KYUNGPOOK Math. J. 56(2016), 951-964
http://dx.doi.org/10.5666/KMJ.2016.56.3.951
pISSN 1225-6951 eISSN 0454-8124
© Kyungpook Mathematical Journal

On Concircular Curvature Tensor with respect to the Semi-symmetric Non-metric Connection in a Kenmotsu Manifold

Abdul Haseeb
Department of Mathematics, Faculty of Science, Jazan University, Jazan, Kingdom of Saudi Arabia
e-mail : malikhaseeb80@gmail.com

Abstract. The objective of the present paper is to study some new results on concircular curvature tensor in a Kenmotsu manifold with respect to the semi-symmetric non-metric connection.

1. Introduction

In 1969, S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension [14]. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He showed that they can be divided into three classes: (1) homogeneous normal contact Riemannian manifolds with $c > 0$, (2) global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if c = 0 and (3) a warped product space $R \times_f C$ if $c > 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. K. Kenmotsu [12] characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. Kenmotsu manifolds have been studied by various authors such as G. Pathak and U. C. De [13], J. B. Jun, U. C. De and G. Pathak [11], A. Yildiz et al. [17] and many others.

In 1924, the idea of semi-symmetric linear connection on a differentiable manifold was introduced by A. Friedmann and J. A. Schouten [8]. In 1930, E. Bartolotti [2] gave a geometrical meaning of such a connection. In 1932, H. A. Hayden [10] introduced semi-symmetric metric connection in Riemannian manifolds and this was studied systematically by K. Yano [15]. The semi-symmetric non-metric connection in a Riemannian manifold have been studied by N. S. Agashe and M. R.

Received May 1, 2015; revised July 6, 2016; accepted July 25, 2016.
2010 Mathematics Subject Classification: 53C05, 53D15, 53C25.
Key words and phrases: Kenmotsu manifold, semi-symmetric non-metric connection, η-Einstein manifold, concircular curvature tensor, concircularly flat manifold, concircularly symmetric manifold.

Chafle [1], U. C. De and S. C. Biswas [7], L. S. Das, M. Ahmad and A. Haseeb [6], S. K. Chaubey and R. H. Ojha [5] and others. Recently, A. Haseeb, M. A. Khan and M. D. Siddiqi studied an ϵ-Kenmotsu manifold with a semi-symmetric metric connection and obtained some new noteworthy results on this manifold [9].

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y],$$
$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is said to be *symmetric* if its torsion tensor T vanishes, otherwise it is *non-symmetric*. The connection ∇ is said to be *metric connection* if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is *non-metric*. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

A linear connection ∇ is said to be *semi-symmetric connection* if its torsion tensor T is of the form

$$T(X,Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form.

Semi-symmetric connections play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jaruselam or Mekka or the North pole, then this displacement is semi-symmetric and metric [8].

Motivated by the above studies, in this paper we obtain some new results on concircular curvature tensor in a Kenmotsu manifold with respect to the semi-symmetric non-metric connection. The paper is organized as follows : In Section 2, we give a brief introduction of a Kenmotsu manifold and define semi-symmetric non-metric connection. In Section 3, we find the curvature tensor, Ricci tensor and scalar curvature in a Kenmotsu manifold with respect to the semi-symmetric non-metric connection. Section 4 deals with the study of concircular curvature tensor in a Kenmotsu manifold with respect to the semi-symmetric non-metric connection and we establish the relation between concircular curvature tensors with respect to the connections ∇ and $\bar{\nabla}$. In Section 5, we show that the Kenmotsu manifold satisfying the condition $\bar{C}^*(\xi, X)\bar{S} = 0$ is an η-Einstein manifold. Section 6 is devoted to the study of concircularly flat and ξ-concircularly flat Kenmotsu manifolds with respect to the connection $\bar{\nabla}$. In Section 7, we study concircularly symmetric Kenmotsu manifolds with respect to the semi-symmetric non-metric connection.

