Contact CR-Warped product Submanifolds in Cosymplectic Manifolds

Mehmet Atçeken

Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpasa University, Tokat, Turkey

e-mail : mehmet.atceken382@gmail.com

ABSTRACT. The aim of this paper is to study the geometry of contact CR-warped product submanifolds in a cosymplectic manifold. We search several fundamental properties of contact CR-warped product submanifolds in a cosymplectic manifold. We also give necessary and sufficient conditions for a submanifold in a cosymplectic manifold to be contact CR-(warped) product submanifold. After then we establish a general inequality between the warping function and the second fundamental for a contact CR-warped product submanifold in a cosymplectic manifold and consider contact CR-warped product submanifold in a cosymplectic manifold which satisfy the equality case of the inequality and some new results are obtained.

1. Introduction

It is well known that the notations of warped product are widely used in differential geometry as well as physics. The study of warped product manifolds was initiated by R.L. Bishop and B. O'Neill with differential geometric point of view[9]. After then several papers appeared which have dealt with various geometric aspects of warped product submanifolds[references and their references].

CR-warped product was first introduced by B-Y. Chen. Recently, he studied warped product CR-submanifolds in Kaehler manifolds and shown that there exist no warped product CR-submanifolds in the form $M_\perp \times_f M_T$ in Kaehler manifolds. Therefore he considered warped product CR-submanifolds in the form $M_T \times_f M_\perp$ called CR-warped product by reversing factor manifolds. He established a relationship between the warping function f and the second fundamental form of CR-warped product submanifold in Kaehler manifolds[4, 3].

I. Hasegawa and I. Mihai obtained a similarly inequality for the squared norm

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of the second fundamental form in terms of the warping function for contact CR-warped products in Sasakian manifolds and some applications were derived[10].

In [5], Authors studied contact CR-warped product submanifolds in Kenmotsu space forms.

The notion of a contact CR-(warped) product submanifolds of cosymplectic manifolds have not been to be widely used in the literature and in fact that papers directly related to the problem are scarce so far. So I would like to study the geometry of contact CR-warped product submanifolds in a cosymplectic manifold.

In this paper, we consider contact CR-warped product submanifolds which are in the form $M = M_T \times_f M_\perp$ in a cosymplectic manifold $\bar{M}$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively. We obtain a sharp estimation for the squared norm of the second fundamental form and the warping function for a contact CR-warped product submanifold of cosymplectic manifold $\bar{M}$. We research necessary and sufficient conditions that inequality case to be equality case and we derive results that product manifolds to be totally geodesic, totally umbilical, minimal and real space form.

2. Preliminaries

Let $\bar{M}$ be $2m + 1$-dimensional almost contact manifold with an almost contact structure (φ, ξ, η), i.e., ξ is a global vector field, φ is a $(1,1)$-type tensor field and η is a 1-form on $\bar{M}$ such that

$$(2.2) \qquad \varphi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta o \varphi = 0,$$

for any $X \in \Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ denotes the set differentiable vector fields on $\bar{M}$.

The almost contact manifold is called an almost contact metric manifold if there exists a Riemannian metric g satisfying;

$$(2.3) \qquad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any $X, Y \in \Gamma(T\bar{M})$. Clearly, in this case, η is dual of ξ, i.e., $\eta(X) = g(X, \xi)$, for any $X \in \Gamma(T\bar{M})$.

The fundamental 2-form Ψ on $\bar{M}$ is defined as $\Psi(X, Y) = g(X, \varphi Y)$, for any $X, Y \in \Gamma(T\bar{M})$. The $\bar{M}$ is called an almost cosymplectic manifold if η and Ψ are closed, i.e., $d\eta = 0$ and $d\Psi = 0$, where d is the exterior differential operator. Also, an almost contact metric manifold is called normal if $N_\varphi + d\eta \otimes \xi = 0$, where N_φ is the Nijenhuis tensor field which is defined by $N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$. If $\bar{M}$ is almost cosymplectic and normal, $\bar{M}$ is said to be cosymplectic manifold. It is well known that an almost contact metric manifold is cosymplectic if and only if $\bar{\nabla}\varphi = 0$, where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$. These manifolds are locally a product of a Kaehler manifold and a real line or a circle[8].

