Characterizations for Real Hypersurfaces in Complex and Quaternionic Space Forms Related to the Normal Jacobi Operator

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ABSTRACT. In this paper we give some non-existence theorems for parallel normal Jacobi operator of real hypersurfaces in real, complex and quaternionic space forms, respectively.

1. Introduction

It is known that there do not exist real hypersurfaces with parallel curvature tensor in quaternionic projective spaces. Motivated by this result, Pérez and Suh ([11]) gave a classification of real hypersurfaces in quaternionic projective spaces whose curvature tensor is parallel in the direction of certain 3-dimensional distribution

Let $(\tilde{M}, \tilde{g})$ be a Riemannian manifold. The Jacobi operator $\tilde{R}_X$ for any tangent vector field X at $x \in \tilde{M}$ defined by

$$(\tilde{R}_X Y)(x) = (\tilde{R}(Y, X)X)(x)$$

for any $Y \in T_x \tilde{M}$, becomes a self adjoint endomorphism of the tangent bundle $T\tilde{M}$ of $\tilde{M}$, where $\tilde{R}$ denotes a Riemannian curvature tensor of $(\tilde{M}, \tilde{g})$. That is, the Jacobi operator satisfies $\tilde{R}_X \in \text{End}(T_x \tilde{M})$ and is symmetric in the sense of $\tilde{g}(\tilde{R}_X Y, Z) = \tilde{g}(\tilde{R}_X Z, Y)$ for any vector fields Y and Z on $\tilde{M}$.

Related the Riemannian curvature tensor $\bar{R}$ defined on Kaehler manifold $\bar{M}$, let us consider the following notion, namely, normal Jacobi operator.

Let M be a real hypersurface in Kaehler manifold $\bar{M}$. It means that there exists only one unit normal vector to $M \subset \bar{M}$, which is denoted by $N \in T_x \bar{M}$,

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$x \in \bar{M}$. For this normal vector and the Riemannian curvature tensor $\bar{R}$ of $\bar{M}$, we obtain $\bar{R}_N \in \mathrm{End}(T_x\bar{M})$. The Jacobi operator $\bar{R}$ is said to be a nomal Jacobl operator.Since the tangent vector space T_xM at $x \in M$ is a subset of $T_x\bar{M}$ at $x \in M \subset \bar{M}$, let us consider the normal Jacobi operator $\bar{R}_N$ restrict to T_xM, that is, $\bar{R}_N X = \bar{R}(X, N)N$ for any tangent vector field X on M.

Actually, related to the commuting problem with the shape operator for real hypersurfaces M in quaternionic projective space $\mathbb{H}P^m$ or in quaternionic hyperbolic space $\mathbb{H}H^m$, Berndt ([1]) has introduced the notion of normal Jacobi operator $\bar{R}_N \in \mathrm{End}(T_xM)$, $x \in M$, where $\bar{R}$ denotes the Riemannian curvature tensor of the ambient spaces $\mathbb{H}P^m$ or $\mathbb{H}H^m$, respectively. He ([1]) also has shown that the *curvature adaptedness*, that is, the normal Jacobi operator commutes with the shape operator A, is equivalent to the fact that the distributions $\mathfrak{D}$ and $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are invariant by the shape operator A of M, where $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$. Moreover he gave a complete classification of curvature adapted real hypersurfaces in a quaternionic projective space $\mathbb{H}P^m$ and a quaternionic hyperbolic space $\mathbb{H}H^m$, respectively.

We say that the normal Jacobi operator $\bar{R}_N \in \mathrm{End}(T_xM)$ is *parallel* on M if the covariant derivative is of the normal Jacobi operator $\bar{R}_N$ identically vanishes, that is, $\nabla_X \bar{R}_N = 0$ for any tangent vector field X on M. It means that the eigenspaces of the normal Jacobi operator are parallel along any curve γ in M. Here the eigenspaces of the normal Jacobi operator $\bar{R}_N$ are said to be parallel along any curve γ if they are invariant with respect to any parallel displacement along γ.

