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Hermite-Hadamard-Fejér Type Inequalities for Harmonically Quasi-convex Functions via Fractional Integrals

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Abstract. In this paper, some Hermite-Hadamard-Fejér type integral inequalities for harmonically quasi-convex functions in fractional integral forms have been obtained.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

is well known in the literature as Hermite-Hadamard's inequality [5].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [4], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1.1):

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then the inequality*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx \tag{1.2}$$

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holds, where $g : [a, b] \to \mathbb{R}$ *is nonnegative, integrable and symmetric to* $a + b/2$*.*

For some results which generalize, improve and extend the inequalities (1.1) and (1.2) see [1, 6, 7, 16, 18].

We recall the following inequality and special functions which are known as Beta and hypergeometric function respectively:

$$\beta(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\ \ x, y > 0,$$

$$\begin{aligned} {}_2F_1(a, b; c; z) &= \frac{1}{\beta(b, c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt, \\ c &> b > 0, |z| < 1 \,(\text{see [13]}). \end{aligned}$$

Lemma 1.2.([15, 20]) *For* $0 < \alpha \leq 1$ *and* $0 \leq a < b$ *we have* $|a^\alpha - b^\alpha| \leq (b-a)^\alpha$.

The following definitions and mathematical preliminaries of fractional calculus theory are used further in this paper.

Definition 1.3.([13]) Let $f \in L[a, b]$. The Riemann-Liouville integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt,\ \ x > a$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt,\ \ x < b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ and $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [3, 8, 9, 17, 19, 20].

Definition 1.4.([21]) A function $f : I \subseteq (0, \infty) \to [0, \infty)$ is said to be harmonically quasi-convex, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \sup\{f(x), f(y)\}$$

for all $x, y \in I$ and $t \in [0, 1]$.

In [11], İşcan defined the so-called harmonically convex functions and established following Hermite-Hadamard type inequality for them as follows:

Definition 1.5. Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq tf(y) + (1-t)f(x) \tag{1.3}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (1.3) is reversed, then f is said to be harmonically concave.

Theorem 1.6.([11]) *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ then the following inequalities holds:*

$$f\left(\frac{2ab}{a+b}\right) \leq \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \leq \frac{f(a)+f(b)}{2}. \tag{1.4}$$

In [10], İşcan and Wu presented a Hermite-Hadamard type inequality for harmonically convex functions in fractional integral forms as follows:

Theorem 1.7. *Let $f : I \subset (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right) &\leq \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c} J^{\alpha}_{1/a-}(f\circ h)(1/b) \\ +J^{\alpha}_{1/b+}(f\circ h)(1/a) \end{array}\right\} \\ &\leq \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.5}$$

with $\alpha > 0$ and $h(x) = 1/x$.

In [14] Latif et al. gave the following definition:

Definition 1.8. A function $g : [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $2ab/a+b$, if

$$g(x) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for all $x \in [a,b]$.

In [2] Chan and Wu presented a Hermite-Hadamard-Fejér inequality for harmonically convex functions as follows:

Theorem 1.9. *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ and $g : [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a+b$, then*

$$\begin{aligned} f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx &\leq \int_a^b \frac{f(x)g(x)}{x^2}dx \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx. \end{aligned} \tag{1.6}$$

In [12] İşcan and Kunt presented a Hermite–Hadamard-Fejér type inequality for harmonically convex functions in fractional integral forms and established following identity as follows:

Theorem 1.10. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a harmonically convex function with $a<b$ and $f\in L[a,b]$. If $g:[a,b]\rightarrow\mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a+b$, then the following inequalities for fractional integrals holds:*

$$
\begin{aligned}
(1.7)\qquad & f\left(\frac{2ab}{a+b}\right)\left[J_{1/b+}^{\alpha}\left(g\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(g\circ h\right)(1/b)\right] \\
\leq\ & \left[J_{1/b+}^{\alpha}\left(fg\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(fg\circ h\right)(1/b)\right] \\
\leq\ & \frac{f(a)+f(b)}{2}\left[J_{1/b+}^{\alpha}\left(g\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(g\circ h\right)(1/b)\right]
\end{aligned}
$$

