KYUNGPOOK Math. J. 56(2016), 861-866
http://dx.doi.org/10.5666/KMJ.2016.56.3.861
pISSN 1225-6951 eISSN 0454-8124


## On Left SF-Rings and Strongly Regular Rings

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Abstract. A ring R called left SF if its simple left modules are flat. Regular rings are known to be left SF-rings. However, till date it is unknown whether a left SF-ring is necessarily regular. In this paper, we prove the strong regularity of left (right) complement bounded left SF-rings. We also prove the strong regularity of a class of generalized semi-commutative left SF-rings.

### 1. Introduction

Throughout this paper, $R$ denotes an associative ring with identity and all modules are unital. The symbols $Z(_RR)$, $(Z(R_R))$, $J(R)$, respectively stand for left (right) singular ideal and the Jacobson radical of $R$. For any $a \in R$, $l(a)$ $(r(a))$ stands for the left (right) annihilator of $a$. By an ideal, we mean a two sided ideal. As usual, a *reduced ring* is a ring without non-zero nilpotent elements. Following [9], $R$ is *strongly left* (*right*) *bounded* if every non-zero left (right) ideal of $R$ contains a non-zero ideal of $R$. $R$ is *left* (*right*) *complement bounded* [1] if every non zero complement left (right) ideal of $R$ contains a non zero ideal of $R$. $R$ is *left* (*right*) *duo* if every left ideal of $R$ is an ideal. $R$ is *left* (*right*) *quasi duo* if every maximal left (right) ideal of $R$ is an ideal. $R$ is *semicommutative* if $l(a)$ is an ideal of $R$ for every $a \in R$. It is known that $R$ is semicommutative if and only if $r(a)$ is an ideal of $R$ for every $a \in R$. $R$ is (*von Neumann*) *regular* if for every $a \in R$, there exists

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* Corresponding Author.
Received September 8, 2015; revised February 19, 2016; accepted March 11, 2016.
2010 Mathematics Subject Classification: 16D25; 16E50; 16D40 .
Key words and phrases: Left SF-rings, von Neumann regular rings, strongly regular rings, left complement bounded rings.

some $b \in R$ such that $a = aba$. $R$ is *strongly regular* if for every $a \in R$, there exists some $b \in R$ such that $a = a^2b$. It is known that a ring $R$ is strongly regular if and only if $R$ is a reduced regular ring. Following [2], $R$ is a *left* (*right*) *SF-ring* if simple left (right) $R$-modules are flat. It is well known that regular rings are left (right) SF-rings. Ramamurthy in [2], initiated the study of left (right) SF-rings and of the question whether a left (right) SF-ring is necessarily regular. Since then, left (right) SF-rings have been extensively studied by many authors and the regularity of left (right) SF-rings which satisfy certain additional conditions is proved (see [2], [3], [4], [5], [6], [7], [8], [9], [10], [11]. However, the question still remains open.

This paper presents some new results on the strong regularity of left SF-rings.

## 2. Main Results

**Lemma 2.1.**([3], Proposition 3.2) *Let $R$ be a left SF-ring and $I$ be an ideal of $R$. Then $R/I$ is also a left SF-ring.*

**Lemma 2.2.**([3], Theorem 4.10) *A left quasi duo left SF-ring is strongly regular.*

**Lemma 2.3.**([3], Remark 3.13) *A reduced left SF-ring is strongly regular.*

**Lemma 2.4.**([3], Lemma 3.14) *Let $L$ be a left ideal of a ring $R$. Then $R/L$ is a flat left $R$-module if and only if for each $a \in L$, there exists $b \in L$ such that $a = ab$.*

**Lemma 2.5.**([4], Lemma 1.1 (2)) *If $R$ is a left SF-ring, then $Z(R_R) \subseteq J(R)$.*

**Theorem 2.6.** *The following conditions are equivalent for a ring $R$:*

(1) *$R$ is a strongly regular ring.*

(2) *$R$ is a left complement bounded left SF-ring.*

(3) *$R$ is a right complement bounded left SF-ring.*

*Proof.* That (1) $\Longrightarrow$ (2), (3) is well known.

(2) $\Longrightarrow$ (1). By Lemma 2.1, $R/Z(_RR)$ is a left SF-ring. Suppose $a^2 \in Z(_RR)$ such that $a \notin Z(_RR)$. If $Rr(a) + Z(_RR) = R$, then $a = ba + \sum r_i t_i a$, where $b \in Z(_RR)$, $r_i \in R$, $t_i \in r(a)$. If for some $i$, $t_i a \notin Z(_RR)$, then $l(t_i a)$ is not an essential left ideal of $R$. Therefore every complement of $l(t_i a)$ in $R$ is non-zero. Let $L$ be a complement of $l(t_i a)$ in $R$. Then $l(t_i a) \cap L = 0$, $L \neq 0$. By hypothesis, $L$ contains a non-zero ideal $I$ of $R$. Let $0 \neq u \in I$. Now, $(t_i a)^2 = t_i(at_i)a = 0$ so that $t_i a \in l(t_i a)$. Therefore we have

$$ut_i a \in l(t_i a) \cap I \subseteq l(t_i a) \cap L = 0,\ u \in l(t_i a) \cap L = 0.$$