If a cosymplectic manifold $\bar{M}$ has constant φ-sectional curvature, then it is called a cosymplectic space form and denoted by $\bar{M}(c)$. Then the Riemannian

curvature tensor $\bar{R}$ of $\bar{M}(c)$ is given by

$$
\begin{aligned}
\bar{R}(X,Y)Z &= \frac{c}{4}\{g(\varphi Y, \varphi Z)X - g(\varphi X, \varphi Z)Y + \eta(Y)g(X,Z)\xi \\
&\quad - \eta(X)g(Y,Z)\xi + g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y + 2g(X, \varphi Y)\varphi Z\},
\end{aligned}
$$

(2.4)

for any $X, Y, Z \in \Gamma(T\bar{M})$[8].

Now, let $\bar{M}$ be an isometrically immersed submanifold in a cosymplectic manifold $\bar{M}$. Then the formulas of Gauss and Weingarten for M in $\bar{M}$ are given by

$$
(2.5) \qquad \bar{\nabla}_X Y = \nabla_X Y + h(X,Y)
$$

and

$$
(2.6) \qquad \bar{\nabla}_X V = -A_V X + \nabla_X^\perp V,
$$

for any vector fields X, Y tangent to M and V normal to M, where ∇ denotes the induced Levi-Civita connection on M, $\nabla^\perp$ is the normal connection, A_V is the shape operator of M with respect to V and h is the second fundamental form of M in $\bar{M}$. h and A are related by

$$
(2.7) \qquad g(h(X,Y),V) = g(A_V X, Y)
$$

for any $X, Y \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$[1].

Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space $T_x M$, $x \in M$. The mean curvature vector H of M is defined by

$$
(2.8) \qquad H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i).
$$

The submanifold M is called totally geodesic, minimal and totally umbilical in $\bar{M}$ if $h = 0$, $H = 0$ and $h(X,Y) = g(X,Y)H$, respectively, for any $X, Y \in \Gamma(TM)$.

Also we put

$$
(2.9) \qquad h_{ij}^r = g(h(e_i, e_j), e^r) \quad and \quad \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)),
$$

where $\{e^r\}$, $1 \le r \le s$, are orthonormal basis vector fields of $(TM^\perp)$.

Furthermore, the equations of Gauss and Codazzi are, respectively, given by

$$
(2.10) \qquad (\bar{R}(X,Y)Z)^T = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X
$$

and

$$
(2.11) \qquad (\bar{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z),
$$

for any $X, Y, Z \in \Gamma(T\bar{M})$, where $(\bar{R}(X,Y)Z)^T$ and $(\bar{R}(X,Y)Z)^\perp$ denote the tangent and normal components of $\bar{R}(X,Y)Z$, respectively, R is the Riemannian curvature tensor of M. Also the covariant derivative of h is defined by

$$(2.12) \qquad (\bar{\nabla}_X h)(Y,Z) = \nabla_X^\perp h(Y,Z) - h(\nabla_X Y, Z) - h(\nabla_X Z, Y).$$

For any $X \in \Gamma(TM)$, we can write

$$(2.13) \qquad\qquad\qquad \varphi X = tX + nX,$$

where tX and nX denote the tangential and normal components of φX, respectively. In the same way, for any vector field V normal to M, we put

$$(2.14) \qquad\qquad\qquad \varphi V = BV + CV,$$

where BV and CV denote the tangential and normal components of φV, respectively.

A submanifold M is said to be invariant if n is identically zero. On the other hand, M is said to be anti-invariant submanifold if t is identically zero.

For a contact CR-submanifolds, the above definition has been generalized as follows.

For submanifolds tangent to the structure vector field ξ, there are different classes of submanifolds. We mention the following.

1.) A submanifold M tangent to ξ is called an invariant submanifold if φ preserves any tangent space of M, i.e., $\varphi(T_x M) \subseteq T_x M$, for each $x \in M$.

2.) A submanifold M tangent to ξ is called an anti-invariant submanifold if φ maps any tangent space of M into the normal space, that is, $\varphi(T_x M) \subseteq T_x^\perp M$, for each $x \in M$.

3.) A submanifold M tangent to ξ is called contact CR-submanifold if it admits an invariant distribution $\xi \in D$ whose orthogonal complementary distribution $D^\perp$ is anti-invariant, that is, $TM = D \oplus D^\perp$ with $\varphi D_x \subseteq D_x$ and $\varphi D_x^\perp \subseteq T_x^\perp M$ for each $x \in M$.

