency (PTE), Quality Factor (Q), Transient Circuit Theory, Wireless Power Transmission (WPT).

I. INTRODUCTION

Wireless power transfer has long been a topic of interest in the scientific community. Kurs et al. [1] used coupled-mode theory (CMT) to analyze the characteristics of a wireless power transfer system (WPTS) using the near field. Following this, many research papers on non-radiative magnetically coupled WPTSs were published [2–5], which demonstrated the potential to deliver power more efficiently than traditional inductive systems. Also, there have been several reports on the theoretical analysis of wireless power transmission (WPT) using CMT to propose or test multi-coil inductive links, which can considerably increase the power transfer efficiency (PTE) at large coupling distances [1, 6]. Alternatively, the design and optimization of inductive power transfers have been studied from a circuit

perspective [7, 8]. However, in-depth comparison between CMT and circuit-based theory, especially in transient mode, is still lacking.

To clarify the relationship between these two theories, we analyzed the WPTS theoretically, using CMT in the time-domain analysis [9] and compared it with the result using equivalent circuit analysis [10]. We found that CMT-based transient solutions are not exactly the same as those from the circuit-based solutions in predicting energy exchanges between resonators. However, CMT-based analysis gives an approximate analytical transient solution that can be used to provide qualitative insight into the energy transfer mechanism in the WPTS.

Using a Fourier transformation of the transient CMT, the efficiency of the WPTS using either a power or voltage source at

resonant frequency can be derived easily. It matches well with the efficiency based on steady-state equivalent circuit theory.

The critical coupling coefficient is derived in the time domain with the resonator quality factor (Q). A criterion is suggested to distinguish inductive coupled and magnetic resonant coupled WPT in terms of critical distance corresponding to critical coupling.

II. THE COUPLING OF TWO RESONATOR MODES

1. Analysis of a WPTS Using CMT in the Time Domain

The structure of the proposed WPTS is illustrated in Fig. 1. The transmitting and receiving antennas consist of multi-turn spiral coils. Each coil acts as a high Q RLC tank resonator. The two coils are coupled by mutual inductance, which is a function of the geometry of the coils and the distance between them. One of the resonators transfers the energy supplied by the source to the load of the other resonator.

A time-domain solution for the energy exchange between two coupled resonators may be useful in understanding the coupling mechanism between them. Consider the equation for the mode amplitudes a_1 and a_2 of two coupled lossy resonators with natural frequencies ω_1 and ω_2, respectively [11]:

$$\frac{da_1}{dt} = \left(j\omega_1 - \frac{1}{\tau_1} - \frac{1}{\tau_{ext_1}} \right) a_1 + \kappa_{12} a_2$$

$$\frac{da_2}{dt} = \left(j\omega_2 - \frac{1}{\tau_2} - \frac{1}{\tau_{ext_2}} \right) a_2 + \kappa_{21} a_1 \tag{1}$$

The variable $a(t)$ is defined so that the energy stored in the resonator is $|a(t)|^2$ and where $1/\tau$ is its intrinsic decay rate due to absorption and radiated losses [1]. κ_{12} and κ_{21} are the coupling coefficients. Using two general solutions for Eq. (1), one finds the time-dependent solution of Eq. (1) due to unit energy input to one of two resonators.

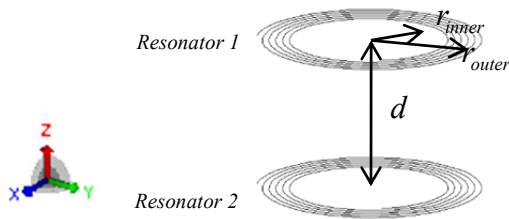### 2. Derivation of PTE in the WPTS Using CMT

Fig. 1. Structure of the proposed wireless power transfer system (WP-TS) in meters.