2. Kenmotsu Manifolds

An n-dimensional smooth manifold (M, g) $(n = 2m + 1 > 1)$ is said to be an almost contact metric manifold [3], if it admits a $(1,1)$ tensor field ϕ, a structure vector field ξ, a 1-form η and the Riemannian metric g which satisfy

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$g(X,\xi) = \eta(X), \quad \eta(\phi X) = 0, \quad \phi(\xi) = 0, \quad \eta(\xi) = 1, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y) \tag{2.3}$$

for arbitrary vector fields X and Y on M.

An almost contact metric manifold $M(\phi,\xi,\eta,g)$ is said to be *Kenmotsu manifold* if ([12])

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \phi)(Y) = -g(X,\phi Y) - \eta(Y)\phi X, \tag{2.5}$$

where ∇ denotes the Riemannian connection of g.

Also the following relations hold in a Kenmotsu manifold [12]:

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{2.6}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi,X)Y = -R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.8}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.9}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{2.10}$$

$$S(\xi,\xi) = -(n-1), \quad \textit{i.e.,} \quad Q\xi = -(n-1)\xi, \tag{2.11}$$

where $g(QX,Y) = S(X,Y)$. It yields to

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y). \tag{2.12}$$

Example. We consider the three dimensional manifold $M = [(x,y,z) \in R^3 \mid z \neq 0]$, where (x,y,z) are the cartesian coordinates in R^3. Choosing the vector fields

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = -z\frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric define by

$$g(e_1,e_3)=g(e_2,e_3)=g(e_2,e_2)=0, \qquad g(e_1,e_1)=g(e_2,e_2)=g(e_3,e_3)=1.$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_3)$ for any vector field Z on TM. Let ϕ be the $(1,1)$ tensor field defined by $\phi(e_1)=-e_2, \quad \phi(e_2)=e_1, \quad \phi(e_3)=0$. Then by the linearity property of ϕ and g, we have

$$\phi^2 Z=-Z+\eta(Z)e_3, \qquad \eta(e_3)=1 \quad \text{and} \quad g(\phi Z,\phi W)=g(Z,W)-\eta(Z)\eta(W)$$

for any vector fields Z,W on M.
Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1,e_2]=0, \qquad [e_1,e_3]=e_1, \qquad [e_2,e_3]=e_2.$$

The Riemannian connection ∇ with respect to the metric g is given by

$$2g(\nabla_X Y,Z)=Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)+g([X,Y],Z)-g([Y,Z],X)+g([Z,X],Y).$$

From above equation which is known as Koszul's formula, we have

$$\begin{aligned}
&\nabla_{e_1}e_3=e_1, \qquad \nabla_{e_2}e_3=e_2, \quad \nabla_{e_3}e_3=0,\\
&\nabla_{e_1}e_2=0, \qquad \nabla_{e_2}e_2=-e_3, \quad \nabla_{e_3}e_2=0,\\
&\nabla_{e_1}e_1=-e_3, \quad \nabla_{e_2}e_1=0, \qquad \nabla_{e_3}e_1=0.
\end{aligned}$$

Using the above relations, for any vector field X on M, we have

$$\nabla_X\xi=X-\eta(X)\xi$$

for $\xi\in e_3$. Hence the manifold M under consideration is a Kenmotsu manifold of dimension three.

Definition 2.1([16]). A Kenmotsu manifold M is said to be an *η-Einstein manifold* if its Ricci tensor S of type (0,2) is of the form

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y), \tag{2.13}$$

where a and b are smooth functions on M. If $a=0$, then the manifold will be called a *special type of η-Einstein manifold*.

Definition 2.2([4]). The *concircular curvature tensor* C^* in a Kenmotsu manifold M is defined by

$$C^*(X,Y)Z=R(X,Y)Z-\frac{r}{n(n-1)}(g(Y,Z)X-g(X,Z)Y), \tag{2.14}$$

where R and r are the Riemannian curvature tensor and the scalar curvature of the manifold respectively.

Definition 2.3. A Kenmotsu manifold M is said to be *symmetric* if

$$(\nabla_W R)(X,Y)Z = 0 \tag{2.15}$$

for all vector fields X, Y, Z, W on M, where R is the curvature tensor with respect to ∇.