Related to this notion, there are some results for normal Jacobi operator $\bar{R}_N$ defined on real hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$. It is known that $G_2(\mathbb{C}^{m+2})$ which consists of all complex two dimensional linear subspaces in $\mathbb{C}^{m+2}$ becomes the unique compact irreducible Riemannian symmetric space equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ (see Berndt and Suh [2]). In [4], Jeong, Kim and Suh have proved a non-existence theorem for Hopf hypersurfaces in complex two-plane Grassmannians $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator as follows:

Theorem A. *There does not exist any connected Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with parallel normal Jacobi operator.*

Moreover, as the weaker condition for parallel normal Jacobi operator some geometer studied various notion, namely, $\mathfrak{D}$-parallel, $\mathfrak{F}$-parallel, Codazzi type, semi-parallel and so on ([3], [5], [7], [8] and [10]).

Related to such a parallelism of the normal Jacobi operator, in subsection 2.2 we prove a non-existence theorem for real hypersurfaces in non-flat complex space forms as follows:

Theorem 1.1. *There does not exist any real hypersurface in non-flat complex space forms with parallel normal Jacobi operator.*

We will give a complete proof of Theorem 1.1 in subsection 2.2.

Moreover, we consider real hypersurfaces in non-flat quaternionic space forms with parallel normal Jacobi operator in subsection 2.3. Therefore, we assert the

following:

Theorem 1.2. *There does not exist any real hypersurface in a quaternionic projective space $\mathbb{H}P^m$, $m \geq 2$, with parallel normal Jacobi operator.*

Theorem 1.3. *There does not exist any real hypersurface in a quaternionic hyperbolic space $\mathbb{H}H^m$, $m \geq 2$, with parallel normal Jacobi operator and constant principal curvatures.*

In subsection 2.3 we will also give a complete proof of Theorem 1.2 and 1.3.

2. Parallelism of Normal Jacobi Operator for Real Hypersurfaces in Space Forms

In this section we want to derive the parallel normal Jacobi operator from the curvature tensor $\bar{R}(X, Y)Z$ of real, complex and quaternionic space forms, respectively.

2.1 Parallelism of Normal Jacobi Operator for Real Hypersurfaces in Real Space Forms

In this subsection we consider the notion of parallel normal Jacobi operator for real hypersurfaces in real space forms.

The Riemannian curvature tensor $\bar{R}$ of a real space form $\mathbb{R}M^m(c)$ with constant sectional curvature c is of the form

$$\bar{R}(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}$$

for any vector fields X, Y and Z on $\mathbb{R}M^m(c)$. Then we consider a real hypersurface M in real space form with parallel normal Jacobi operator $\bar{R}_N$, that is, $\nabla_X \bar{R}_N = 0$ for any vector field X on M. Then first of all, the normal Jacobi operator $\bar{R}_N$ is defined by

$$\bar{R}_N(X) = \bar{R}(X, N)N = cX.$$

Now let us consider a covariant derivative of the normal Jacobi operator $\bar{R}_N$ along the direction X. Then it is given by

$$\begin{aligned}
(\nabla_X \bar{R}_N)Y &= \nabla_X(\bar{R}_N Y) - \bar{R}_N \nabla_X Y \\
&= (Xc)Y = 0.
\end{aligned}$$

Remark. The normal Jacobi operator for every hypersurfaces in real space forms satisfy parallel normal Jacobi operator.

2.2 Parallelism of Normal Jacobi Operator for Real Hypersurfaces in Complex Space Forms

Let M be a real hypresurface in a complex m-dimensional complex space form $\mathbb{C}M^m(c)$, $c \neq 0$, $m \geq 3$, and let N be a unit normal vector field on a neighborhood of a point p in M. Let us denote by J the almost complex structure of $\mathbb{C}M^m(c)$. For

any local vector field X on a neighborhood of a point p in M, the transformation of X and N under J can be given by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, while η and ξ denote a 1-form and a tangent vector field on a neighborhood of a point p in M, respectively. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. They satisfy the following

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Moreover, the Reeb vector field ξ is said to be *Hopf* if it is invariant by the shape operator A. The 1-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is a *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the *Hopf foliation* of M is totally geodesic. Let ∇ be the Levi-Civita connection of M. The covariant derivative of the structure vector is given by

$$\nabla_X \xi = \phi A X.$$

Real hypersurfaces with constant holomorphic sectional curvature of complex space forms $\mathbb{C}M^m(c)$, $c \neq 0$, $m \geq 3$, have been classified by Kimura in [6] when $c > 0$, i.e., in the complex projective space $\mathbb{C}P^m(c)$, and by Pérez and Ortega [9], and Sohn and Suh ([12]) when $c < 0$, i.e., in the complex hyperbolic space $\mathbb{C}H^m(c)$. The Riemannian curvature tensor $\bar{R}$ of complex space form $\mathbb{C}M^m(c)$ of constant holomorphic sectional curvature c is given by

