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Szász-Kantorovich Type Operators Based on Charlier Polynomials

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Abstract. In the present article, we study some approximation properties of the Kantorovich type generalization of Szász type operators involving Charlier polynomials introduced by S. Varma and F. Taşdelen (Math. Comput. Modelling, 56 (5-6) (2012) 108-112). First, we establish approximation in a Lipschitz type space, weighted approximation theorems and A-statistical convergence properties for these operators. Then, we obtain the rate of approximation of functions having derivatives of bounded variation.

1. Introduction

Szász ([31]) constructed the following linear positive operators

$$S_n(f;x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \tag{1.1}$$

where $x \in [0, \infty)$ and $f(x)$ is a continuous function on $[0, \infty)$ whenever the above sum converges uniformly. Butzer ([7]) defined and studied an integral modification of the operators S_n. Several researchers have studied approximation properties of these operators and their iterates (cf. [6, 13, 16, 24, 25, 32, 35]).

Jakimovski and Leviatan ([21]) introduced a generalization of Szász operators involving the Appell polynomials and studied some approximation properties of these operators. Varma et al. ([33]) constructed a generalization of Szász operators defined by means of the Brenke type polynomials and studied convergence properties of these operators using the Korovkin theorem and the order of convergence by using the classical second order modulus of continuity and Peetre's

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K-functional. Altomare et al. ([4]) defined a new kind of generalization of Szász-Mirakjan-Kantorovich operators and studied the rate of convergence by means of suitable moduli of smoothness. Very recently, Agrawal et al. ([2]) introduced a Kantorovich type generalization of the q-Bernstein-Schurer operators and gave some approximation properties of these operators. In [34], Varma and Taşdelen introduced a link between discrete orthogonal polynomials and certain linear positive operators. They have defined Szász type operators involving Charlier polynomials. These polynomials ([18]) have the generating functions of the form

$$e^t\left(1-\frac{t}{a}\right)^u=\sum_{k=0}^{\infty}C_k^{(a)}(u)\frac{t^k}{k!},\ \ |t|<a, \tag{1.2}$$

where $C_k^{(a)}(u)=\sum_{r=0}^{k}\binom{k}{r}(-u)_r\left(\frac{1}{a}\right)^r$ and $(m)_0=1, (m)_j=m(m+1)\cdots(m+j-1)$ for $j\geq 1$.

Varma and Taşdelen ([34]) defined the Szász type operators involving Charlier polynomials as

$$L_n(f;x,a)=e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}f\left(\frac{k}{n}\right),\quad a>1, x\geq 0.$$

Further, they considered Kantorovich type generalization of the operators $L_n(f;x,a)$ for a function $f\in\tilde{C}[0,\infty):=\left\{f\in C[0,\infty):|F(x)|=\left|\int_0^x f(s)ds\right|\leq Ke^{Bx}, B\in\mathbb{R}\right.$ and $\left.K\in\mathbb{R}^+\right\}$ as follows :

$$L_{n,a}^*(f;x)=ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)dt, \tag{1.3}$$

where $a>1$ and $x\geq 0$ and studied the uniform convergence of $L_{n,a}^*(f;x)$ to f on each compact subset of $[0,\infty)$ and the degree of approximation in terms of the classical modulus of continuity.

The purpose of this paper is to establish some more approximation properties of the operators $L_{n,a}^*$ such as weighted approximation, A-statistical convergence and approximation of functions with a derivative of bounded variation. The outline of paper is as follows.

In Section 2, we present some moment estimates and a result needed to study approximation of functions with derivatives of bounded variation. In Section 3, we discuss the main results of the paper wherein we establish approximation in a Lipschitz type space, weighted approximation theorems and A-statistical convergence properties for the operators $L_{n,a}^*$. Lastly, we obtain the rate of convergence for functions having a derivative of bounded variation on every finite subinterval of $[0,\infty)$, for these operators.

2. Preliminaries

In this section we collect some properties and examples of Charlier polynomials and some results about the operators $L^*_{n,a}$ useful in the sequel.

Since the Charlier polynomials play substantial role in the definition of the operators given by (1.3), we mention below some examples and properties of these non-classical polynomials:

Example 2.1. $C_0^{(a)}(u) = 1, \quad C_1^{(a)}(u) = 1 - \dfrac{u}{a}, \quad C_2^{(a)}(u) = 1 - \dfrac{u}{a^2}(1+2a) + \dfrac{u^2}{a^2}$ and $C_3^{(a)}(u) = 1 - \dfrac{u}{a^3}(3a^2+3a+2) + \dfrac{3u^2}{a^3}(a+1) - \dfrac{u^3}{a^3}$ etc.

Proposition 2.2.([8], Ch.VI, p.170) *For the function $C_k^{(a)}(u)$, there hold the following:*

(i) *$C_k^{(a)}(u)$ is a polynomial in u of degree k with the coefficient of u^k as* $\left(\dfrac{-1}{a}\right)^k$,

(ii) *$C_k^{(a)}(u)$ can be expressed in terms of Laguerre polynomials $L_k^{(u-k)}(a)$ as*

$$C_k^{(a)}(u) = k!\left(\frac{-1}{a}\right)^k L_k^{(u-k)}(a), \quad \textit{where} \quad L_k^{(\alpha)}(a) = \sum_{r=0}^{k} \binom{k+\alpha}{k-r} \frac{(-a)^r}{r!},$$

(iii) *$C_k^{(a)}(u)$ satisfies the recursion relation*

$$-aC_{k+1}^{(a)}(u) = (u-k-a)C_k^{(a)}(u) + kC_{k-1}^{(a)}(u), \quad k \geq 1,$$