with $\alpha>0$ and $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Lemma 1.11.([12]) *Let $f:I\subset(0,\infty)\rightarrow\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $g:[a,b]\rightarrow\mathbb{R}$ is integrable and harmonically symmetric with respect to $2ab/a+b$, then the following equality for fractional integrals holds:*

$$
\begin{aligned}
(1.8)\qquad & \frac{f(a)+f(b)}{2}\left[J_{1/b+}^{\alpha}\left(g\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(g\circ h\right)(1/b)\right] \\
& -\left[J_{1/b+}^{\alpha}\left(fg\circ h\right)(1/a)+J_{1/a-}^{\alpha}\left(fg\circ h\right)(1/b)\right] \\
=\ & \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\left[\begin{array}{c}\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\alpha-1}\left(g\circ h\right)(s)ds \\ -\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}\left(g\circ h\right)(s)ds\end{array}\right]\left(f\circ h\right)'(t)dt
\end{aligned}
$$

with $\alpha>0$ and $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

In this paper, we give some new inequalities connected with the right-hand side of Hermite-Hadamard-Fejér type integral inequality for harmonically quasi-convex function in fractional integrals.

2. Main Results

Throughout this section, we write $\left\Vert g\right\Vert_{\infty}=\sup_{t\in[a,b]}\left\vert g(t)\right\vert$, for the continuous function $g:[a,b]\rightarrow\mathbb{R}$.

Theorem 2.1. *Let $f:I\subset(0,\infty)\rightarrow\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $\left\vert f'\right\vert$ is harmonically quasi-convex on $[a,b]$, $g:[a,b]\rightarrow\mathbb{R}$ is continuous and harmonically symmetric with respect to $2ab/a+b$,*

then the following inequality for fractional integrals holds:

$$(2.1)\qquad \begin{vmatrix} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right] \end{vmatrix}$$
$$\leq \frac{\|g\|_{\infty}\,ab\,(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} C_1(\alpha)\sup\{|f'(a)|,|f'(b)|\}$$

where

$$C_1(\alpha)=\begin{bmatrix} \frac{b^{-2}}{\alpha+1}\ {}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{\alpha+1}\ {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right) \\ +\frac{4(a+b)^{-2}}{\alpha+1}\ {}_2F_1\left(2,\alpha+1;\alpha+2;\frac{b-a}{b+a}\right) \end{bmatrix}$$

with $0<\alpha\leq 1$ *and* $h(x)=1/x$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Proof. From Lemma 1.11 we have

$$(2.2)\qquad \begin{vmatrix} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right] \end{vmatrix}$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_{\frac{1}{b}}^{\frac{1}{a}}\begin{vmatrix} \int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\alpha-1}(g\circ h)(s)ds \\ -\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds \end{vmatrix}\left|(f\circ h)'(t)\right|dt.$$

Since g is harmonically symmetric with respect to $2ab/a+b$, using Definition 1.8 we have $g\left(\frac{1}{x}\right)=g\left(\frac{1}{(\frac{1}{a})+(\frac{1}{b})-x}\right)$ for all $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

$$(2.3)\qquad \left|\int_{\frac{1}{b}}^{t}\left(\frac{1}{a}-s\right)^{\alpha-1}(g\circ h)(s)ds-\int_{t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds\right|$$
$$=\left|\int_{\frac{1}{a}+\frac{1}{b}-t}^{\frac{1}{a}}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds+\int_{\frac{1}{a}}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds\right|$$
$$=\left|\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)ds\right|$$
$$\leq\begin{cases} \int_{t}^{\frac{1}{a}+\frac{1}{b}-t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds\ , & t\in\left[\frac{1}{b},\frac{a+b}{2ab}\right] \\ \int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds\ , & t\in\left[\frac{a+b}{2ab},\frac{1}{a}\right] \end{cases}.$$