Definition 2.4. A Kenmotsu manifold M is said to be *concircularly symmetric* if

$$(\nabla_W C^*)(X,Y)Z = 0 \tag{2.16}$$

for all vector fields X, Y, Z, W on M, where C^* is the concircular curvature tensor defined by (2.14).

A linear connection $\bar{\nabla}$ in M is called *semi-symmetric connection* if its torsion tensor

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y], \tag{2.17}$$

satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y. \tag{2.18}$$

Further, a semi-symmetric connection is called *semi-symmetric non-metric connection* [1] if

$$\begin{aligned}(\bar{\nabla}_X g)(Y,Z) &= \bar{\nabla}_X g(Y,Z) - g(\bar{\nabla}_X Y, Z) - g(Y, \bar{\nabla}_X Z)\\ &= -\eta(Y)g(X,Z) - \eta(Z)g(X,Y).\end{aligned} \tag{2.19}$$

Let M be an n-dimensional Kenmotsu manifold and ∇ be the Levi-Civita connection on M, the semi-symmetric non-metric connection $\bar{\nabla}$ on M is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X. \tag{2.20}$$

3. Curvature Tensor on a Kenmotsu Manifold with respect to the Semi-symmetric Non-metric Connection

The curvature tensor $\bar{R}$ of semi-symmetric non-metric connection $\bar{\nabla}$ in M is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{3.1}$$

Using Equations (2.6) and (2.20) in (3.1), we get

$$\bar{R}(X,Y)Z = R(X,Y)Z + [g(X,Z)Y - g(Y,Z)X] + 2[\eta(Y)X - \eta(X)Y]\eta(Z), \tag{3.2}$$

where

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is the Riemannian curvature tensor of the connection ∇.

Contracting X in (3.2), we get

$$\bar{S}(Y,Z) = S(Y,Z) - (n-1)g(Y,Z) + 2(n-1)\eta(Y)\eta(Z), \tag{3.3}$$

where $\bar{S}$ and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇, respectively on M.

This gives

$$\bar{Q}Y = QY - (n-1)Y + 2(n-1)\eta(Y)\xi. \tag{3.4}$$

Contracting again Y and Z in (3.3), it follows that

$$\bar{r} = r - n^2 + 3n - 2, \tag{3.5}$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively on M.

Lemma 3.1. *Let M be an n-dimensional Kenmotsu manifold with respect to the semi-symmetric non-metric connection. Then*

$$\bar{R}(X,Y)\xi = 0, \tag{3.6}$$

$$\bar{R}(\xi,X)Y = 2[\eta(X)\eta(Y)\xi - g(X,Y)\xi], \tag{3.7}$$

$$\bar{S}(X,\xi) = 0, \tag{3.8}$$

$$\bar{Q}\xi = 0, \tag{3.9}$$

$$\bar{S}(\phi X,\phi Y) = \bar{S}(X,Y). \tag{3.10}$$

Proof. By taking $Z = \xi$ in (3.2) and using (2.7) and (2.2), we get (3.6). (3.7) follows from (3.2) and (2.8). By using (2.10) and (3.3) we have (3.8). From (2.11) and (3.4), we get (3.9). (3.10) follows, by considering $X = \phi X$ and $Y = \phi Y$ in (3.3) and using (2.2), (2.3) and (2.12). □

Thus we are in a position to prove the following theorem.

Theorem 3.2. *In a Kenmotsu manifold with respect to the semi-symmetric non-metric connection the curvature condition $\bar{R}(\xi,Y).\bar{S} = 0$ is satisfied.*

Proof. In order to prove the theorem (3.2), we consider the following expression

$$\bar{R}(X,Y).\bar{S} = \bar{S}(\bar{R}(X,Y)U,V) + \bar{S}(U,\bar{R}(X,Y)V). \tag{3.11}$$

Taking $X = \xi$ in (3.11), we have

$$\bar{R}(\xi, Y).\bar{S} = \bar{S}(\bar{R}(\xi, Y)U, V) + \bar{S}(U, \bar{R}(\xi, Y)V). \tag{3.12}$$