(2.2.3) $$\bar{R}(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX$$
$$- g(JX,Z)JY - 2g(JX,Y)JZ\}.$$

By putting $Y = Z = N$ in (2.2.3), we write the normal Jacobi operator $\bar{R}_N$, which is given by

$$\bar{R}_N(X) = \bar{R}(X,N)N$$
$$= \frac{c}{4}\{X + 3\eta(X)\xi\}.$$

Now let us consider a covariant derivative of the normal Jacobi operator $\bar{R}_N$ along the direction X. Then it is given by

$$(\nabla_X \bar{R}_N)Y = \nabla_X(\bar{R}_N Y) - \bar{R}_N \nabla_X Y$$
$$= \frac{3c}{4}\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\}.$$

Let us consider a real hypersurface M in complex space forms $\mathbb{C}M^m(c)$, $c \neq 0$, $m \geq 2$, with parallel normal Jacobi operator $\bar{R}_N$, that is, $(\nabla_X \bar{R}_N)Y = 0$ for any vector fields X and Y on M. Then it is given by

(2.2.4) $$g(\phi AX, Y)\xi + \eta(Y)\phi AX = 0.$$

Lemma 2.1. *Let M be a real hypersurface in complex space forms $\mathbb{C}M^m(c)$, $c \neq 0$, $m \geq 3$, with parallel normal Jacobi operator. Then the Reeb vector ξ becomes a Hopf vector.*

proof. By putting $X = Y = \xi$ into (2.2.4), we obtain $\phi A\xi = 0$. From this, if we apply the structure tensor ϕ, we have

$$0 = \phi^2 A\xi = -A\xi + \eta(A\xi)\xi.$$

We obtain $A\xi = \eta(A\xi)\xi$. This means that a real hypersurface M satisfying parallel normal Jacobi operator becomes a Hopf hypersurface. $\square$

From (2.2.4), if we apply the structure tensor ϕ, we have

$$0 = \eta(Y)\phi^2 AX = \eta(Y)(-AX + \eta(AX)\xi).$$

From this, by putting $Y = \xi$, we have

$$0 = \eta(\xi)(-AX + \eta(AX)\xi)$$
$$= -AX + \eta(AX)\xi.$$

We obtain $AX = \eta(AX)\xi$. And by using the result of Lemma 2.1, that is, $A\xi = \alpha\xi$, where $\alpha = \eta(A\xi) = g(A\xi, \xi)$, we have $AX = \alpha\eta(X)\xi$. And if $X = \xi$, then $AX = \alpha\xi$ and if $X \perp \xi$, then $AX = 0$. This means that the rank of A is less than 2.

On the other hand, in order to prove Theorem 1 in the introduction we introduce the following result ([13]):

Let M be a real hypersurface of $\mathbb{C}M^m(c)$, $c \neq 0$, $m \geq 3$, which satisfies $t(p) \leq 2$ for all $p \in M$. Then M is ruled.

Here, if p is a point of M, the rank of A at p is called the type number of M at p, and it will be denoted by $t(p)$. Thus by using the above theorem we know that M is ruled. This means that the structure vector ξ can not become an eigenvector (see also [13]). But by the result of Lemma 2.1, it makes a contradiction. This gives a complete proof of Theorem 1.1 in the introduction.

2.3 Parallelism of Normal Jacobi Operator for Real Hypersurfaces in Quaternionic Space Forms

In this subsection, we want to investigate real hypersurfaces in non-flat quaternionic space forms with parallel normal Jacobi operator. Let N be a unit local normal vector field on a real hypersurface M of quaternionic space form and $\xi_\nu = J_\nu N$, $\nu = 1, 2, 3$, be the structure vector fields on M, where $\{J_\nu\}_{\nu=1,2,3}$ is a canonical local basis of the quaternionic Kähler structure $\mathfrak{J}$ of quaternionic space form. The vector field $J_\nu X$ can be decomposed as $J_\nu X = \phi_\nu X + \eta_\nu(X)N$ for any tangent vector X of M in quaternionic space form.