(iv) *$C_k^{(a)}(u)$ satisfies the discrete orthogonality property*

$$\sum_{u=0}^{\infty} \omega(u) C_m^{(a)}(u) C_n^{(a)}(u) = a^n (n!) \quad \delta_{mn},$$

where $\omega(u) = \dfrac{e^{-a}a^u}{u!}$ *and δ_{mn} is the Kronecker delta.*

Lemma 2.3. *For the operators $L^*_{n,a}(f;x)$, we have*

(i) $L^*_{n,a}(1;x) = 1,$

(ii) $L^*_{n,a}(t;x) = x + \dfrac{3}{2n},$

(iii) $L^*_{n,a}(t^2;x) = x^2 + \dfrac{x}{n}\left(4 + \dfrac{1}{a-1}\right) + \dfrac{10}{3n^2},$

(iv) $L^*_{n,a}(t^3;x) = x^3 + \dfrac{x^2}{n}\left(\dfrac{15}{2} + \dfrac{3}{a-1}\right) + \dfrac{x}{n^2}\left(\frac{23}{2} + \dfrac{5}{a-1} + \dfrac{2}{(a-1)^2}\right) + \dfrac{37}{4n^3},$

(v) $$L_{n,a}^*(t^4;x) = x^4 + \frac{2x^3}{n}\left(6 + \frac{3}{a-1}\right) + \frac{x^2}{n^2}\left(63 - \frac{24}{a-1} + \frac{11}{(a-1)^2}\right) + \frac{x}{n^3}\left(98 + \frac{59}{a-1} - \frac{16}{(a-1)^2} + \frac{6}{(a-1)^3}\right) + \frac{151}{n^4}.$$

Proof. The proofs of the parts (i), (ii) and (iii) are given in ([34], Lemma 2). The moments $L_{n,a}^*(t^3;x)$ and $L_{n,a}^*(t^4;x)$ can be computed following the same idea of proof of ([34], Lemma 2). □

Lemma 2.4. *The central moments for the operators $L_{n,a}^*(f;x)$ are given by*

(i) $L_{n,a}^*(t-x;x) = \dfrac{3}{2n}$;

(ii) $L_{n,a}^*((t-x)^2;x) = \dfrac{a}{(a-1)}\dfrac{x}{n} + \dfrac{10}{3n^2}$;

(iii) $L_{n,a}^*((t-x)^3;x) = \dfrac{x}{n^2}\left(\dfrac{3}{2} + \dfrac{5}{a-1} + \dfrac{2}{(a-1)^2}\right) + \dfrac{37}{4n^3}$;

(iv) $L_{n,a}^*((t-x)^4;x) = \dfrac{x^2}{n^2}\left(37 - \dfrac{44}{a-1} + \dfrac{3}{(a-1)^2}\right) + \dfrac{x}{n^3}\left(61 + \dfrac{59}{a-1} - \dfrac{16}{(a-1)^2} + \dfrac{6}{(a-1)^3}\right) + \dfrac{151}{n^4}$.

Remark 2.5. From Lemma 2.4, for each $x \in [0,\infty), \eta(a) > 1$ and n sufficiently large, we have

$$L_{n,a}^*(|t-x|;x) \leq \left(L_{n,a}^*((t-x)^2;x)\right)^{1/2} \leq \sqrt{\frac{\eta(a)x}{n}},$$

where $\eta(a)$ is some positive constant depending on a.

The operators $L_{n,a}^*(f;x)$ also admit the integral representation

$$L_{n,a}^*(f;x) = \int_0^\infty K_n^*(x,t)f(t)dt \tag{2.1}$$

and $K_n^*(x,t) := ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum\limits_{k=0}^{\infty} \dfrac{C_k^{(a)}(-(a-1)nx)}{k!}\chi_{n,k}(t)$, where $\chi_{n,k}(t)$ is the characteristic function of the interval $\left[\dfrac{k}{n}, \dfrac{k+1}{n}\right]$ with respect to $[0,\infty)$.

Lemma 2.6. *For a fixed $x \in (0,\infty)$ and sufficiently large n, we have*

(i) $\beta_n^*(x,y) = \int_0^y K_n^*(x,t)dt \leq \dfrac{\eta(a)x}{n(x-y)^2}$, $0 \leq y < x$,

(ii) $1 - \beta_n^*(x,z) = \int_z^\infty K_n^*(x,t)dt \leq \dfrac{\eta(a)x}{n(z-x)^2}$, $x < z < \infty$.

Proof. (i) Using Remark 2.5, we get

$$\begin{aligned}
\beta_n^*(x,y) &= \int_0^y K_n^*(x,t)dt \le \int_0^y \left(\frac{x-t}{x-y}\right)^2 K_n^*(x,t)dt \\
&= L_{n,a}^*((t-x)^2;x)(x-y)^{-2} \\
&\le \frac{\eta(a)x}{n(x-y)^2}.
\end{aligned}$$

The assertion (ii) can be proved in a similar manner hence the details are omitted. □

In what follows, let $C_B[0,\infty)$ be the space of all real valued bounded and uniformly continuous functions f on $[0,\infty)$, endowed with the norm $||f||_{C_B[0,\infty)} = \sup\limits_{x\in[0,\infty)} |f(x)|$.

Further, let us define the following Peetre's K-functional:

$$K_2(f,\delta) = \inf_{g\in W^2}\{||f-g||_{C_B[0,\infty)} + \delta||g''||_{C_B[0,\infty)}\},\ \delta > 0,$$

where $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$ and the norm

$$\| f \|_{W^2} = \| f \|_{C_B[0,\infty)} + \| f' \|_{C_B[0,\infty)} + \| f'' \|_{C_B[0,\infty)} .$$

By ([9], p.177, Theorem 2.4) there exists an absolute constant $M > 0$ such that

$$K_2(f,\delta) \le M\left\{\omega_2(f,\sqrt{\delta}) + \min(1,\delta) \| f \|_{C_B[0,\infty)}\right\}, \tag{2.2}$$

where the second order modulus of smoothness is defined as

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<|h|\le\sqrt{\delta}} \sup_{x\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|.$$

The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined as

$$\omega(f,\delta) = \sup_{0<|h|\le\delta} \sup_{x\in[0,\infty)} |f(x+h) - f(x)|.$$

3. Degree of Approximation

In this section we establish approximation properties in several settings. For the reader's convenience we split up this section in more subsections.