Using (3.7), we obtain

$$\bar{R}(\xi, Y).\bar{S} = 2\eta(Y)\eta(U)\bar{S}(\xi, V) - g(Y, U)\bar{S}(\xi, V) + 2\eta(Y)\eta(V)\bar{S}(\xi, U) - g(Y, V)\bar{S}(\xi, U). \tag{3.13}$$

By making use of (3.8) in (3.13), we get

$$\bar{R}(\xi, Y).\bar{S} = 0. \tag{3.14}$$

This completes the proof. □

4. Concircular Curvature Tensor on a Kenmotsu Manifold with respect to the Semi-symmetric Non-metric Connection

Analogous to the Definition 2.2, the concircular curvature tensor $\bar{C}^*$ on a Kenmotsu manifold with respect to the semi-symmetric non-metric connection is given by

$$\bar{C}^*(X, Y)Z = \bar{R}(X, Y)Z - \frac{\bar{r}}{n(n-1)}(g(Y, Z)X - g(X, Z)Y), \tag{4.1}$$

where $\bar{R}$ and $\bar{r}$ are the curvature tensor and scalar curvature with respect to the connection $\bar{\nabla}$, respectively.

By making use of (3.2) and (3.5) in the above equation, we obtain

$$\bar{C}^*(X, Y)Z = C^*(X, Y)Z + \frac{2}{n}(g(X, Z)Y - g(Y, Z)X) + 2(\eta(Y)X - \eta(X)Y)\eta(Z), \tag{4.2}$$

where $C^*(X, Y)Z$ is the concircular curvature tensor with respect to the connection ∇ and is given by (2.14). The equation (4.2) is the relation between the concircular curvature tensors with respect to the connections $\bar{\nabla}$ and ∇.

Interchanging X and Y in (4.2), we have

$$\bar{C}^*(Y, X)Z = C^*(Y, X)Z + \frac{2}{n}(g(Y, Z)X - g(X, Z)Y) + 2(\eta(X)Y - \eta(Y)X)\eta(Z). \tag{4.3}$$

On adding (4.2) and (4.3) and using the fact that $R(X, Y)Z + R(Y, X)Z = 0$, we get

$$\bar{C}^*(X, Y)Z + \bar{C}^*(Y, X)Z = 0. \tag{4.4}$$

From (3.2), (4.1) and first Bianchi identity $R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$ with respect to ∇, we obtain

$$\bar{C}^*(X, Y)Z + \bar{C}^*(Y, Z)X + \bar{C}^*(Z, X)Y = 0. \tag{4.5}$$

Equation (4.4) (resp.,(4.5)) shows that in a Kenmotsu manifold with respect to the semi-symmetric non-metric connection the concircular curvature tensor is skew-symmetric (resp., cyclic).

Next, taking inner product of (4.1) with ξ , we have

$$\eta(\bar{C}^*(X,Y)Z) = \eta(\bar{R}(X,Y)Z) - \frac{\bar{r}}{n(n-1)}(g(Y,Z)\eta(X) - g(X,Z)\eta(Y)). \tag{4.6}$$

Using (3.2) and (3.5) in (4.6), we find

$$\eta(\bar{C}^*(X,Y)Z) = (2 + \frac{r - n^2 + 3n - 2}{n(n-1)})[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \tag{4.7}$$

5. Kenmotsu Manifolds with respect to the Semi-symmetric Non-metric Connection Satisfying the Curvature Condition $\bar{C}^*(\xi,X).\bar{S} = 0$

Now we consider a Kenmotsu manifold with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ satisfying the condition

$$\bar{C}^*(\xi,X).\bar{S} = 0. \tag{5.1}$$

Then we have

$$\bar{S}(\bar{C}^*(\xi,X)Y,\xi) + \bar{S}(Y,\bar{C}^*(\xi,X)\xi) = 0. \tag{5.2}$$

By virtue of (2.2), (3.5), (3.7) and (4.1), we have

$$\bar{C}^*(\xi,X)Y = 2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi - \frac{r - n^2 + 3n - 2}{n(n-1)}(g(X,Y)\xi - \eta(Y)X), \tag{5.3}$$

$$\bar{C}^*(\xi,X)\xi = \frac{r - n^2 + 3n - 2}{n(n-1)}(X - \eta(X)\xi). \tag{5.4}$$