$$a_1(t) = \left[a_1(0)\cos\Omega t + \left(\frac{\kappa_{12}a_2(0) + a_1(0)\left(\sqrt{\Omega^2 - \kappa_{12}^2}\right)}{\Omega} \right) \sin\Omega t \right]$$

$$\cdot e^{\frac{j(\omega_1+\omega_2) - \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_{ext_1}} + \frac{1}{\tau_{ext_2}}\right)}{2} t} \tag{2}$$

$$a_2(t) = \left[a_2(0)\cos\Omega t + \left(\frac{\kappa_{21}a_1(0) - a_2(0)\left(\sqrt{\Omega^2 - \kappa_{12}^2}\right)}{\Omega} \right) \sin\Omega t \right]$$

$$\cdot e^{\frac{j(\omega_1+\omega_2) - \left(\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_{ext_1}} + \frac{1}{\tau_{ext_2}}\right)}{2} t}$$

$$where, \quad \Omega = \sqrt{\left(\frac{j(\omega_1 - \omega_2) - \left(\frac{1}{\tau_1} - \frac{1}{\tau_2} + \frac{1}{\tau_{ext_1}} - \frac{1}{\tau_{ext_2}}\right)}{2} \right)^2 + \kappa_{12}^2} \tag{3}$$

In steady state, the field amplitude inside the resonator is held constant, namely, $a(t) = Ae^{-j\omega t}$. The time-averaged extracted power can be derived using the constant field amplitude $|A|$ in the resonator and the decay rate [1]. Using the energy conservation principle, the total power is equal to the powers delivered from source to system.

$$P_{total} = P_1 + P_2 + P_{Load} = \frac{1}{\tau_1}|A_1|^2 + \frac{1}{\tau_2}|A_2|^2 + \frac{1}{\tau_{Load}}|A_2|^2 \tag{4}$$

where A_1, A_2 in Eq. (4) represent the field amplitude of each resonator. By Fourier transforming Eqs. (2) and (3), we can derive the field amplitude of each resonator at a given frequency, ω, as Eqs. (5) and (6).

$$F_{1\_voltage}(\omega) = \frac{a_1(0)}{\left(\Omega^2_{voltage} + \left(\frac{\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_{ext_1}} + \frac{1}{\tau_{ext_2}}}{2} + j(\omega - \omega_0) \right)^2 \right) \cdot \left(\frac{1}{\tau_2} + \frac{1}{\tau_{ext_2}} + j(\omega - \omega_0) \right)} \tag{5}$$

$$F_{2\_voltage}(\omega) = \frac{a_1(0)\cdot\kappa_{12}}{\Omega^2_{voltage} + \left(\frac{\frac{1}{\tau_1} + \frac{1}{\tau_2} + \frac{1}{\tau_{ext_1}} + \frac{1}{\tau_{ext_2}}}{2} + j(\omega - \omega_0) \right)^2}$$

$$where, \quad \Omega_{voltage} = \sqrt{\left(\frac{\dfrac{1}{\tau_1} - \dfrac{1}{\tau_2} + \dfrac{1}{\tau_{ext_1}} - \dfrac{1}{\tau_{ext_2}}}{2} \right)^2 + \kappa_{12}^2}$$

(6)

Using the field amplitude and decay rate of each resonator, the PTE in WPTS is

$$\eta_{voltage\_CMT} = \frac{P_{Load}}{P_{in}}$$

$$= \frac{\left| \dfrac{1}{\tau_{ext_2}} \right| \left| F_{2\_voltage}(\omega) \right|^2}{\left| \dfrac{1}{\tau_1} \right| \left| F_{1\_voltage}(\omega) \right|^2 + \left| \dfrac{1}{\tau_2} \right| \left| F_{2\_voltage}(\omega) \right|^2 + \left| \dfrac{1}{\tau_{ext_2}} \right| \left| F_{2\_voltage}(\omega) \right|^2}$$

(7)