3.1. Lipschitz-type space

Let us consider the Lipschitz-type space with two parameters [29]: for $a_1, a_2 > 0$, we define

$$Lip_M^{(a_1,a_2)}(\alpha) := \left\{ f \in C[0,\infty) : |f(t) - f(x)| \le M \frac{|t-x|^\alpha}{(t + a_1x^2 + a_2x)^{\frac{\alpha}{2}}}; x, t \in (0,\infty) \right\},$$

where M is a positive constant and $\alpha \in (0,1]$.

Theorem 3.1. *Let $f \in Lip_M^{(a_1,a_2)}(\alpha)$. Then, for all $x > 0$, we have*

$$|L_{n,a}^*(f;x) - f(x)| \le M \left(\frac{u_n^a(x)}{(a_1x^2 + a_2x)} \right)^{\frac{\alpha}{2}},$$

where $u_n^a(x) = L_{n,a}^*((t-x)^2; x)$.

Proof. First, we prove the theorem for the case $\alpha = 1$. We may write

$$\begin{aligned}
&|L_{n,a}^*(f;x) - f(x)| \\
&\le ne^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)| dt \\
&\le Mne^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} \frac{|t-x|}{\sqrt{t + a_1x^2 + a_2x}} dt.
\end{aligned}$$

Using the fact that $\frac{1}{\sqrt{t + a_1x^2 + a_2x}} < \frac{1}{\sqrt{a_1x^2 + a_2x}}$, and the Cauchy-Schwarz inequality, the above inequality implies that

$$\begin{aligned}
&|L_{n,a}^*(f;x) - f(x)| \\
&\le \frac{Mn}{\sqrt{a_1x^2 + a_2x}} e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |t-x| dt \\
&= \frac{M}{\sqrt{a_1x^2 + a_2x}} L_{n,a}^*(|t-x|; x) \le M \left(\sqrt{\frac{u_n^a(x)}{a_1x^2 + a_2x}} \right).
\end{aligned}$$

Thus, the result holds for $\alpha = 1$. Now, let $0 < \alpha < 1$, then applying the Hölder inequality with $p = \frac{1}{\alpha}$ and $q = \frac{1}{1-\alpha}$, we have

$$\begin{aligned}
&|L_{n,a}^*(f;x) - f(x)| \\
&\le ne^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!} \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)| dt \\
&\le \left\{ e^{-1}\left(1 - \frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!} \left(n \int_{\frac{k}{n}}^{\frac{k+1}{n}} |f(t) - f(x)| dt \right)^{\frac{1}{\alpha}} \right\}^{\alpha}
\end{aligned}$$

$$\leq \left\{ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|f(t)-f(x)|^{\frac{1}{\alpha}}dt\right\}^{\alpha}$$

$$\leq M\left\{ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{|t-x|}{\sqrt{t+a_1x^2+a_2x}}dt\right\}^{\alpha}$$

$$\leq \frac{M}{(a_1x^2+a_2x)^{\frac{\alpha}{2}}}\left\{ne^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\int_{\frac{k}{n}}^{\frac{k+1}{n}}|t-x|dt\right\}^{\alpha}$$

$$\leq \frac{M}{(a_1x^2+a_2x)^{\frac{\alpha}{2}}}(L^*_{n,a}(|t-x|;x))^{\alpha} \leq M\left(\frac{u_n^a(x)}{(a_1x^2+a_2x)}\right)^{\frac{\alpha}{2}}.$$

This completes the proof. □

3.2. Weighted approximation

Let $H_\phi[0,\infty)$ be the space of all functions f defined on $[0,\infty)$ with the property that $|f(x)| \leq M_f\phi(x)$, where M_f is a positive constant depending only on f and $\phi(x) = 1+x^2$ is a weight function. Let $C_\phi[0,\infty)$ be the subspace of $H_\phi[0,\infty)$ of all continuous functions with the norm $\| f \|_\phi = \sup_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}$ and $C^*_\phi[0,\infty) = \left\{f \in C_\phi[0,\infty) : \lim_{x\to\infty}\frac{|f(x)|}{1+x^2} < \infty\right\}$. The usual modulus of continuity of f on $[0,b]$ is defined as

$$\omega_b(f,\delta) = \sup_{|t-x|\leq\delta}\ \sup_{x,t\in[0,b]} |f(t)-f(x)|.$$

In what follows, let $\|\cdot\|_{C[0,d]}$ denote the sup-norm over $[0,d], d>0$.

Theorem 3.2. *Let $f \in C_\phi[0,\infty)$. Then, we have*

$$\|L^*_{n,a}(f)-f\|_{C[0,b]} \leq 4M_f(1+b^2)u_n^a(b) + 2\omega_{b+1}(f,\sqrt{u_n^a(b)}),$$

where $u_n^a(b) = \dfrac{ab}{n(a-1)} + \dfrac{10}{3n^2}$.

Proof. From [17], for $x \in [0,b]$ and $t \geq 0$, we have

$$|f(t)-f(x)| \leq 4M_f(1+x^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{b+1}(f,\delta), \delta > 0.$$

Hence applying Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|L^*_{n,a}(f;x)-f(x)|\\
&\leq 4M_f(1+x^2)L^*_{n,a}((t-x)^2;x)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}L^*_{n,a}(|t-x|;x)\right)\\
&\leq 4M_f(1+x^2)u_n^a(x)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{u_n^a(x)}\right)\\
&\leq 4M_f(1+b^2)u_n^a(b)+\omega_{b+1}(f,\delta)\left(1+\frac{1}{\delta}\sqrt{u_n^a(b)}\right).
\end{aligned}$$

Choosing $\delta=\sqrt{u_n^a(b)}$, we get the desired result. □

Next we give a theorem to approximate all functions in $C_\phi[0,\infty)$. This type of result is discussed in [14] for locally integrable functions.