From (5.2), (5.3) and (5.4), we have

$$\bar{S}[2\eta(X)\eta(Y)\xi - 2g(X,Y)\xi - \frac{r - n^2 + 3n - 2}{n(n-1)}(g(X,Y)\xi - \eta(Y)X),\xi]$$
$$+\bar{S}[Y,\frac{r - n^2 + 3n - 2}{n(n-1)}(X - \eta(X)\xi)] = 0,$$

which on using (3.8), reduces to

$$\frac{r - n^2 + 3n - 2}{n(n-1)}\bar{S}(X,Y) = 0. \tag{5.5}$$

By using (3.3), (5.5) takes the form

$$\frac{r - n^2 + 3n - 2}{n(n-1)}(S(X,Y) - (n-1)g(X,Y) + 2(n-1)\eta(X)\eta(Y)) = 0. \tag{5.6}$$

This implies that either the scalar curvature of M is $r = n^2 - 3n + 2$ or $S = (n-1)g(X,Y) - 2(n-1)\eta(X)\eta(Y)$. The converse part is trivial.

Thus we can state the following theorem.

Theorem 5.1. *The concircular curvature tensor with respect to the semi-symmetric non-metric connection in a Kenmotsu manifold satisfies $\bar{C}^*(\xi, X).\bar{S} = 0$ if and only if either M has the scalar curvature $n^2 - 3n + 2$ or M is an η-Einstein manifold.*

6. Concircularly flat and ξ-concircularly Flat Kenmotsu Manifolds with respect to the Semi-symmetric Non-metric Connection

Firstly, we assume that the manifold M with respect to the semi-symmetric non-metric connection is concircularly flat, that is, $\bar{C}^*(X,Y)Z = 0$. Then from (4.1), it follows that

$$\bar{R}(X,Y)Z = \frac{\bar{r}}{n(n-1)}(g(Y,Z)X - g(X,Z)Y). \tag{6.1}$$

Taking inner product of the above equation with ξ, we have

$$g(\bar{R}(X,Y)Z, \xi) = \frac{\bar{r}}{n(n-1)} g((g(Y,Z)X - g(X,Z)Y), \xi). \tag{6.2}$$

In view of (2.2), (2.9), (3.2) and (3.5), we get

$$(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\frac{r + (n-1)(n+2)}{n(n-1)} = 0. \tag{6.3}$$

This implies that either the scalar curvature of M is $r = -(n-1)(n+2)$ or

$$g(Y,Z)\eta(X) - g(X,Z)\eta(Y) = 0. \tag{6.4}$$

Putting $Y = \xi$ in (6.4) and using (2.2), it follows that

$$\eta(X)\eta(Z) - g(X,Z) = 0. \tag{6.5}$$

Taking $X = QX$ in (6.5), and using (2.10), we get

$$S(X,Z) = -(n-1)\eta(X)\eta(Z). \tag{6.6}$$

Hence we can state the following theorem.

Theorem 6.1. *For a concircularly flat Kenmotsu manifold with respect to the semi-symmetric non-metric, either the scalar curvature is $-(n-1)(n+2)$ or the manifold is a special type of η-Einstein manifold.*

Secondly, we assume that the manifold M with respect to the semi-symmetric non-metric connection is ξ-concircularly flat, that is, $\bar{C}^*(X,Y)\xi = 0$. For this taking $Z = \xi$ in (6.1), we have