In this paper, we are concern with case 3.) as general case. We denote the orthogonal complementary distribution of $\varphi D^\perp$ in $T^\perp M$ by ν, then we have

$$(2.15) \qquad\qquad\qquad T^\perp M = \varphi D^\perp \oplus \nu.$$

We can easily to see that ν is an invariant subbundle with respect to φ.

3. Contact CR-Warped Product Submanifolds in a Cosymplectic Manifold

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and f is a positive definite differentiable function on M_1. The warped product of manifolds M_1 and M_2 is the Riemannian manifold

$$(2.4) \qquad\qquad M = M_1 \times_f M_2 = (M_1 \times M_2, g),$$

where $g = g_1 + f^2 g_2$. A warped product manifold $M = M_1 \times_f M_2$ is characterized by the fact that M_1 and M_2 are totally geodesic and totally umbilical submanifolds of M, respectively.

We recall the general formulae on a warped product

$$(2.5) \qquad \nabla_Z X = \nabla_X Z = (X \ln f) Z,$$

for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, where ∇ denote the Levi-Civita connection on M[2].

Definition 3.1. A warped product submanifold $M_1 \times_f M_2$ of a cosymplectic manifold $\bar{M}$, with M_1 is a $(2p+1)$-dimensional invariant submanifold tangent to ξ and M_1 is a q-dimensional anti-invariant submanifold of $\bar{M}$, is said to be a contact CR-warped product submanifold and we shall denote by $M_T \times_f M_\perp$ in the rest of this paper.

The following theorems characterize the contact CR-warped product submanifolds in cosymplectic manifolds.

Theorem 3.2. *Let M be a warped product submanifold of a cosymplectic manifold $\bar{M}$. Then M is a contact CR-warped product submanifold of $\bar{M}$ if and only if $nt = 0$.*

Proof. Let M be a contact CR-warped product submanifold of a cosymplectic manifold $\bar{M}$. Then we denote by P and Q the projections on the distributions D and $D^\perp$, respectively. Then we have

$$(2.6) \qquad P + Q = I, \quad P^2 = P, \quad Q^2 = Q \quad and \quad PQ = QP = 0.$$

For any $X \in \Gamma(TM)$, we can write

$$X = PX + QX.$$

From (2.13) and taking into account of D and $D^\perp$ being invariant and anti-invariant, respectively, we have

$$(2.7) \qquad \varphi X = \varphi PX + \varphi QX = tPX + nQX.$$

Since D is an invariant distribution, we get

$$(2.8) \qquad QtP = 0, \quad nP = 0.$$

Again, from (2.13), we also have

$$\varphi QX = tQX + nQX.$$

Moreover, the invariant of D and the anti-invariant of $D^\perp$ lead to

$$(2.9) \qquad tP = t \quad and \quad tQ = 0$$

by $Q = I - P$. The ambient space $\bar{M}$ is a cosymplectic manifold and $\xi \in \Gamma(D)$, we arrive at results

$$(2.10) \qquad t^2 = -I + \eta \otimes \xi - Bn \quad and \quad Cn + nt = 0.$$

Here, applying P from to right to the second equation (2.10) and taking into account of (2.9) and (2.8), we conclude

$$(2.11) \qquad nt = 0,$$

which is equivalent to

$$(2.12) \qquad Cn = 0.$$

Conversely, (2.11) is satisfied. Then for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^\perp)$, we have

$$g(X, BV) \quad = \quad -g(nX, V)$$

and

$$g(X, \varphi BV) \quad = \quad g(\varphi nX, V)$$
$$g(X, tBV) \quad = \quad g(CnX, V) = 0,$$

which gives $tB = 0$ for $Cn = 0$. The ambient space $\bar{M}$ is a cosymplectic manifold, by direct calculations, we get

$$(2.13) \qquad nB + C^2 = -I \quad and \quad tB + BC = 0.$$

Thus we have $BC = 0$. Now, applying the operators t and C from the right to the first equations of (2.10) and (2.13), respectively, we get

$$(2.14) \qquad t^3 + t = 0 \quad and \quad C^3 + C = 0.$$

We now set

$$(2.15) \qquad t^2 = -P + \eta \otimes \xi \quad and \quad Q = I - P,$$

then it is easily seen that

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q \quad and \quad PQ = QP = O.$$