Theorem 3.3. *For each $f\in C_\phi[0,\infty)$ and $\beta>0$, we have*

$$\lim_{n\to\infty}\sup_{x\in[0,\infty)}\frac{|L^*_{n,a}(f;x)-f(x)|}{(1+x^2)^{1+\beta}}=0.$$

Proof. For any fixed $x_0>0$,

$$\begin{aligned}
\sup_{x\in[0,\infty)}\frac{|L^*_{n,a}(f;x)-f(x)|}{(1+x^2)^{1+\beta}} &\leq \sup_{x\leq x_0}\frac{|L^*_{n,a}(f;x)-f(x)|}{(1+x^2)^{1+\beta}}+\sup_{x\geq x_0}\frac{|L^*_{n,a}(f;x)-f(x)|}{(1+x^2)^{1+\beta}}\\
&\leq \parallel L^*_{n,a}(f)-f\parallel_{C[0,x_0]}+\parallel f\parallel_\phi\sup_{x\geq x_0}\frac{|L^*_{n,a}(1+t^2;x)|}{(1+x^2)^{1+\beta}}\\
&\quad+\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\beta}},\\
&= I_1+I_2+I_3,\ \text{say.}
\end{aligned}\tag{3.1}$$

Since $|f(x)|\leq ||f||_\phi(1+x^2)$, we have

$$I_3=\sup_{x\geq x_0}\frac{|f(x)|}{(1+x^2)^{1+\beta}}\leq\sup_{x\geq x_0}\frac{||f||_\phi}{(1+x^2)^{\beta}}\leq\frac{||f||_\phi}{(1+x_0^2)^\beta}.$$

Let $\epsilon>0$ be arbitrary. In view of ([34], Theorem 3) there exists $n_1\in\mathbb{N}$ such that

$$\begin{aligned}
\parallel f\parallel_\phi\frac{|L^*_{n,a}(1+t^2;x)|}{(1+x^2)^{1+\beta}} &< \frac{1}{(1+x^2)^{1+\beta}}\parallel f\parallel_\phi\left((1+x^2)+\frac{\epsilon}{3||f||_\phi}\right),\quad\forall n\geq n_1\\
&< \frac{\|f\|_\phi}{(1+x^2)^\beta}+\frac{\epsilon}{3},\forall n\geq n_1.
\end{aligned}\tag{3.2}$$

Hence, $||f||_\phi \sup_{x \geq x_0} \dfrac{|L^*_{n,a}(1+t^2;x)|}{(1+x^2)^{1+\beta}} < \dfrac{||f||_\phi}{(1+x_0^2)^\beta} + \dfrac{\epsilon}{3}, \ \forall n \geq n_1.$

Thus, $I_2 + I_3 < \dfrac{2||f||_\phi}{(1+x_0^2)^\beta} + \dfrac{\epsilon}{3}, \ \forall n \geq n_1.$

Now, let us choose x_0 to be so large that $\dfrac{\|f\|_\phi}{(1+x_0^2)^\beta} < \dfrac{\epsilon}{6}$.

Then,

$$I_2 + I_3 < \frac{2\epsilon}{3}, \ \forall n \geq n_1. \tag{3.3}$$

By Theorem 3.2, there exists $n_2 \in \mathbb{N}$ such that

$$I_1 = \| L^*_{n,a}(f) - f \|_{C[0,x_0]} < \frac{\epsilon}{3}, \ \forall n \geq n_2. \tag{3.4}$$

Let $n_0 = \max(n_1, n_2)$. Then, combining (3.1)-(3.4)

$$\sup_{x \in [0,\infty)} \frac{|L^*_{n,a}(f;x) - f(x)|}{(1+x^2)^{1+\beta}} < \epsilon, \ \ \forall n \geq n_0.$$

This completes the proof. □

Let $f \in C^*_\phi[0,\infty)$. The weighted modulus of continuity is defined as :

$$\Omega(f;\delta) = \sup_{x \in [0,\infty), 0<h\leq\delta} \frac{|f(x+h) - f(x)|}{1+(x+h)^2},$$

was defined by Yüksel and Ispir in [36].

Lemma 3.4([36]). *Let $f \in C^*_\phi[0,\infty)$, then*

(i) *$\Omega(f,\delta)$ is a monotone increasing function of δ,*

(ii) $\lim_{\delta \to 0^+} \Omega(f;\delta) = 0$,

(iii) *for each $m \in \mathbb{N}, \Omega(f, m\delta) \leq m\Omega(f;\delta)$,*

(iv) *for each $\lambda \in [0,\infty), \Omega(f;\lambda\delta) \leq (1+\lambda)\Omega(f;\delta)$.*

Theorem 3.5. *Let $f \in C^*_\phi[0,\infty)$. Then there exists a positive constant $M(a)$ depending on a such that*

$$\sup_{x \in (0,\infty)} \frac{|L^*_{n,a}(f;x) - f(x)|}{(1+x^2)^{\frac{5}{2}}} \leq M(a)\Omega\left(f; n^{-1/2}\right). \tag{3.5}$$

Proof. For $t > 0, x \in (0,\infty)$ and $\delta > 0$, by the definition of $\Omega(f;\delta)$ and Lemma 3.4, we can write

$$\begin{aligned} |f(t) - f(x)| &\leq (1 + (x + |x - t|)^2)\Omega(f; |t - x|) \\ &\leq 2(1 + x^2)(1 + (t - x)^2)\left(1 + \frac{|t - x|}{\delta}\right)\Omega(f;\delta). \end{aligned}$$

Since $L^*_{n,a}$ is linear and positive, we have

$$\begin{aligned} |L^*_{n,a}(f;x) - f(x)| &\leq 2(1 + x^2)\Omega(f;\delta) \\ &\quad \times\left\{1 + L^*_{n,a}((t - x)^2; x) + L^*_{n,a}\left((1 + (t - x)^2)\frac{|t - x|}{\delta}; x\right)\right\}. \end{aligned} \tag{3.6}$$