$$\bar{R}(X,Y)\xi = \frac{\bar{r}}{n(n-1)}(\eta(Y)X - \eta(X)Y). \tag{6.7}$$

In view of (3.5) and (3.6), we have

$$0 = \frac{r - n^2 + 3n - 2}{n(n-1)}(\eta(Y)X - \eta(X)Y). \tag{6.8}$$

Taking $Y = \xi$ in (6.8) and using (2.2), we get

$$0 = \frac{r - n^2 + 3n - 2}{n(n-1)}(X - \eta(X)\xi). \tag{6.9}$$

Taking inner product of (6.9) with U, we have

$$0 = \frac{r - n^2 + 3n - 2}{n(n-1)}(g(X,U) - \eta(X)\eta(U)). \tag{6.10}$$

Now taking $X = QX$ in (6.10), we have

$$0 = \frac{r - n^2 + 3n - 2}{n(n-1)}(g(QX,U) - \eta(QX)\eta(U)). \tag{6.11}$$

By making use (2.10) and the fact that $S(X,U) = g(QX,U)$, we find

$$0 = \frac{r - n^2 + 3n - 2}{n(n-1)}(S(X,U) + (n-1)\eta(X)\eta(U)). \tag{6.12}$$

This implies that either the scalar curvature of M is $r = (n-1)(n-2)$ or

$$S(X,U) = -(n-1)\eta(X)\eta(Y). \tag{6.13}$$

Hence we can state the following theorem.

Theorem 6.2. *For a ξ-concircularly flat Kenmotsu manifold with respect to the semi-symmetric non-metric, either the scalar curvature is $(n-1)(n-2)$ or the manifold is a special type of η-Einstein manifold.*

Next, taking $Z = \xi$ in (4.2) and using (2.2), we get

$$\bar{C}^*(X,Y)\xi = C^*(X,Y)\xi + \frac{2}{n}(g(X,\xi)Y - g(Y,\xi)X) + 2(\eta(Y)X - \eta(X)Y)\eta(\xi). \tag{6.14}$$

Making use of (2.2), it follows that

$$\bar{C}^*(X,Y)\xi = C^*(X,Y)\xi + 2(1-\frac{1}{n})(\eta(Y)X - \eta(X)Y). \tag{6.15}$$

Since $\eta(X)Y - \eta(Y)X = R(X,Y)\xi \neq 0$, in a Kenmotsu manifold, in general, then we have the following theorem.

Theorem 6.3. *ξ-concircularly flatness in Kenmotsu manifolds with respect to the semi-symmetric non-metric connection and the Levi-Civita connection are not equivalent.*

7. Concircularly symmetric Kenmotsu manifolds with respect to the semi-symmetric non-metric connection

Using (2.14), we can write

$$\begin{aligned}(\bar{\nabla}_W\bar{C}^*)(X,Y)Z &= (\nabla_W\bar{C}^*)(X,Y)Z + \eta(\bar{C}^*(X,Y)Z)W \\ &-[\eta(X)\bar{C}^*(W,Y)Z + \eta(Y)\bar{C}^*(X,W)Z + \eta(Z)\bar{C}^*(X,Y)W].\end{aligned} \tag{7.1}$$

Now differentiating (4.2) with respect to W, we have

$$\begin{aligned}(\nabla_W\bar{C}^*)(X,Y)Z &= (\nabla_W C^*)(X,Y)Z + 2[(\nabla_W\eta)(Y)X - (\nabla_W\eta)(X)Y]\eta(Z) \\ &+ 2[\eta(Y)X - \eta(X)Y](\nabla_W\eta)Z.\end{aligned} \tag{7.2}$$

Making use of (2.6) in (7.2), we obtain

$$\begin{aligned}(\nabla_W\bar{C}^*)(X,Y)Z &= (\nabla_W C^*)(X,Y)Z - 2[\eta(X)g(W,Z)Y \\ &+ \eta(Z)g(W,X)Y - \eta(Y)g(W,Z)X - \eta(Z)g(W,Y)X] \\ &+ 4\eta(W)\eta(Y)\eta(Z)X - 4\eta(W)\eta(X)\eta(Z)Y.\end{aligned} \tag{7.3}$$

From (7.1) and (7.3), we get

$$\begin{aligned}(\bar{\nabla}_W\bar{C}^*)(X,Y)Z &= (\nabla_W C^*)(X,Y)Z - 2[\eta(X)g(W,Z)Y + \eta(Z)g(W,X)Y \\ &- \eta(Y)g(W,Z)X - \eta(Z)g(W,Y)X] + 4\eta(W)\eta(Y)\eta(Z)X \\ &- 4\eta(W)\eta(X)\eta(Z)Y + \eta(\bar{C}^*(X,Y)Z)W - [\eta(X)\bar{C}^*(W,Y)Z \\ &+ \eta(Y)\bar{C}^*(X,W)Z + \eta(Z)\bar{C}^*(X,Y)W].\end{aligned} \tag{7.4}$$