This is a contradiction to $u \neq 0$. Hence it follows that $t_i a \in Z(_RR)$ for all $i$ so that $\sum r_i t_i a \in Z(_RR)$ and hence $a \in Z(_RR)$ which is a contradiction. Thus $Rr(a) + Z(_RR) \neq R$ and so there exists a maximal left ideal $M$ of $R$ containing $Rr(a) + Z(_RR)$. Since $a^2 \in Z(_RR) \subseteq M$ and $R$ is a left SF-ring, by Lemma 2.4, there exists some $c \in M$ such that $a^2 = a^2c$. This implies that $a - ac \in r(a) \subseteq M$

yielding $a \in M$. Hence again by Lemma 2.4, there exists $d \in M$ such that $a = ad$. Therefore $1 - d \in r(a) \subseteq M$, whence $1 \in M$, a contradiction to $M \neq R$. Thus $R/Z(_RR)$ is reduced so that by Lemma 2.3, $R/Z(_RR)$ is strongly regular.

Suppose there exists $0 \neq a \in Z(_RR)$. If $r(a) + Z(_RR) = R$ then $1 = b + c$ for some $b \in r(a)$, $c \in Z(_RR)$. Then $a = a(b + c) = ac$. Since $c \in Z(_RR)$, $l(c)$ is an essential left ideal of $R$. Hence $l(c) \cap Ra \neq 0$. Therefore there exist $0 \neq ra \in Ra$ such that $rac = 0$. So $ra = rac = 0$, a contradiction. Therefore $r(a) + Z(_RR) \neq R$. Then there exists a maximal right ideal $K$ of $R$ such that $r(a) + Z(_RR) \subseteq K$. Since $R/Z(_RR)$ is strongly regular, $K/Z(_RR)$ is an ideal of $R/Z(_RR)$. Hence $K$ is an ideal of $R$. Thus there exists a maximal left ideal $L$ of $R$ such that $Z(_RR) + r(a) \subseteq K \subseteq L$. Since $R$ is left $SF$ and $a \in Z(_RR) \subseteq L$, there exists some $d \in L$ such that $a = ad$. Then $1 - d \in r(a) \subseteq L$. Then $1 = (1 - d) + d \in L$, a contradiction. Hence $Z(_RR) = 0$. Therefore $R$ is strongly regular.

(3) $\Longrightarrow$ (1). $R/J(R)$ is a left SF-ring by Lemma 2.1. Let $a^2 \in J(R)$ such that $a \notin J(R)$. Suppose $Rr(a) + J(R) = R$. Then $1 = c + \sum r_i t_i$, where $c \in J(R)$, $r_i \in R$, $t_i \in r(a)$. This yields $a = ca + \sum r_i t_i a$. Now for each $i$, $(t_i a)^2 = t_i(at_i)a = 0$. Suppose $t_i a \notin J(R)$. Then by Lemma 2.5, $t_i a \notin Z(R_R)$ and so $r(t_i a)$ is not an essential right ideal of $R$. Let $K$ be a complement of $r(t_i a)$ in $R$. Clearly $K \neq 0$ and $r(t_i a) \cap K = 0$. By hypothesis, $K$ contains a non-zero ideal $I$ of $R$. Let $0 \neq u \in I$. Thus

$$t_i a u \in r(t_i a) \cap I \subseteq r(t_i a) \cap K = 0,\ u \in r(t_i a) \cap K = 0.$$

This contradicts that $u \neq 0$. Hence $t_i a \in J(R)$ for all $i$. Therefore $\sum r_i t_i a \in J(R)$ yielding $a \in J(R)$, a contradiction to $a \notin J(R)$. Therefore $Rr(a) + J(R) \neq R$, and so there exists a maximal left ideal $M$ of $R$ containing $Rr(a) + J(R)$. Since $R$ is a left SF-ring and $a^2 \in J(R) \subseteq M$, by Lemma 2.4, there exists some $c \in M$ such that $a^2 = a^2 c$, that is $a - ac \in r(a) \subseteq M$, whence $a \in M$. Hence, again by Lemma 2.4, there exists some $d \in M$ such that $a = ad$. Then $1 - d \in r(a) \subseteq M$, so that $1 \in M$, contradicting $M \neq R$. Therefore $R/J(R)$ is reduced. Therefore by Lemma 2.3, $R/J(R)$ is strongly regular. As a strongly regular ring is left duo, we see that $R$ is left quasi duo. Thus by Lemma 2.2, $R$ is strongly regular. □

**Corollary 2.7([6]).** *A left SF-ring whose complement right ideals are ideals is strongly regular.*

**Corollary 2.8([8]).** *A left SF-ring whose complement left ideals are ideals is strongly regular.*

**Corollary 2.9([9]).** *A strongly left (right) bounded left SF-ring is strongly regular.*

**Definition 2.10.** Following [11], a left ideal $L$ of a ring $R$ is a *generalized weak ideal (GW-ideal)*, if for every $a \in L$, there exists some $n > 0$ such that $a^n R \subseteq L$. A right ideal $K$ of $R$ is defined similarly to be a $GW$-ideal.