These show that P and Q are orthogonal projections and they define orthogonal distributions such as D and $D^\perp$, respectively. By using $\varphi\xi = t\xi = n\xi = 0$, (2.14) and (2.15), we obtain

$$tP = t \quad and \quad tQ = 0.$$

On the other hand, the skew-symmetric of t and symmetric of Q lead to

$$Qt = 0 \quad and \quad QPt = 0.$$

Moreover, from the first equation of (2.15), we get

$$nP = 0.$$

These equations show that the distributions D and $D^\perp$ are invariant and anti-invariant distributions with respect to φ, respectively. Furthermore, since $P\xi = \xi$ and $Q\xi = 0$, the invariant distribution D contain ξ. This completes the proof. $\square$

Theorem 3.3. *Let M be a contact CR-submanifold of a cosymplectic manifold $\bar{M}$. Then M is a contact CR-warped product submanifold if and only if the shape operator A of M satisfies*

(2.16) $$A_{\varphi Z} X = (\varphi X(\mu)) Z, \quad X \in \Gamma(D), \quad Z \in \Gamma(D^\perp),$$

for some function μ on M satisfying $Z(\mu) = 0$.

Proof. Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a cosymplectic manifold $\bar{M}$. Then for all $X \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\perp)$, since the ambient space $\bar{M}$ is cosymplectic manifold, we have

(2.17) $$\varphi \nabla_X Z + \varphi h(X, Z) = -A_{\varphi Z} X + \nabla_X^\perp \varphi Z.$$

By taking the inner product of (2.17) by φY, for any $Y \in \Gamma(TM_T)$, we find

(2.18) $$g(\nabla_X Z, Y) = -g(A_{\varphi Z} X, \varphi Y) = -g(h(X, \varphi Y), \varphi Z).$$

Furthermore, since $M = M_T \times_f M_2$ is a warped product and M_1 is a totally geodesic submanifold in M, we get

$$g(h(X, \varphi Y), \varphi Z) = 0.$$

It follows that $A_{\varphi D^\perp} D \in \Gamma(D^\perp)$. Thus for any $W \in TM_\perp$, we have

$$\begin{aligned}
g(A_{\varphi Z} X, W) &= g(h(X, W), \varphi Z) = g(\bar{\nabla}_W X, \varphi Z) = -g(\bar{\nabla}_W \varphi X, Z) \\
&= -g(\nabla_W \varphi X, Z) = -(\varphi X \ln f) g(Z, W),
\end{aligned}$$

which implies that $A_{\varphi Z} X = (-\varphi X \ln f) Z$. Here, setting $\mu = \ln(1/f)$ and considering f is a function on M_T, we get desired result.

Conversely, we suppose that M is a contact CR-warped product submanifold of a cosymplectic manifold $\bar{M}$ satisfying (2.16). From (2.16), we obtain

(2.19) $$g(h(X, Y), \varphi Z) = 0, \quad i.e., \quad g(h(D, D), \varphi D^\perp) = 0$$

and

(2.20) $g(h(X, W), \varphi Z) = (\varphi X(\mu)) g(Z, W),$

for any $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$. The condition in (2.19) implies that the invariant distribution D is integrable and it is totally geodesic in M. From [8], we known that the anti-invariant distribution $D^\perp$ of a contact CR-submanifold of a cosymplectic manifold is always integrable, $W(\mu) = 0$, for any $W \in \Gamma(D^\perp)$, imply that each leaf of $D^\perp$ is an extrinsic sphere in M, that is, it is a totally umbilical submanifold with parallel mean curvature vector. Thus M is a locally the warped product $M = M_T \times_f M_\perp$ of a cosymplectic manifold $\bar{M}$, where M_T and $M_\perp$ are invariant and anti-invariant submanifolds of $\bar{M}$, respectively, M_T and $M_\perp$ denote the leaf of D and $D^\perp$, respectively and f is also warping function on M_T.   □

The following theorems characterize the contact CR-warped product submanifols as well as contact CR-product submanifolds in cosymplectic space forms.