In view of (4.1) and (4.7), (7.4) takes the form

(7.5)
$$
\begin{aligned}
(\bar{\nabla}_W \bar{C}^*)(X,Y)Z =& (\nabla_W C^*)(X,Y)Z - 2[\eta(X)g(W,Z)Y + \eta(Z)g(W,X)Y - \eta(Y)g(W,Z)X \\
&- \eta(Z)g(W,Y)X] + 4\eta(W)\eta(Y)\eta(Z)X - 4\eta(W)\eta(X)\eta(Z)Y \\
&+ (2 + \frac{r-n^2+3n+2}{n(n-1)})[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]W \\
&- \eta(X)[C^*(W,Y)Z + \frac{2}{n}(g(W,Z)Y - g(Y,Z)W) + 2(\eta(Y)W - \eta(W)Y)\eta(Z)] \\
&- \eta(Y)[C^*(X,W)Z + \frac{2}{n}(g(X,Z)W - g(W,Z)X) + 2(\eta(W)X - \eta(X)W)\eta(Z)] \\
&- \eta(Z)[C^*(X,Y)W + \frac{2}{n}(g(X,W)Y - g(Y,W)X) + 2(\eta(Y)X - \eta(X)Y)\eta(W)].
\end{aligned}
$$

Now differentiating (2.14) with respect to W, we get

$$
(\nabla_W C^*)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{\nabla_W r}{n(n-1)}(g(Y,Z)X - g(X,Z)Y). \tag{7.6}
$$

From (7.5) and (7.6), we obtain

(7.7)
$$
\begin{aligned}
(\bar{\nabla}_W \bar{C}^*)(X,Y)Z =& (\nabla_W R)(X,Y)Z - \frac{\nabla_W r}{n(n-1)}(g(Y,Z)X - g(X,Z)Y) - 2[\eta(X)g(W,Z)Y \\
&+ \eta(Z)g(W,X)Y - \eta(Y)g(W,Z)X - \eta(Z)g(W,Y)X] + 4\eta(W)\eta(Y)\eta(Z)X \\
&- 4\eta(W)\eta(X)\eta(Z)Y + (2 + \frac{r-n^2+3n+2}{n(n-1)})[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]W \\
&- \eta(X)[C^*(W,Y)Z + \frac{2}{n}(g(W,Z)Y - g(Y,Z)W) + 2(\eta(Y)W - \eta(W)Y)\eta(Z)] \\
&- \eta(Y)[C^*(X,W)Z + \frac{2}{n}(g(X,Z)W - g(W,Z)X) + 2(\eta(W)X - \eta(X)W)\eta(Z)] \\
&- \eta(Z)[C^*(X,Y)W + \frac{2}{n}(g(X,W)Y - g(Y,W)X) + 2(\eta(Y)X - \eta(X)Y)\eta(W)]
\end{aligned}
$$

from which we have

(7.8)
$$
(\bar{\nabla}_W \bar{C}^*)(X,Y)Z = (\nabla_W R)(X,Y)Z - \frac{\nabla_W r}{n(n-1)}(g(Y,Z)X - g(X,Z)Y) + F(X,Y,Z,W),
$$

where

(7.9)
$$
\begin{aligned}
F(X,Y,Z,W) =& -2[\eta(X)g(W,Z)Y + \eta(Z)g(W,X)Y - \eta(Y)g(W,Z)X - \eta(Z)g(W,Y)X] \\
&+ 4\eta(W)\eta(Y)\eta(Z)X - 4\eta(W)\eta(X)\eta(Z)Y \\
&+ (2 + \frac{r-n^2+3n+2}{n(n-1)})[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]W \\
&- \eta(X)[C^*(W,Y)Z + \frac{2}{n}(g(W,Z)Y - g(Y,Z)W) + 2(\eta(Y)W - \eta(W)Y)\eta(Z)] \\
&- \eta(Y)[C^*(X,W)Z + \frac{2}{n}(g(X,Z)W - g(W,Z)X) + 2(\eta(W)X - \eta(X)W)\eta(Z)] \\
&- \eta(Z)[C^*(X,Y)W + \frac{2}{n}(g(X,W)Y - g(Y,W)X) + 2(\eta(Y)X - \eta(X)Y)\eta(W)].
\end{aligned}
$$