If we use (2.3) in (2.2), we have

$$\begin{aligned}
&\left|\begin{array}{c}\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]\\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\end{array}\right|\\
&\leq \frac{1}{\Gamma(\alpha)}\left[\begin{array}{c}\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_t^{\frac{1}{a}+\frac{1}{b}-t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds\right)\left|(f\circ h)'(t)\right|dt\\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left|\left(s-\frac{1}{b}\right)^{\alpha-1}(g\circ h)(s)\right|ds\right)\left|(f\circ h)'(t)\right|dt\end{array}\right]\\
&\leq \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c}\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_t^{\frac{1}{a}+\frac{1}{b}-t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\left|(f\circ h)'(t)\right|dt\\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\left|(f\circ h)'(t)\right|dt\end{array}\right]\\
&\leq \frac{\|g\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{c}\int_{\frac{1}{b}}^{\frac{a+b}{2ab}}\left(\int_t^{\frac{1}{a}+\frac{1}{b}-t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\frac{1}{t^2}\left|f'(\frac{1}{t})\right|dt\\ +\int_{\frac{a+b}{2ab}}^{\frac{1}{a}}\left(\int_{\frac{1}{a}+\frac{1}{b}-t}^{t}\left(s-\frac{1}{b}\right)^{\alpha-1}ds\right)\frac{1}{t^2}\left|f'(\frac{1}{t})\right|dt\end{array}\right].
\end{aligned}$$

Setting $t=\frac{ub+(1-u)a}{ab}$ and $dt=\left(\frac{b-a}{ab}\right)du$ gives

$$\begin{aligned}
&\left|\begin{array}{c}\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]\\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\end{array}\right|\\
&\leq \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c}\int_0^{\frac{1}{2}}\frac{(1-u)^\alpha-u^\alpha}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du\\ +\int_{\frac{1}{2}}^{1}\frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du\end{array}\right].
\end{aligned} \tag{2.4}$$

Since $|f'|$ is harmonically quasi-convex on $[a,b]$, we have

$$\left|f'(\frac{ab}{ub+(1-u)a})\right|\leq \sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}. \tag{2.5}$$

If we use (2.5) in (2.4), we have

$$\begin{aligned}
&\left|\begin{array}{c}\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right]\\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\end{array}\right|\\
&\leq \frac{\|g\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}\\
&\quad\times\left[\int_0^{\frac{1}{2}}\frac{(1-u)^\alpha-u^\alpha}{(ub+(1-u)a)^2}du+\int_{\frac{1}{2}}^{1}\frac{u^\alpha-(1-u)^\alpha}{(ub+(1-u)a)^2}du\right].
\end{aligned} \tag{2.6}$$

Using Lemma 1.2, we have

$$
\begin{aligned}
(2.7)\quad & \int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2}du + \int_{\frac{1}{2}}^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2}du \\
= & \int_0^1 \frac{u^\alpha - (1-u)^\alpha}{(ub+(1-u)a)^2}du + 2\int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2}du \\
= & \int_0^1 \frac{u^\alpha}{(ub+(1-u)a)^2}du - \int_0^1 \frac{(1-u)^\alpha}{(ub+(1-u)a)^2}du + 2\int_0^{\frac{1}{2}} \frac{(1-u)^\alpha - u^\alpha}{(ub+(1-u)a)^2}du \\
\leq & \int_0^1 \frac{u^\alpha}{(ub+(1-u)a)^2}du - \int_0^1 \frac{(1-u)^\alpha}{(ub+(1-u)a)^2}du + 2\int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub+(1-u)a)^2}du.
\end{aligned}
$$

Calculating the following integrals, we have

$$
\begin{aligned}
(2.8)\quad & \int_0^1 \frac{u^\alpha}{(ub+(1-u)a)^2}du - \int_0^1 \frac{(1-u)^\alpha}{(ub+(1-u)a)^2}du + 2\int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub+(1-u)a)^2}du \\
= & \int_0^1 \frac{(1-u)^\alpha}{(ua+(1-u)b)^2}du - \int_0^1 \frac{u^\alpha}{(ua+(1-u)b)^2}du + \int_0^1 \frac{(1-u)^\alpha}{\left(\frac{u}{2}b+\left(1-\frac{u}{2}\right)a\right)^2}du \\
= & \int_0^1 (1-u)^\alpha b^{-2}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2}du - \int_0^1 u^\alpha b^{-2}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2}du \\
& + \int_0^1 v^\alpha \left(\frac{a+b}{2}\right)^{-2}\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2}dv \\
= & \left[\begin{array}{c} \frac{b^{-2}}{\alpha+1}\ {}_2F_1\left(2,1;\alpha+2;1-\frac{a}{b}\right) \\ -\frac{b^{-2}}{\alpha+1}\ {}_2F_1\left(2,\alpha+1;\alpha+2;1-\frac{a}{b}\right) \\ +\frac{4(a+b)^{-2}}{\alpha+1}\ {}_2F_1\left(2,\alpha+1;\alpha+2;\frac{b-a}{b+a}\right) \end{array}\right] \\
= & C_1(\alpha).
\end{aligned}
$$