3. Validation of the Time–Domain Solution of WPTS Using CMT

In order to show the validity of the time-domain solution of the WPTS using CMT, the equivalent circuit of the same WPTS is used as shown in Fig. 2. The total energy stored in each resonator is obtained by calculating the sum of electric energy in the capacitor and the magnetic energy in the inductor. In Figs. 3 and 4, we compare the results obtained by CMT and the equivalent circuit for two different conditions of the Q and coupling coefficient when the unit energy is input into one of the resonators at the time $t = 0$. The transient solution of energy in each resonator in the equivalent circuit is obtained using a circuit simulator with the parameters listed in Table 1 [10]. Two results match well in the case of weak coupling and high Q, as shown in Fig. 3. However, in the case of strong coupling and low Q, the transient energy variation using the equivalent circuit has a more complicated shape and a shorter period than that of CMT, as shown in Fig. 4, even though the overall response looks similar. This discrepancy is due to the resonator Qs being assumed to have fixed values in CMT despite actually being frequency-dependent. Additionally, the coupling coefficient between the two resonators is assumed to be frequency-independent in CMT, but it was found to vary according to the fre-
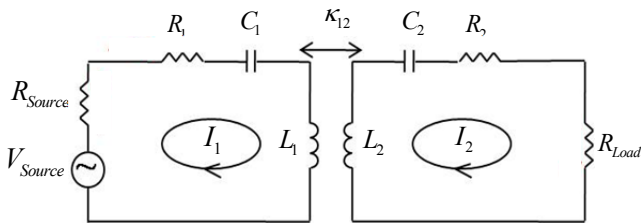

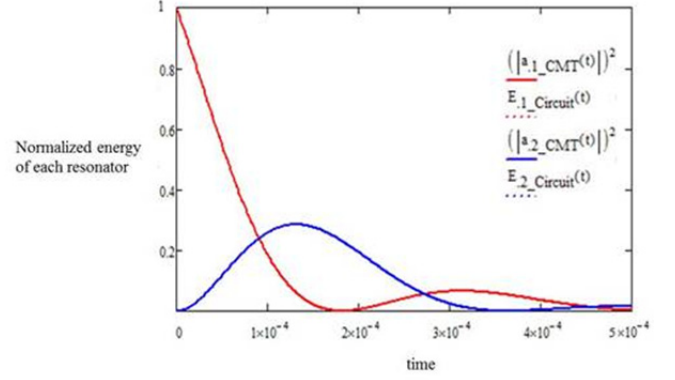Fig. 2. Equivalent circuit model of the wireless power transfer system (WPTS) structure.

Fig. 3. Normalized energy stored in each resonator as a function of time, which is calculated using coupled-mode theory (CMT) and the equivalent circuit in the case of weak coupling and high Q.
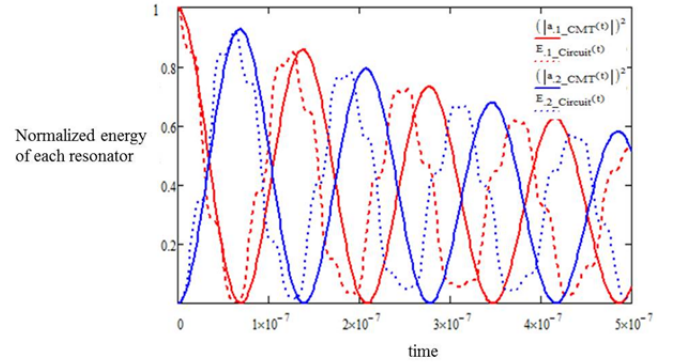

Fig. 4. Normalized energy stored in each resonator as a function of time, which is calculated using coupled-mode theory (CMT) and the equivalent circuit in the case of strong coupling and low Q.

Table 1. Specification parameters for the transient solution of energy in each resonator in the equivalent circuit [12]

Resonator specification	Weak coupling, high Q	Strong coupling, low Q
Inductance (μH)	96.2	0.962
Series resistance (Ω)	0.76	1.08
Resonant capacitance (pF)	0.812	80.2
Coupling coefficient (k_{12})	0.153×10^{-3}	0.4
Quality factor (Q)	1.4×10^4	100.74
Coupling rate (κ_{12})	8652	22.6×10^6
Source, load resistance (Ω)	0	
Resonant frequency (MHz)	18	

quency. The reason why the two solutions match well in the former case (Fig. 3) is that the effect of the Q and coupling coefficient is small in the transient behavior of the WPTS when the coupling is weak and Q is high.

The PTE derived from Fig. 2 at a given frequency, ω, using equivalent circuit theory is Eq. (8) [8].