**Definition 2.11.** We call a ring $R$ a *left* (*right*) *GW-semicommutative ring* if $l(a)\,(r(a))$ is a GW-ideal of $R$ for every $a \in R$.

Clearly a semicommutative ring is a left (right) GW-semicommutative. However the converse is not true. This is shown by the following example.

**Example 2.12.**
Take $R = \left\{ \begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} : a,\, a_i \in \mathbb{Z}_2,\, i = 1,2,3,4,5,6 \right\}$.

Then every non-unit of $R$ is nilpotent. Hence every left (right) ideal of $R$ is a GW-ideal which implies that $R$ is left (right) GW-semicommutative.

Let $d = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then it is easy to see that

$\begin{pmatrix} a & a_1 & a_2 & a_3 \\ 0 & a & a_4 & a_5 \\ 0 & 0 & a & a_6 \\ 0 & 0 & 0 & a \end{pmatrix} d = 0$ if and only if $a = 0,\ a_4 = 0,\ a_1 = a_2$. Thus

$l(d) = \left\{ \begin{pmatrix} 0 & b_1 & b_1 & b_2 \\ 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & b_4 \\ 0 & 0 & 0 & 0 \end{pmatrix} : b_i \in \mathbb{Z}_2, i = 1,2,3,4 \right\} = L$ (say). Now

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in L, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in R$$

but
$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \notin L.$$

Therefore $R$ is not semicommutative.

**Theorem 2.13.** *The following conditions are equivalent for a ring $R$:*

(1) *$R$ is strongly regular.*

(2) *$R$ is a left GW-semicommutative left SF-ring.*

(3) *$R$ is a right GW-semicommutative left SF-ring.*

*Proof.* It is known that (1) $\Longrightarrow$ (2), (3) .

(2) $\Longrightarrow$ (1). Let $a^2 \in J(R)$ such that $a \notin J(R)$. Suppose $Rr(a) + J(R) = R$. Then $1 = c + \sum r_i t_i$, where $c \in J(R)$, $r_i \in R$, $t_i \in r(a)$. This yields $a = ca + \sum r_i t_i a$. If $t_i a \notin J(R)$ for some $i$, then $t_i a \notin M$ for some maximal right ideal $M$ of $R$. This implies that $M + t_i aR = R$ which yields $x + t_i ay = 1$ for some $x \in M$, $y \in R$. Now for each $i$, $(t_i a)^2 = t_i(at_i)a = 0$ so that $t_i a \in l(t_i a)$ and hence $yt_i a \in l(t_i a)$. As $R$ is left GW-semicommutative, $l(t_i a)$ is a GW-ideal of $R$. Therefore there exists a positive integer $n$ such that $(yt_i a)^n y \in l(t_i a)$. Hence

$$(1 - x)^{n+1} = (t_i ay)^{n+1} = t_i a(yt_i a)^n y \in l(t_i a).$$

Hence $(1 - x)^{n+1} t_i a = 0$. This together with $x \in M$ implies that $t_i a \in M$, a contradiction to $t_i a \notin M$. Hence $t_i a \in J(R)$ for all $i$. Proceeding as in (3) $\Longrightarrow$ (1) of Theorem 2.6, we can prove that $R$ is strongly regular.

(3) $\Longrightarrow$ (1). Let $0 \neq b \in R$ such that $b^2 = 0$. Then $r(b) \neq R$, and so there exists a maximal right ideal $K$ of $R$ such that $r(b) \subseteq K$. If $Rr(b) \nsubseteq K$, then $rs \notin K$ for some $r \in R$, $s \in r(b)$. So $K + rsR = R$, which yields $x + rst = 1$ for some $x \in K$, $t \in R$. Since $R$ is right GW-semicommutative, $r(b)$ is a GW-ideal of $R$. Since $str \in r(b)$, there exists a positive integer $n$ such that $r(str)^n \in r(b) \subseteq K$, and therefore

$$(1 - x)^{n+1} = (rst)^{n+1} = r(str)^n st \in r(b).$$

This yields $1 \in K$, a contradiction to $K \neq R$. Thus $Rr(b) \subseteq K \neq R$. Let $M$ be a maximal left ideal of $R$ containing $Rr(b)$. Since $R$ is a left SF-ring and $b \in r(b) \subseteq Rr(b) \subseteq M$, by Lemma 2.4, there exists some $c \in M$ such that $b = bc$. Then $1 - c \in r(b) \subseteq M$, whence $1 \in M$. This contradicts $M \neq R$. Therefore $R$ is reduced, and so by Lemma 2.3, $R$ is strongly regular. □

**Corollary 2.14([7]).** *A semicommutative left SF-ring is strongly regular.*

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