Now let us define a distribution $\mathfrak{D}$ by $\mathfrak{D}(x) = \{X \in T_x M : X \perp \xi_\nu(x), \nu = 1, 2, 3\}$, $x \in M$, of a real hypersurface M in quaternionic space forms, which is orthogonal to the structure vector fields ξ_1, ξ_2, ξ_3 and is invariant with respect

to the structure tensors ϕ_1, ϕ_2, ϕ_3 and by $\mathfrak{D}^\perp = \text{Span}\{\xi_1, \xi_2, \xi_3\}$ its orthogonal complement in TM. A canonical local basis $\{J_\nu\}_{\nu=1,2,3}$ of quaternionic Kähler structure $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo 3. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\bar{\nabla}$ of quaternionic space forms for a canonical local basis $\{J_\nu\}_{\nu=1,2,3}$ of $\mathfrak{J}$ there exist three one-forms $\{q_\nu\}_{\nu=1,2,3}$ such that $\bar{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2}$ for all vector fields X on quaternionic space form. The quaternionic Kähler structure $\mathfrak{J}$ induces an almost contact 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1, 2, 3$ on M. Also, the following identities can be proved in a straightforward method.

$$\phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2},$$
$$\phi\xi_\nu = \phi_\nu\xi, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X),$$
$$\phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu,$$
$$\phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}.$$

The tensors $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ satisfy the following

$$\phi_\nu{}^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \phi_\nu\xi_\nu = 0$$
$$\eta_\nu(\phi_\nu X) = 0, \quad \eta_\nu(\xi_\nu) = 1.$$

Now we want to derive the parallel normal Jacobi operator from the curvature tensor $\bar{R}(X, Y)Z$ of non-flat quaternionic space forms.

A quaternionic space form with constant quaternionic sectional curvature $c \in \mathbb{R}$ is a connected quaternionic Kähler manifold $\bar{M}$ with the property that the Riemannian sectional curvature is equal to c for all tangent 2-planes span $\{X, JX\}$ with any unit tangent vector $X \in T_p\bar{M}$, $J \in \mathfrak{J}_p$, $p \in \bar{M}$. The standard models of quaternionic space forms are the quaternionic projective space $\mathbb{H}P^m(c)$ $(c > 0)$, the quaternionic space $\mathbb{H}^m$ $(c = 0)$ and the quaternionic hyperbolic space $\mathbb{H}H^m(c)$ $(c < 0)$. The Riemannian curvature tensor $\bar{R}$ of a quaternionic space form $\mathbb{H}M^m(c)$ with constant quaternionic sectional curvature c is of the form

$$\bar{R}(X,Y)Z = \frac{c}{4}\Big[g(Y,Z)X - g(X,Z)Y$$
$$+ \sum_{\nu=1}^{3}\{g(J_\nu Y, Z)J_\nu X - g(J_\nu X, Z)J_\nu Y - 2g(J_\nu X, Y)J_\nu Z\}\Big]$$

for every canonical local basis $\{J_1, J_2, J_3\}$ of a quaternionic Kähler structure $\mathfrak{J}$. Then first of all, we obtain the normal Jacobi operator $\bar{R}_N$ from the curvature tensor $\bar{R}(X, Y)Z$ of non-flat quaternionic space forms $\mathbb{H}M^m(c)$, which is given by

$$\bar{R}_N(X) = \bar{R}(X, N)N$$
$$= \frac{c}{4}\{X + 3\sum_{i=1}^{3}\eta_i(X)\xi_i\}.$$

Now let us consider a real hypersurface M in non-flat quaternionic space forms $\mathbb{H}M^m(c)$ with parallel normal Jacobi operator $\bar{R}_N$, that is, $\nabla_X \bar{R}_N = 0$ for any vector field X on M. Then it is given by

$$(2.3.1) \qquad (\nabla_X \bar{R}_N)Y = \frac{3c}{4} \Bigg\{ \sum_{i=1}^{3} g(q_{i+2}(X)\xi_{i+1} - q_{i+1}(X)\xi_{i+2} + \phi_i AX, Y)\xi_i$$
$$+ \sum_{i=1}^{3} \eta_i(Y)(q_{i+2}(X)\xi_{i+1} - q_{i+1}(X)\xi_{i+2} + \phi_i AX) \Bigg\}.$$