Table 2. Values of PTE calculated by CMT and equivalent circuit model

Theory	CMT (%)	Equivalent circuit theory (%)
Weak coupling, high Q	6.60	6.60
Strong coupling, low Q	95.12	95.12

$$\eta_{voltage\_circuit} = \frac{P_{Load}}{P_{in}} = \frac{\left|I_{2\_voltage}\right|^2 \cdot R_{Load}}{\left|I_{1\_voltage}\right| \cdot V_{Source}} \cdot 100$$

$$where, \quad I_{2\_voltage} = \frac{jX_{21}}{\left(R_1 + j\omega_0 L_1 + \frac{1}{j\omega_0 C_1}\right)\left(R_2 + R_{Load} + j\omega_0 L_2 + \frac{1}{j\omega_0 C_2}\right) - (jX_{21})^2}$$

$$I_{1\_voltage} = \frac{\left(R_1 + R_{Load} + j\omega_0 L_1 + \frac{1}{j\omega_0 C_1}\right)}{jX_{21}} \cdot \left|I_{2\_voltage}\right| \tag{8}$$

As seen above, two cases are used to verify the derived PTE equations. The values of PTE calculated by CMT and equivalent circuit theory at a resonant frequency match well (see Table 2). Even though the time-domain CMT solution is somewhat inaccurate in predicting the transient energy storage in each resonator, it appears to be useful enough to allow for an understanding of the energy exchange process qualitatively between the two resonators. Also, this CMT analysis can be applied to two asymmetric resonator cases because the decay rates are defined for internal and external loss independently.

III. CRITERION FOR DISTINGUISHING INDUCTIVE COUPLED AND MAGNETIC RESONANCE COUPLED WPTS

1. Derivations of the Critical Coupling Coefficient

The critical coupling coefficient, $k_{critical}$, is defined as the coupling value at which S_{21} is at a maximum. This coupling value actually corresponds to the coupling value at the farthest distance between the two resonators where maximum power efficiency is still achievable [8].

As shown in Figs. 3 and 4, the two resonators in the WPTS exchange energy continuously, starting from the unit input energy in resonator 1. The energy goes back and forth between the two resonators until all the input energy is dissipated in the WPTS. The critical coupling coefficient, $k_{critical}$, introduced by Sample et al. [8] can be applied to the transient response of the WPTS. It is defined by specifying the maximum return energy after the first resonator transmits all the initial energy. It is found that the critical coupling condition from Sample et al. [8] corresponds to the condition that the maximum return energy at resonator 1 is approximately 5% of its initial value. Considering Eq. (1), one period of the normalized energy stored in each

resonator as a function of time is

$$1\,T(1\,period) = \frac{\pi}{\sqrt{\left(\frac{j(\omega_1 - \omega_2) - \left(\frac{1}{\tau_1} - \frac{1}{\tau_2} + \frac{1}{\tau_{ext_1}} - \frac{1}{\tau_{ext_2}}\right)}{2}\right)^2 + \kappa_{12}^2}} \tag{9}$$

Using Eq. (9), the critical coupling coefficient can be achieved when the maximum return energy after one period is 5% (i.e., e^{-3} times) of initial transmitted energy

$$\left|a_1(1T)\right|^2 = \left|a_1(0)\right|^2 \cdot e^{-3} \tag{10}$$

Using Eqs. (2) and (3), the equation of $k_{critical,CMT}$ is

$$k_{critical} = \sqrt{\left(\frac{\left(\frac{1}{Q_1} + \frac{1}{Q_2} + \frac{1}{Q_{ext_1}} + \frac{1}{Q_{ext_2}}\right) \cdot \pi}{3}\right)^2 - \left(\frac{\frac{1}{Q_{ext_1}} - \frac{1}{Q_{ext_2}}}{2}\right)^2} \tag{11}$$

where $Q_{1,2}$ and $Q_{ext1,2}$ represent unloaded and loaded quality factors of each resonator at resonant frequency.