Theorem 3.4. *Let M be a submanifold of a cosymplectic space form $\bar{M}(c)$ with $c \neq 0$. Then M is a contact CR-(warped)product submanifold if and only if the maximal invariant subspaces $D_x = T_x M \cap \varphi(T_x M)$, for each $x \in M$, define a non-trivial differentiable distribution D on M such that*

(2.21) $\bar{K}(D, D, D^\perp, D^\perp) = 0,$

where $D^\perp$ denotes the orthogonal complementary distribution in M and $\bar{K}$ denotes the Riemannian Christoffel curvature tensor of $\bar{M}$.

Proof. Let M be a contact CR-product submanifold of $\bar{M}$. By using (2.4), we get

$$\bar{K}(X, Y, Z, W) = g(\bar{R}(X, Y)Z, W) = 0,$$

for any $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$, that is, (2.21) is satisfied.

Conversely, we assume that the maximal invariant subspace D_x of $T_x M$ define a non-trivial differentiable distribution on M such that (2.21) holds. Then we have

$$\bar{K}(X, \varphi X, Z, W) = g(\bar{R}(X, \varphi X)Z, W) = \frac{c}{2} g(\varphi X, \varphi X) g(Z, \varphi W) = 0,$$

for any $X \in \Gamma(D)$ and $Z, W \in \Gamma(D^\perp)$. So we conclude $g(Z, \varphi W) = 0$ because $c \neq 0$ and $D \neq \{0\}$. It follows that $\varphi D^\perp$ is orthogonal to $D^\perp$. on the other hand, since D is an invariant distribution, we also have

$$g(X, \varphi Z) = -g(\varphi X, Z) = 0,$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$, which implies that $\varphi D^\perp$ is orthogonal to D. Furthermore, one find $g(\xi, \varphi Z) = 0$ for any $Z \in \Gamma(D^\perp)$. Finally, we reach $\varphi D_x^\perp \subset T_x M^\perp$, for each $x \in M$, that is, $D_x^\perp$ is an anti-invariant distribution on M and M becomes a contact CR-(warped) product submanifold.   □

Theorem 3.5. *Let M be a submanifold tangent to ξ of a cosymplectic space form $\bar{M}(c)$ with $c \neq 0$. Then M is a contact CR-(warped) product submanifold if and only if anti-invariant subspaces $D_x^\perp \subset T_x M$, for each $x \in M$, on M such that*

$$(2.22) \qquad \bar{K}(D, \varphi D, \nu, D) = 0,$$

where ν is defined as (2.15) and D is orthogonal complementary distribution of $D^\perp$ in M.

Proof. Let M be a contact CR-product submanifold of a cosymplectic space form $\bar{M}(c)$. From (2.4), we obtain

$$\bar{K}(X, \varphi Y, V, Z) = g(\bar{R}(X, \varphi Y)V, Z) = -\frac{c}{2}g(\varphi X, \varphi Y)g(\varphi V, Z) = 0,$$

for any $X, Y, Z \in \Gamma(D)$ and $V \in \Gamma(\nu)$, that is, (2.22) is satisfied.

Conversely, we suppose that anti-invariant subspaces $D_x^\perp \subset T_x M$, for each $x \in M$, define a non-trivial differentiable distribution $D^\perp$ on M such that (2.22) holds. From (2.4) we obtain

$$(2.23) \qquad \bar{K}(X, \varphi X, V, X) = \frac{c}{2}g(\varphi X, \varphi X)g(\varphi X, V) = 0,$$

for any $X \in \Gamma(D)$ and $V \in \Gamma(\nu)$. Thus (2.23) implies that φD is orthogonal to ν. Since $D^\perp$ is an anti-invariant distribution and using (2.3), we also get φD is orthogonal to ξ and $\varphi D^\perp$. So we mean that $\varphi D_x \subset T_x M$ and $\varphi D_x = D_x$, for each $x \in M$, that is, D is an invariant distribution and M becomes a contact CR-product submanifold. The proof is complete. $\qquad \square$

Now, we state the following estimation of the squared norm of the second fundamental form for a contact CR-warped product submanifolds in a cosymplectic manifold by the following theorems.