From Lemma 2.4 (ii), we have

$$L^*_{n,a}((t - x)^2; x) \leq M_1(a)\frac{(1 + x^2)}{n}, \tag{3.7}$$

where $M_1(a)$ is some positive constant depending on a. Formally applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} &L^*_{n,a}\left((1 + (t - x)^2)\tfrac{|t-x|}{\delta}; x\right) \\ &\leq \frac{1}{\delta}\sqrt{L^*_{n,a}((t - x)^2; x)} + \frac{1}{\delta}\sqrt{L^*_{n,a}((t - x)^4; x)}\sqrt{L^*_{n,a}((t - x)^2; x)}. \end{aligned} \tag{3.8}$$

By using Lemma 2.4 (ii), there exists a positive constant $M_2(a)$ depending on a such that

$$\sqrt{\left(L^*_{n,a}(t - x)^4; x)\right)} \leq M_2(a)\frac{(1 + x^2)}{n}. \tag{3.9}$$

Collecting the estimates (3.6)-(3.9) and taking $M(a) = 2(1+M_1(a)+\sqrt{M_1(a)}+M_2(a)\sqrt{M_1(a)})$, $\delta = \frac{1}{\sqrt{n}}$, we get the required result (3.5). □

3.3. A-statistical convergence

Let $A = (a_{nk}), (n, k \in \mathbb{N})$, be a non-negative infinite summability matrix. For a given sequence $x := (x_k)$, the A-transform of x denoted by $Ax : ((Ax)_n)$ is defined as

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k,$$

provided the series converges for each n. A is said to be regular if $\lim_n (Ax)_n = L$ whenever $\lim_n x_n = L$. The sequence $x = (x_n)$ is said to be a A-statistically convergent to L i.e. $st_A - \lim_n x_n = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|x_k - L| \geq \epsilon} a_{nk} = 0$.

Replacing A by C_1, the Cesáro matrix of order one, the A-statistical convergence reduces to the statistical convergence. Similarly, if we take $A = I$, the identity matrix, then A-statistical convergence coincides with the ordinary convergence. It is to be noted that the concept of $A-$statistical convergence may also be given in normed spaces. Many researchers have studied the statistical convergence of different types of operators (cf. [5, 10, 11, 12, 15, 20, 26, 27, 30]). In the following result we prove a weighted Korovkin theorem via A-statistical convergence.

Throughout this section, let us assume that $e_i(t) = t^i, i = 0, 1, 2$.

Theorem 3.6. *Let (a_{nk}) be a non-negative regular infinite summability matrix and $x \in [0, \infty)$. Let $\phi_\gamma \geq 1$ be a continuous function such that*

$$\lim_{x \to \infty} \frac{\phi(x)}{\phi_\gamma(x)} = 0.$$

Then, for all $f \in C^*_\phi[0, \infty)$, we have

$$st_A - \lim_n \| L^*_{n,a}(f) - f \|_{\phi_\gamma} = 0.$$

Proof. From ([12], p. 195, Th. 6), it is enough to show that

$$st_A - \lim_n \| L^*_{n,a}(e_i) - e_i \|_\phi = 0.$$

From Lemma 2.3, we get

$$st_A - \lim_n \| L^*_{n,a}(e_0) - e_0 \|_\phi = 0.$$

Again by using Lemma 2.3, we have

$$\| L^*_{n,a}(e_1) - e_1 \|_\phi = \frac{3}{2n} \sup_{x \in [0,\infty)} \frac{1}{1 + x^2} \leq \frac{3}{2n}.$$

For $\epsilon > 0$, we define the following sets:

$$\begin{aligned} S &:= \{n : \| L^*_{n,a}(e_1) - e_1 \|_\phi \geq \epsilon\} \\ S_1 &:= \{n : \frac{3}{2n} \geq \epsilon\}, \end{aligned}$$

which yields us $S \subseteq S_1$, hence for all $n \in \mathbb{N}$, we have $\sum_{k \in S} a_{nk} \leq \sum_{k \in S_1} a_{nk}$.

Therefore, we get $st_A - \lim_n \parallel L^*_{n,a}(e_1) - e_1 \parallel_\phi = 0$.
Similarly, we have

$$\begin{aligned} \parallel L^*_{n,a}(e_2) - e_2 \parallel_\phi &\leq \frac{1}{n}\left(4 + \frac{1}{a-1}\right) \sup_{x\in[0,\infty)} \frac{x}{1+x^2} + \frac{10}{3n^2} \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\ &\leq \frac{1}{n}\left(4 + \frac{1}{a-1}\right) + \frac{10}{3n^2}. \end{aligned} \tag{3.10}$$

Now, we define the following sets:

$$\begin{aligned} T &:= \left\{ n : \parallel L^*_{n,a}(e_2) - e_2 \parallel_\phi \geq \epsilon \right\}, \\ T_1 &:= \left\{ n : \frac{1}{n}\left(4 + \frac{1}{a-1}\right) \geq \frac{\epsilon}{2} \right\}, \\ T_2 &:= \left\{ n : \frac{10}{3n^2} \geq \frac{\epsilon}{2} \right\}. \end{aligned}$$

In view of (3.10), it is clear that $T \subseteq T_1 \cup T_2$, which yields us

$$\sum_{k\in T} a_{nk} \leq \sum_{k\in T_1} a_{nk} + \sum_{k\in T_2} a_{nk}.$$

Thus, we get $st_A - \lim_n \parallel L^*_{n,a}(e_2) - e_2 \parallel_\phi = 0$.
Similarly, from Lemma 2.4, we have

$$st_A - \lim_n \|L^*_{n,a}((e_1 - xe_0)^j)\|_\phi = 0, \quad j = 0, 1, 2, 3, 4. \tag{3.11}$$ □

Next, we prove a Voronovskaja type theorem for the operators $L^*_{n,a}$.

Theorem 3.7. *Let $A = (a_{nk})$ be a nonnegative regular infinite summability matrix. Then, for every $f \in C^*_\phi[0,\infty)$ such that $f', f'' \in C^*_\phi[0,\infty)$, we have*

$$st_A - \lim_{n\to\infty} n\left(L^*_{n,a}(f;x) - f(x)\right) = \frac{3}{2} f'(x) + \frac{1}{2}\frac{ax}{(a-1)} f''(x),$$

uniformly with respect to $x \in [0, E], (E > 0)$.