2. Simulation Results and Discussion

Consider the WPTS shown in Fig. 1 where the transmitting and receiving resonators have an inner radius of 0.2 m and an outer radius of 0.3 m. All coils are made of 2-mm-diameter copper and spiral inward 5.7 turns. The series-connected variable capacitors are used to tune the system to 10.03 MHz.

The equivalent circuit of the WPTS shown in Fig. 2 has the circuit parameters $R_1 = R_2 = 0.296\ \Omega$, $L_1 = L_2 = 13.6\ \mu H$, $C_1 = C_2 = 18.51$ pF, $Q = 2897$ at a resonant frequency of 10.03 MHz.

The coupling coefficient, k_{12}, varies with the distance between two loop resonators. The coupling becomes weaker as the distance between the two resonators is longer. We expect there is a distance, called critical distance, $d_{critical}$, at which the coupling coefficient is $k_{critical}$. When the distance between the two resonators is shorter than the critical distance ($d < d_{critical}$), the system is said to be over-coupled, and when the distance between the two resonators is longer than the critical distance ($d > d_{critical}$), the system is under-coupled [8]. For the WPTS with the parameters shown above, $R_{Load} = 50\ \Omega$ and $R_{Source} = 0\ \Omega$. The critical distance between the two resonators is shown to be at $d_{critical} = 0.47$ m, and $k_{critical}$ is 0.048, at which point transmitting and receiving resonators exchange energy in maximum power efficiency.

Fig. 5 shows the energy stored in the transmitting resonator when k_{12} is less than, equal to, and larger than $k_{critical,CMT}$. The
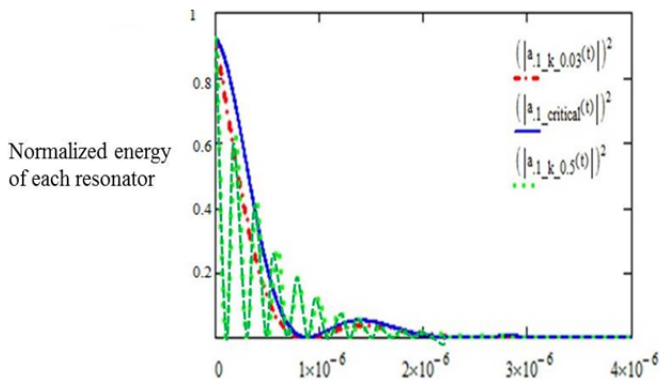
Fig. 5. Normalized energy stored in the transmitting resonator as a function of time in three cases ($k_{12} = 0.03$, $k_{12} = k_{critical}$, and $k_{12} = 0.5$).

$k_{critical,CMT}$ derived by Eq. (9) is 0.054, which is very close to $k_{critical}$. Using this derived $k_{critical,CMT}$, the maximum return energy is 5% of the initial transmitted energy.

Even though there are many articles that have mentioned inductive coupled WPTS and magnetic resonance coupled WPTS, it seems that all the WPTSs composed of two magnetic resonators (coils) have the same characteristics with respect to the distance between two resonators. They become over-coupled, critically coupled, and then under-coupled WPTSs as the distance between the two resonators becomes larger. Therefore, there seems to be no difference in the power transfer mechanism in magnetic WPTSs.

However, from a practical point of view, a WPTS may be called inductive coupled if the critical distance, $d_{critical}$, is quite short (for example, less than 1 cm), whereas it may be called magnetic resonance coupled if the critical distance, $d_{critical}$, is somewhat long (for example, longer than 5 cm).

IV. Conclusion

A WPTS is theoretically analyzed in a time-domain using CMT when unit energy is applied to one of two resonators. By Fourier transforming the transient CMT, the equation for the PTE of the WPTS at resonant frequency is derived. The CMT transient solutions are compared with the equivalent circuit-based solutions. We found that the inaccuracy of CMT in the WPTS analysis of a time-domain is due to the assumption of the frequency-independent Q and coupling coefficient of the resonators in the CMT model. Based on the time-domain CMT solution, the critical coupling coefficient is derived using the Q of the WPTS. The critical distance corresponding to critical coupling is suggested as a criterion for distinguishing inductive coupled and magnetic resonance coupled WPTSs.

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