From (2.3.1) we know that a real hypersurface M in non-flat quaternionic space forms $\mathbb{H}M^m(c)$ with parallel normal Jacobi operator $\bar{R}_N$ satisfies the following

$$(2.3.2) \qquad 0 = \sum_{i=1}^{3} g(q_{i+2}(X)\xi_{i+1} - q_{i+1}(X)\xi_{i+2} + \phi_i AX, Y)\xi_i$$
$$+ \sum_{i=1}^{3} \eta_i(Y)(q_{i+2}(X)\xi_{i+1} - q_{i+1}(X)\xi_{i+2} + \phi_i AX)$$

for any vector fields X and Y on M in $\mathbb{H}M^m(c)$.

Lemma 2.2. *Let M be a real hypersurface in non-flat quaternionic space forms with parallel normal Jacobi operator. Then $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$.*

Proof. By putting $Y = \xi_1$ into (2.3.2), we obtain

$$(2.3.3) \qquad\qquad g(AX, \xi_3)\xi_2 - g(AX, \xi_2)\xi_3 + \phi_1 AX = 0.$$

From this, if we apply the structure tensor ϕ_1, we have

$$0 = g(AX, \xi_3)\xi_3 + g(AX, \xi_2)\xi_2 - AX + \eta_1(AX)\xi_1.$$

We obtain $AX = \sum_{i=1}^{3} g(AX, \xi_i)\xi_i$. This means that $AX \in \mathfrak{D}^{\perp}$ for any tangent vector field X on M. From this, taking an inner product with any $Y \in \mathfrak{D}$, we have $g(AX, Y) = 0$. So we get $g(AX, Y) = 0$ for any vectors $X \in \mathfrak{D}^{\perp}$ and $Y \in \mathfrak{D}$, that is, $g(A\mathfrak{D}, \mathfrak{D}^{\perp}) = 0$. This gives a complete proof of our Lemma. $\qquad\square$

We recall next Theorems due to Berndt ([1]) which will be used in the proof of our theorems.

Theorem 2.3. *Let M be a connected curvature-adapted real hypersurface in $\mathbb{H}P^m$, $m \geq 2$. Then M is congruent to an open part of one of the following real hypersurfaces in $\mathbb{H}P^m$:*

(A) *a tube of some radius r, $0 < r < \dfrac{\pi}{2}$ around the canonically (totally geodesic) embedded quaternionic projective space $\mathbb{H}P^k$ for some $k \in \{0, 1, \cdots, m-1\}$.*

(B) *a tube of some radius r, $0 < r < \dfrac{\pi}{4}$ around the canonically (totally geodesic) embedded complex projective space $\mathbb{C}P^m$.*

Conversely, each of these model spaces is curvature-adapted in $\mathbb{H}P^m$.

Theorem 2.4 *Let M be a connected curvature-adapted real hypersurface in $\mathbb{H}H^m$, $m \geq 2$, with constant principal curvatures. Then M is congruent to an open part of one of the following real hypersurfaces in $\mathbb{H}H^m$:*

(A_0) *a horosphere in $\mathbb{H}H^m$.*

  (A) *a tube of some radius $r \in \mathbb{R}^+$ around the canonically (totally geodesic) embedded quaternionic hyperbolic space $\mathbb{H}H^k$ for some $k \in \{0, 1, \cdots, m-1\}$.*

  (B) *a tube of some radius $r \in \mathbb{R}^+$ around the canonically (totally geodesic) emheobedded complex hyperbolic space $\mathbb{C}H^m$.*

Conversely, each of these model spaces is curvature-adapted in $\mathbb{H}H^m$ and its principal curvatures are constant.

Taking $Y = \xi_1$ in (2.3.2) we have

(2.3.4) $$g(AX, \xi_3)\xi_2 - g(AX, \xi_2)\xi_3 + \phi_1 AX = 0.$$

Now first let us consider for real hypersurfaces in quaternionic projective space $\mathbb{H}P^m$. From Lemma 2.2 and together with Theorem 2.3 we know that any real hypersurfaces in quaternionic projective space $\mathbb{H}P^m$ with parallel normal Jacobi operator are congruent to real hypersurfaces of type (A) or of type (B). So we check for these cases whether it satisfies the parallel normal Jacobi operator or not, respectively.