Proof. Let $f, f', f'' \in C^*_\phi[0,\infty)$. For each $x \geq 0$, define a function

$$\Theta(t,x) = \begin{cases} \dfrac{f(t) - f(x) - (t-x)f'(x) - \frac{1}{2}(t-x)^2 f''(x)}{(t-x)^2} & if\ t \neq x \\ 0, & if\ t = x. \end{cases}$$

Then

$$\Theta(x,x) = 0 \text{ and } \Theta(\cdot,x) \in C^*_\phi[0,\infty).$$

Thus, we have

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+(t-x)^2\Theta(t,x).$$

Operating by $L^*_{n,a}$ on the above equality, we obtain

$$\begin{aligned}&n\left(L^*_{n,a}(f;x)-f(x)\right)\\&=f'(x)nL^*_{n,a}((t-x);x)+\frac{1}{2}f''(x)nL^*_{n,a}((t-x)^2;x)+nL^*_{n,a}((t-x)^2\Theta(t,x);x).\end{aligned}$$

In view of Lemma 2.4, we get

$$st_A-\lim_{n\to\infty} nL^*_{n,a}((t-x);x)=\frac{3}{2}, \tag{3.12}$$

$$st_A-\lim_{n\to\infty} nL^*_{n,a}((t-x)^2;x)=\frac{ax}{(a-1)}, \tag{3.13}$$

and

$$st_A-\lim_{n\to\infty} n^2L^*_{n,a}((t-x)^4;x)=x^2\left(37-\frac{44}{a-1}+\frac{3}{(a-1)^2}\right), \tag{3.14}$$

uniformly with respect to $x\in[0,E]$.

Applying Cauchy-Schwarz inequality, we have

$$nL^*_{n,a}((t-x)^2\Theta(t,x);x)\le\sqrt{n^2L^*_{n,a}((t-x)^4;x)}\sqrt{L^*_{n,a}(\Theta^2(t,x);x)}.$$

Let $\eta(t,x)=\Theta^2(t,x)$, we observe that $\eta(x,x)=0$ and $\eta(\cdot,x)\in C^*_\phi[0,\infty)$. It follows from [12] that

$$st_A-\lim_{n\to\infty} L^*_{n,a}(\Theta^2(t,x);x)=st_A-\lim_{n\to\infty} L^*_{n,a}(\eta(t,x);x)=\eta(x,x)=0,$$

uniformly with respect to $x\in[0,E]$. Hence, using (3.14), we obtain

$$st_A-\lim_{n\to\infty} nL^*_{n,a}((t-x)^2\Theta(t,x);x)=0, \tag{3.15}$$

uniformly in $x\in[0,E]$. Combining (3.12), (3.13) and (3.15), we get the desired result. □

Now, we obtain the rate of A-statistical convergence for the operators $L^*_{n,a}$ with the help of Peetre's K-functional.

Theorem 3.8. *Let $f\in W^2$. Then, we have*

$$st_A-\lim_{n}\|L^*_{n,a}(f)-f\|_{C_B[0,\infty)}=0. \tag{3.16}$$

Proof. By our hypothesis, from Taylor's expansion we find that

$$L^*_{n,a}(f;x) - f(x) = f'(x)L^*_{n,a}((e_1 - x);x) + \frac{1}{2}f''(\chi)L^*_{n,a}((e_1 - x)^2;x);$$

where χ lies between t and x. Thus, we get

$$\begin{aligned} \|L^*_{n,a}(f) - f\|_{C_B[0,\infty)} &\leq \|f'\|_{C_B[0,\infty)}\|L^*_{n,a}((e_1 - \cdot), \cdot)\|_{C_B[0,\infty)} \\ &+ \|f''\|_{C_B[0,\infty)}\|L^*_{n,a}((e_1 - \cdot)^2, \cdot)\|_{C_B[0,\infty)} \\ &= C_1 + C_2, \qquad \text{say.} \end{aligned} \tag{3.17}$$

Using (3.11) for $\epsilon > 0$, we have

$$\lim_n \sum_{k \in \mathbb{N} : C_1 \geq \frac{\epsilon}{2}} a_{nk} = 0,$$

$$\lim_n \sum_{k \in \mathbb{N} : C_2 \geq \frac{\epsilon}{2}} a_{nk} = 0.$$

From (3.17), we may write

$$\sum_{k \in \mathbb{N} : \|L^*_{k,a}(f) - f\|_{C_B[0,\infty)} \geq \epsilon} a_{nk} \leq \sum_{k \in \mathbb{N} : C_1 \geq \frac{\epsilon}{2}} a_{nk} + \sum_{k \in \mathbb{N} : C_2 \geq \frac{\epsilon}{2}} a_{nk}.$$

Hence taking the limit as $n \to \infty$, we get the desired result. □

Now we give an estimate of the rate of convergence by means of $\omega_2(f, \delta)$.

Theorem 3.9. *Let $f \in C_B[0,\infty)$, we have*

$$\|L^*_{n,a}(f) - f\|_{C_B[0,\infty)} \leq M\omega_2(f, \sqrt{\delta_{n,a}}),$$

where $\delta_{n,a} = \|L^*_{n,a}((e_1 - \cdot), \cdot)\|_{C_B[0,\infty)} + \|L^*_{n,a}((e_1 - \cdot)^2, \cdot)\|_{C_B[0,\infty)}$.