Theorem 3.6. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a cosymplectic manifold $\bar{M}$ such that M_T is a $(2p + 1)$-dimensional invariant submanifold tangent to ξ and $M_\perp$ is a q-dimensional anti-invariant submanifold of $\bar{M}$. Then*
1.) *The squared norm of the second fundamental form h of M satisfies*

$$(2.24) \qquad \|h\|^2 \geq 2q\|\nabla \ln f\|^2,$$

where, $\nabla \ln f$ is the gradient of $\ln f$.
2.) *If the equality sign of (2.24) holds identically, then M_T is totally geodesic invariant submanifold and $M_\perp$ is a totally umbilical anti-invariant submanifold of $\bar{M}$. In this case, M is a minimal contact CR-warped product submanifold of $\bar{M}$.*

Proof. Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a cosymplectic manifold $\bar{M}$ such that M_T is an invariant submanifold tangent to ξ and M_2 is an anti-invariant submanifold of $\bar{M}$. Then by using (2.3), (2.5) and (2.5), we have

$$
\begin{aligned}
g(h(\varphi X, Z), \varphi W) &= g(\bar{\nabla}_Z \varphi X, \varphi W) \\
&= g(\varphi \bar{\nabla}_Z X, \varphi W) = g(\bar{\nabla}_Z X, W) \\
&= g(\nabla_Z X, W) = (X \ln f) g(Z, W),
\end{aligned}
$$

(2.25)

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\perp)$. In the same way, by taking X for φX in (2.25), we get

(2.26) $$ g(h(X, Z), \varphi W) = (-\varphi X \ln f) g(Z, W). $$

On the other hand, since the ambient space $\bar{M}$ is a cosymplectic manifold, we can easily to see that

(2.27) $$ h(\xi, U) = h(\xi, \xi) = 0 \quad and \quad \xi \ln f = 0, $$

for any $U \in \Gamma(TM)$. Furthermore, if we denote by $h_\perp$ the second fundamental form of $M_\perp$ in $\bar{M}$, then for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\perp)$, making use of (2.5), we have

$$ g(h_\perp(Z, W), X) = g(\nabla_W Z, X) = -g(\nabla_W X, Z) = -(X \ln f) g(Z, W) $$

which is also equivalent to

(2.28) $$ h_\perp(Z, W) = -\nabla(\ln f) g(Z, W). $$

Now, let $\{e_1, e_2, ..., e_p, e_{p+1} = \varphi e_1, e_{p+2} = \varphi e_2, ..., e_{2p} = \varphi e_p, \xi, e^1, e^2, ..., e^q\}$ be a local orthonormal frame of $\Gamma(TM)$ such that e_i and e^j, $1 \leq i \leq p$, $1 \leq j \leq q$, are tangent to M_T and $M_\perp$, respectively. Then we have

$$
\begin{aligned}
\|h\|^2 &= \sum_{i,j=1}^{2p} g(h(e_i, e_j), h(e_i, e_j)) + 2 \sum_{i=1}^{2p} \sum_{r=1}^{q} g(h(e_i, e^r), h(e_i, e^r)) \\
&+ \sum_{r,\ell=1}^{q} g(h(e^r, e^\ell), h(e^r, e^\ell)) + \sum_{i=1}^{2p} g(h(e_i, \xi), h(e_i, \xi)) \\
&+ \sum_{r=1}^{q} g(h(e^r, \xi), h(e^r, \xi)) + g(h(\xi, \xi), h(\xi, \xi)) \\
&= \sum_{i,j=1}^{2p} g(h(e_i, e_j), h(e_i, e_j)) + \sum_{r,\ell=1}^{q} g(h(e^r, e^\ell), h(e^r, e^\ell)) \\
&+ 2 \sum_{i=1}^{2p} \sum_{r=1}^{q} g(h(e_i, e^r), h(e_i, e^r)).
\end{aligned}
$$

(2.29)

Thus, by using (2.25), (2.26) and (2.27), we arrive at

$$\|h\|^2 \geq 2\sum_{i=1}^{2p}\sum_{r=1}^{q} g(h(e_i,e^r),h(e_i,e^r)) \quad = \quad 2\sum_{i=1}^{p}\sum_{r=1}^{q}(e_i \ln f)^2 g(e^r,e^r)$$
$$+ \quad 2\sum_{i=1}^{p}\sum_{r=1}^{q}(\varphi e_i \ln f)^2 g(e^r,e^r),$$

or

$$(2.30) \qquad\qquad\qquad \|h\|^2 \geq 2q\|\nabla \ln f\|^2.$$

This proves the assertion (2.24). If equality sign in (2.30) holds identically, then by using (2.25) and (2.29), we obtain