If the scalar curvature r is constant and $F(X, Y, Z, W) = 0$, then (7.8) reduces to

$$(\bar{\nabla}_W \bar{C}^*)(X, Y)Z = (\nabla_W R)(X, Y)Z. \tag{7.10}$$

Hence we can state the following theorem.

Theorem 7.1. *A Kenmotsu manifold is concircularly symmetric with respect to the semi-symmetric non-metric connection $\bar{\nabla}$ if and only if it is symmetric with respect to Riemannian connection ∇, provided the scalar curvature r is constant and $F(X, Y, Z, W) = 0$.*

Acknowledgement. The author wishes to express his sincere thanks and gratitude to the referee for valuable suggestions towards the improvement of the paper.

References

[1] N. S. Agashe and M. R. Chafle, *A semi-symmetric non-metric connection in a Riemannian manifold*, Indian J. Pure Appl. Math, **23**(1992), 339–409.

[2] E. Bartolotti, *Sulla geometria della variata a connection affine,* Ann. di Mat, **4(8)**(1930), 53–101.

[3] D. E. Blair, *Contact manifolds in Riemannian geometry*, Lecture notes in Mathematics, **509**, Springer-Verlag Berlin-New York, 1976.

[4] D. E. Blair, J. S. Kim and M. M. Tripathi, *On the concircular curvature tensor of a contact metric manifold*, J. Korean Math. Soc, **42(5)**(2005), 883–892.

[5] S. K. Chaubey and R. H. Ojha, *On a semi-symmetric non-metric connection*, Filomat, **25**(2011), 19–27.

[6] L. S. Das, M. Ahmad and A. Haseeb, *On semi-invariant submanifolds of a nearly Sasakian manifold admitting a semi-symmetric non-metric connection,* Journal of Applied Analysis, **17(1)**(2011), 119–130.

[7] U. C. De and S. C. Biswas, *On a type of semi-symmetric non-metric connection on a Riemannian manifold,* Ganita, **48**(1997), 91–94.

[8] A. Friedmann and J. A. Schouten, *Uber die Geometric der halbsymmetrischen Ubertragung,* Math. Z., **21**(1924), 211–223.

[9] A. Haseeb, M. A. Khan and M. D. Siddiqi, *Some more results on an ϵ-Kenmotsu manifold with a semi-symmetric metric connection*, Acta Math. Univ. Comenianae, **85(1)**(2016), 9–20.

[10] H. A. Hayden, *Subspaces of space with torsion,* Proc. London Math. Soc., **34**(1932), 27–50.

[11] J. B. Jun, U. C. De and G. Pathak, *On Kenmotsu manifolds,* J. Korean Math. Soc., **42(3)**(2005), 435–445.

[12] K. Kenmotsu, *A class of almost contact Riemannian manifold,* Tohoku Math. J., **24**(1972), 93–103.

[13] G. Pathak and U. C. De, *On a semi-symmetric connection in a Kenmotsu manifold,* Bull. Calcutta Math. Soc., **94(4)**(2002), 319–324.

[14] S. Tanno, *The automorphism groups of almost contact Riemannian manifolds,* Tohoku Math. J., **21**(1969), 21–38.

[15] K. Yano, *On semi-symmetric metric connections,* Revue Roumaine De Math. Pures Appl., **15**(1970), 1579–1586.

[16] K. Yano and M. Kon, *Structures on Manifolds,* Series in Pure Math., **3**(1984).

[17] A. Yildiz, U. C. De and B. E. Acet, *On Kenmotsu manifolds satisfying certain curvature conditions,* SUT J. Math., **45**(2009), 89–101.