Proof. Let $g \in W^2$, by (3.17), we have

$$\begin{aligned} \|L^*_{n,a}(g) - g\|_{C_B[0,\infty)} &\leq \|L^*_{n,a}((e_1 - \cdot), \cdot)\|_{C_B[0,\infty)}\|g'\|_{C_B[0,\infty)} \\ &\quad + \tfrac{1}{2}\|L^*_{n,a}((e_1 - \cdot)^2, \cdot)\|_{C_B[0,\infty)}\|g''\|_{C_B[0,\infty)} \\ &\leq \delta_{n,a}\|g\|_{W^2}. \end{aligned} \tag{3.18}$$

Using (3.18), for every $f \in C_B[0,\infty)$ and $g \in W^2$, we get

$$\begin{aligned} &\|L^*_{n,a}(f) - f\|_{C_B[0,\infty)} \\ &\leq \|L^*_{n,a}(f) - L^*_{n,a}(g)\|_{C_B[0,\infty)} + \|L^*_{n,a}(g) - g\|_{C_B[0,\infty)} + \|g - f\|_{C_B[0,\infty)} \\ &\leq 2\|g - f\|_{C_B[0,\infty)} + \|L^*_{n,a}(g) - g\|_{C_B[0,\infty)} \\ &\leq 2\|g - f\|_{C_B[0,\infty)} + \delta_{n,a}\|g\|_{W^2}. \end{aligned}$$

Taking the infimum on the right hand side over all $g \in W^2$, we obtain

$$\|L^*_{n,a}(f) - f\|_{C_B[0,\infty)} \leq 2K_2(f, \delta_{n,a}).$$

Using (2.2), we have

$$\|L^*_{n,a}(f) - f\|_{C_B[0,\infty)} \leq M\Big\{\omega_2(f, \sqrt{\delta_{n,a}}) + \min(1, \delta_{n,a})||f||_{C_B[0,\infty)}\Big\}.$$

From (3.11), we get $st_A - \lim\limits_n \delta_{n,a} = 0$, hence $st_A - \omega_2(f, \sqrt{\delta_{n,a}}) = 0$. Therefore, we get the rate of A-statistical convergence of the sequence $L^*_{n,a}(f; x)$ to $f(x)$ in the space $C_B[0,\infty)$. If we take $A = I$, we obtain the ordinary rate of convergence of these operators. □

3.4. Approximation properties on $DBV_\gamma(0,\infty)$

Now, we shall estimate the rate of convergence for the operators $L^*_{n,a}$ for functions with derivatives of bounded variation defined on $(0,\infty)$ at points x where $f'(x+)$ and $f'(x-)$ exist, we shall prove that the operators (1.3) converge to the limit $f(x)$. In this direction, the significant contributions have been made by (cf. [1, 3, 19, 22, 23, 28] etc).

Let $f \in DBV_\gamma(0,\infty)$, $\gamma \geq 0$ be the class of all functions defined on $(0,\infty)$, having a derivative of bounded variation on every finite subinterval of $(0,\infty)$ and $|f(t)| \leq Mt^\gamma$, $\forall\ t > 0$.
We notice that the functions $f \in DBV_\gamma(0,\infty)$ possess a representation

$$f(x) = \int_0^x g(t)dt + f(0),$$

where $g(t)$ is a function of bounded variation on each finite subinterval of $(0,\infty)$.
The following theorem is our main result.

Theorem 3.10. *Let $f \in DBV_\gamma(0,\infty)$. Then, for every $x \in (0,\infty)$ and sufficiently large n, we have*

$$\begin{aligned}
&|L^*_{n,a}(f; x) - f(x)| \\
&\leq \frac{3}{4n}|f'(x+) + f'(x-)| + \sqrt{\frac{\eta(a)x}{n}}\left|\frac{f'(x+) - f'(x-)}{2}\right| \\
&\quad + \frac{\eta(a)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f'_x) + \frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x) + \frac{\eta(a)}{n}\sum_{k=0}^{[\sqrt{n}]}\bigvee_{x}^{x+(x/k)}(f'_x) \\
&\quad + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+(x/\sqrt{n})}(f'_x),
\end{aligned}$$

where $\bigvee_a^b(f'_x)$ *denotes the total variation of* f'_x *on* $[a,b]$ *and* f'_x *is defined by*

$$f'_x(t)=\begin{cases} f'(t)-f'(x-), & 0\le t<x\\ 0, & t=x\\ f'(t)-f'(x+) & x<t<\infty. \end{cases} \tag{3.19}$$

Proof. Since $L^*_{n,a}(1;x)=1$, for every $x\in(0,\infty)$ we get (see(2.1))

$$\begin{aligned} L^*_{n,a}(f;x)-f(x) &= \int_0^\infty K^*_n(x,t)(f(t)-f(x))dt\\ &= \int_0^\infty K^*_n(x,t)\int_x^t f'(u)dudt. \end{aligned} \tag{3.20}$$

For any $f\in DBV_\gamma(0,\infty)$, by (3.19) we may write

$$\begin{aligned} f'(u) &= f'_x(u)+\frac{1}{2}(f'(x+)+f'(x-))+\frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)\\ &\quad +\delta_x(u)[f'(u)-\frac{1}{2}(f'(x+)+f'(x-))], \end{aligned} \tag{3.21}$$

where

$$\delta_x(u)=\begin{cases} 1\ , & u=x\\ 0\ , & u\neq x \end{cases}.$$

Obviously,

$$\int_0^\infty\left(\int_x^t\left(f'(u)-\frac{1}{2}(f'(x+)+f'(x-))\right)\delta_x(u)du\right)K^*_n(x,t)dt=0.$$

We may write

$$\begin{aligned} &\int_0^\infty\left(\int_x^t\frac{1}{2}(f'(x+)+f'(x-))du\right)K^*_n(x,t)dt\\ &=\frac{1}{2}(f'(x+)+f'(x-))\int_0^\infty(t-x)K^*_n(x,t)dt\\ &=\frac{1}{2}(f'(x+)+f'(x-))L^*_{n,a}((t-x);x) \end{aligned}$$

and on an application of Cauchy-Schwarz inequality

$$\begin{aligned} &\int_0^\infty K^*_n(x,t)\left(\int_x^t\frac{1}{2}(f'(x+)-f'(x-))sgn(u-x)du\right)dt\\ &\le\frac{1}{2}\mid f'(x+)-f'(x-)\mid\int_0^\infty|t-x|K^*_n(x,t)dt\\ &\le\frac{1}{2}\mid f'(x+)-f'(x-)\mid L^*_{n,a}(|t-x|;x)\\ &\le\frac{1}{2}\mid f'(x+)-f'(x-)\mid\left(L^*_{n,a}((t-x)^2;x)\right)^{1/2}. \end{aligned}$$