$$(2.31) \qquad\qquad h(TM_T, TM_T) = 0, \quad h(TM_\perp, TM_\perp) = 0,$$

and

$$(2.32) \qquad\qquad h(TM_T, TM_\perp) \in \Gamma(\varphi(TM_\perp)).$$

The first condition in (2.31) implies that M_T is totally geodesic submanifold in $\bar{M}$ because M_T is a totally geodesic submanifold in M. Since $M_\perp$ is a totally umbilical submanifold in M, the second condition in (2.31) and (2.28) imply that $M_\perp$ is totally umbilical submanifold in $\bar{M}$. Moreover by (2.31) and (2.32) it follow that contact CR-warped product submanifold M is a minimal in $\bar{M}$. $\qquad\square$

Theorem 3.7. *Let* $M = M_T \times_f M_\perp$ *be a contact CR-warped product submanifold in cosymplectic space form* $\bar{M}(c)$ *of constant φ-sectional curvature c. If (2.24) equality is satisfied, then we*
1.) M_T is totally geodesic invariant submanifold of $\bar{M}(c)$. Thus M_T is a cosymplectic space form of constant φ-sectional curvature c.
2.) $M_\perp$ is a totally umbilical anti-invariant submanifold of $\bar{M}(c)$. Thus $M_\perp$ is a real space form of sectional curvature $\epsilon = \frac{c}{4} + \|\nabla \ln f\|^2$.
3.) If $q > 1$, then the warping function f satisfies $\|\nabla f\|^2 = \frac{1}{4}(4\epsilon - c)f^2$.

Proof. If (2.24) is satisfied, then M_T is totally geodesic submanifold of $\bar{M}(c)$. So one conclude that M_T is also cosymplectic space form of constant φ-sectional curvature c.

In the same way, if (2.24) is satisfied, then $M_\perp$ is totally umbilical anti-invariant submanifold of $\bar{M}(c)$. If we denote the Riemannian curvature tensor of $M_\perp$ by $R_\perp$, then by direct calculations, we can derive

$$R_\perp(Z,W)U = (\frac{c}{4} + \|\nabla \ln f\|^2)\{g(Z,U)W - g(W,U)Z\},$$

for any $Z, W, U \in \Gamma(TM_\perp)$. This prove that $M_\perp$ is a real space form of sectional curvature $\epsilon = \frac{c}{4} + \|\nabla \ln f\|^2$. Furthermore, if $\dim(M_\perp) > 1$, then the warping function f satisfies the condition $\|\nabla f\|^2 = \frac{1}{4}(4\epsilon - c)f^2$. $\qquad\square$

We note that the square of the length of the mean curvature of a totally umbilical submanifold is constant.

Theorem 3.8. *Let $M = M_T \times_f M_\perp$ be a contact CR-warped product submanifold of a cosymplectic space form $\bar{M}(c)$ such that M_T is a $(2p+1)$-dimensional invariant submanifold tangent to ξ and $M_\perp$ is a q-dimensional anti-invariant submanifold of $\bar{M}$. Then*
1.) *The squared norm of the second fundamental form of M satisfies*

$$(2.33) \qquad \|h\|^2 \geq \frac{cpq}{4} - \frac{q}{2}\|\nabla \ln f\|^2 - \frac{q}{2}\Delta \ln f,$$

where $\Delta \ln f$ denote the Laplacian of $\ln f$.
2.) *The equality sign of (2.33) holds if and only if*
2a.)*M_T is a totally geodesic invariant submanifold of $\bar{M}(c)$. Hence, M_T is cosymplectic space form of constant φ-sectional curvature.*
2b.) *$M_\perp$ is a totally umbilical anti-invariant submanifold of $\bar{M}(c)$. Hence $M_\perp$ is a real space form of sectional curvature $\epsilon = \|\nabla \ln f\|^2 + \frac{c}{4}$.*

Proof. In [7], it was proved that

$$(2.34) \qquad \sum_{i=1}^{p}\sum_{j=1}^{q} \|h_\nu(e_i, e^j)\|^2 = \frac{c.p.q}{4} - \frac{q}{2}\|\nabla \ln f\|^2 - \frac{q}{2}\Delta \ln f,$$

where h_ν denotes the component of h in ν. Thus combining (2.24) and (2.34), we obtain the inequality (2.33). If we consider the equality case of inequality, then form Theorem 3.6 we reach desired results. $\qquad\square$

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