Table 1: The principal curvatures of model spaces and their multiplicities for a real hypersurface in $\mathbb{H}P^m$

	(A)	(B)
λ_1	$\cot(r)$	$\cot(r)$
λ_2	$-\tan(r)$	$-\tan(r)$
α_1	$2\cot(2r)$	$2\cot(2r)$
α_2	-	$-2\tan(2r)$
$m(\lambda_1)$	$4(m-k-1)$	$2(m-1)$
$m(\lambda_2)$	$4k$	$2(m-1)$
$m(\alpha_1)$	3	1
$m(\alpha_2)$	-	2

Check I-1. Type (A) : $T_x M = T_{\alpha_1} \oplus T_{\lambda_1} \oplus T_{\lambda_2}$, $x \in M$.

By taking a unit tangent vector $X \in T_{\lambda_1}$ in (2.3.4), we get $0 = \lambda_1 \phi_1 X$. And since $\lambda_1 = \cot(r)$, $0 < r < \frac{\pi}{2}$, we have $\phi_1 X = 0$. So this makes a contradiction.

Check I-2. Type (B) : $T_x M = T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\lambda_1} \oplus T_{\lambda_2}$, $x \in M$.

By using a unit tangent vector $X \in T_{\lambda_1}$ in (2.3.4), we have $0 = \lambda_1 \phi_1 X$. For the reason of $\lambda_1 = \cot(r)$, $0 < r < \frac{\pi}{4}$, we obtain $\phi_1 X = 0$. Thus this makes a contradiction.

Thus it can be easily checked that the normal Jacobi operator $\bar{R}_N$ for any hypersurfaces of type (A) or of type (B) in Theorem 2.3 does not satisfy parallelism. From this, we complete the proof of our Theorem 1.2 in the introduction.

Next, from Lemma 2.2. and together with Theorem 2.4, we know that any real hypersurfaces in quaternionic hyperbolic space $\mathbb{H}H^m$ with constant principal curvatures and parallel normal Jacobi operator are congruent to real hypersurfaces of types (A_0), (A) or of type (B). So we can check for these cases whether it satisfies the parallel normal Jacobi operator or not, respectively.

Table 2: The principal curvatures of model spaces and their multiplicities for a real hypersurface in $\mathbb{H}H^m$

	(A_0)	(A)	(B)
λ_1	1	$\coth(r)$	$\coth(r)$
λ_2	-	$\tanh(r)$	$\tanh(r)$
α_1	2	$2\coth(2r)$	$2\coth(2r)$
α_2	-	-	$-2\tanh(2r)$
$m(\lambda_1)$	$4(m-1)$	$4(m-k-1)$	$2(m-1)$
$m(\lambda_2)$	-	$4k$	$2(m-1)$
$m(\alpha_1)$	3	3	1
$m(\alpha_2)$	-	-	2

Check II-1. Type (A_0) : $T_x M = T_{\alpha_1} \oplus T_{\lambda_1}$, $x \in M$.

By taking a unit tangent vector $X \in T_{\lambda_1}$ in (2.3.4), we get $\lambda_1 \phi_1 X = 0$. And since $\lambda_1 = 1$, we have $\phi_1 X = 0$. So this makes a contradiction.

Check II-2. Type (A) : $T_x M = T_{\alpha_1} \oplus T_{\lambda_1} \oplus T_{\lambda_2}$, $x \in M$.

By using a unit tangent vector $X \in T_{\lambda_1}$ in (2.3.4), we have $\lambda_1 \phi_1 X = 0$. For the reason of $\lambda_1 = \coth(r)$, $r \in \mathbb{R}^+$, we obtain $\phi_1 X = 0$. Thus this makes a contradiction.

Check II-3. Type (B) : $T_x M = T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\lambda_1} \oplus T_{\lambda_2}$, $x \in M$.

By taking a unit tangent vector $X \in T_{\lambda_1}$ in (2.3.4), we obtain $\lambda_1 \phi_1 X = 0$. According to $\lambda_1 = \coth(r)$, $r \in \mathbb{R}^+$, we get $\phi_1 X = 0$. So this causes a contradiction.

Thus it can be easily checked that the normal Jacobi operator $\bar{R}_N$ for any hypersurfaces of types (A_0), (A) or of type (B) in Theorem 2.4 can not be parallel. From this, we complete the proof of our Theorem 1.3 in the introduction.

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