Using Remark 2.1, and (3.20-3.21) we obtain the following estimate

$$\begin{aligned}&|L^*_{n,a}(f;x)-f(x)|\\ &\le \tfrac{3}{4n}|f'(x+)+f'(x-)|+\tfrac{1}{2}\sqrt{\tfrac{\eta(a)x}{n}}|f'(x+)-f'(x-)|\\ &+\left|\int_0^x\left(\int_x^t f'_x(u)du\right)K^*_n(x,t)dt+\int_x^\infty\left(\int_x^t f'_x(u)du\right)K^*_n(x,t)dt\right|.\end{aligned} \tag{3.22}$$

Now, let

$$A_n(f'_x,x)=\int_0^x\left(\int_x^t f'_x(u)du\right)K^*_n(x,t)dt,$$

and

$$B_n(f'_x,x)=\int_x^\infty\left(\int_x^t f'_x(u)du\right)K^*_n(x,t)dt.$$

To complete the proof, it is sufficient to estimate the terms $A_n(f'_x,x)$ and $B_n(f'_x,x)$. From Lemma 2.6, since $\int_a^b d_t\beta^*_n(x,t)\le 1$ for all $[a,b]\subseteq(0,\infty)$, using integration by parts and applying Lemma 2.6 with $y=x-(x/\sqrt{n})$, we have

$$\begin{aligned}|A_n(f'_x,x)| &= \left|\int_0^x\int_x^t\left(f'_x(u)du\right)d_t\beta^*_n(x,t)\right|\\ &= \left|\int_0^x\beta^*_n(x,t)f'_x(t)dt\right|\\ &\le \left(\int_0^y+\int_y^x\right)|f'_x(t)|\ |\beta^*_n(x,t)|dt\\ &\le \frac{\eta(a)x}{n}\int_0^y\bigvee_t^x(f'_x)(x-t)^{-2}dt+\int_y^x\bigvee_t^x(f'_x)dt\\ &\le \frac{\eta(a)x}{n}\int_0^y\bigvee_t^x(f'_x)(x-t)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^x(f'_x).\end{aligned}$$

By the substitution of $u=x/(x-t)$, we obtain

$$\begin{aligned}\frac{\eta(a)x}{n}\int_0^{x-(x/\sqrt{n})}(x-t)^{-2}\bigvee_t^x(f'_x)dt &= \frac{\eta(a)x}{n}x^{-1}\int_1^{\sqrt{n}}\bigvee_{x-(x/u)}^x(f'_x)du\\ &\le \frac{\eta(a)}{n}\sum_{k=1}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_{x-(x/u)}^x(f'_x)du\\ &\le \frac{\eta(a)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^x(f'_x).\end{aligned}$$

Thus,

$$|A_n(f'_x,x)| \le \frac{\eta(a)}{n}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f'_x). \tag{3.23}$$

Using integration by parts and applying Lemma 2.6 with $z = x + (x/\sqrt{n})$, we have

$$\begin{aligned}
&|B_n(f'_x,x)|\\
&= \left|\int_x^{\infty}\left(\int_x^t f'_x(u)du\right)K_n^*(x,t)dt\right|\\
&= \left|\int_x^{z}\left(\int_x^t f'_x(u)du\right)d_t(1-\beta_n^*(x,t))+\int_z^{\infty}\left(\int_x^t (f'_x)(u)du\right)d_t(1-\beta_n^*(x,t))\right|\\
&= \left|\left[\left(\int_x^t f'_x(u)du\right)(1-\beta_n^*(x,t))\right]_x^z-\int_x^z f'_x(t)(1-\beta_n^*(x,t))dt\right.\\
&\quad\left.+\int_z^{\infty}\left(\int_x^t f'_x(u)du\right)d_t(1-\beta_n^*(x,t))\right|\\
&= \left|\int_x^z f'_x(u)du(1-\beta_n^*(x,z))-\int_x^z f'_x(t)(1-\beta_n^*(x,t))dt\right.\\
&\quad\left.+\left[\int_x^t f'_x(u)du(1-\beta_n^*(x,t))\right]_z^{\infty}-\int_z^{\infty} f'_x(t)(1-\beta_n^*(x,t))dt\right|\\
&= \left|\int_x^z f'_x(t)(1-\beta_n^*(x,t))dt+\int_z^{\infty} f'_x(t)(1-\beta_n^*(x,t))dt\right|\\
&\le \frac{\eta(a)x}{n}\int_z^{\infty}\bigvee_x^t(f'_x)(t-x)^{-2}dt+\int_x^z\bigvee_x^t(f'_x)dt\\
&= \frac{\eta(a)x}{n}\int_{x+(x/\sqrt{n})}^{\infty}\bigvee_x^t(f'_x)(t-x)^{-2}dt+\frac{x}{\sqrt{n}}\bigvee_x^{x+(x/\sqrt{n})}(f'_x).
\end{aligned}$$

By the substitution of $v = x/(t-x)$, we get

$$\begin{aligned}
|B_n(f'_x,x)| &\le \frac{\eta(a)}{n}\int_0^{\sqrt{n}}\bigvee_x^{x+(x/v)}(f'_x)dv+\frac{x}{\sqrt{n}}\bigvee_x^{x+(x/\sqrt{n})}(f'_x)\\
&\le \frac{\eta(a)}{n}\sum_{k=0}^{[\sqrt{n}]}\int_k^{k+1}\bigvee_x^{x+(x/k)}(f'_x)dv+\frac{x}{\sqrt{n}}\bigvee_x^{x+(x/\sqrt{n})}(f'_x)\\
&= \frac{\eta(a)}{n}\sum_{k=0}^{[\sqrt{n}]}\bigvee_x^{x+(x/k)}(f'_x)+\frac{x}{\sqrt{n}}\bigvee_x^{x+(x/\sqrt{n})}(f'_x). \qquad (3.24)
\end{aligned}$$

Collecting the estimates (3.22)-(3.24), we get the required result. This completes the proof of the theorem. □

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