If we use (2.7) and (2.8) in (2.6), we have (2.1). This completes the proof. □

Corollary 2.2. *In Theorem 2.1:*
(1) *If we take* $\alpha = 1$ *we have the following Hermite-Hadamard-Fejér inequality for harmonically quasi-convex functions which is related to the right-hand side of* (1.6)*:*

$$
\begin{aligned}
& \left| \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx \right| \\
\leq & \frac{\|g\|_\infty (b-a)^2}{2} C_1(1) \sup\{|f'(a)|, |f'(b)|\},
\end{aligned}
$$

(2) *If we take* $g(x) = 1$ *we have following Hermite-Hadamard type inequality for harmonically quasi-convex functions in fractional integral forms which is related*

to the right-hand side of (1.5)*:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left\{\begin{array}{c} J^{\alpha}_{1/a-}(f\circ h)(1/b) \\ +J^{\alpha}_{1/b+}(f\circ h)(1/a)\end{array}\right\}\right| \\
\leq\ &\frac{ab(b-a)}{2}C_1(\alpha)\sup\{|f'(a)|,|f'(b)|\},
\end{aligned}$$

(3) *If we take* $\alpha=1$ *and* $g(x)=1$ *we have the following Hermite-Hadamard type inequality for harmonically quasi-convex functions which is related to the right-hand side of* (1.4)*:*

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}dx\right| \\
\leq\ &\frac{ab(b-a)}{2}C_1(1)\sup\{|f'(a)|,|f'(b)|\}.
\end{aligned}$$

Theorem 2.3. *Let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$*, where* $a,b\in I$ *and* $a<b$*. If* $|f'|^q, q\geq 1$*, is harmonically quasi-convex on* $[a,b]$*,* $g:[a,b]\to\mathbb{R}$ *is continuous and harmonically symmetric with respect to* $2ab/a+b$*, then the following inequality for fractional integrals holds:*

$$\begin{aligned}
&\left|\begin{array}{c}\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right]\end{array}\right| \\
\leq\ &\frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}C_2(\alpha)\left[\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right]^{\frac{1}{q}}
\end{aligned}\tag{2.9}$$

where

$$\begin{aligned}
C_2(\alpha) =\ &\frac{b^{-2}}{\alpha+1}\ {}_2F_1\left(2,1;\alpha+2;\tfrac{b-a}{b+a}\right)-\frac{b^{-2}}{\alpha+1}\ {}_2F_1\left(2,\alpha+1;\alpha+2;\tfrac{b-a}{b+a}\right) \\
&+\frac{4(a+b)^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,\alpha+1;\alpha+2;\tfrac{b-a}{b+a}\right),
\end{aligned}$$

with $0<\alpha\leq 1$ *and* $h(x)=1/x$*,* $x\in\left[\frac{1}{b},\frac{1}{a}\right]$*.*

Proof. Using (2.4), power mean inequality and the harmonically quasi-convexity of

$|f'|^q$, it follows that

$$\begin{aligned}
&\left| \begin{array}{c} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(g\circ h)(1/a)+J^{\alpha}_{1/a-}(g\circ h)(1/b)\right] \\ -\left[J^{\alpha}_{1/b+}(fg\circ h)(1/a)+J^{\alpha}_{1/a-}(fg\circ h)(1/b)\right] \end{array} \right| \qquad (2.10)\\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c} \int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du \\ +\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|du \end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{c} \left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|^q du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\left(\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right|^q du\right)^{\frac{1}{q}} \end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\
&\times\left[\begin{array}{c} \left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\left(\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}du\right)^{\frac{1}{q}} \\ +\left(\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right)^{1-\frac{1}{q}} \\ \times\left(\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}du\right)^{\frac{1}{q}} \end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right]^{\frac{1}{q}} \\
&\times\left[\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du+\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du\right].
\end{aligned}$$

Using Lemma 1.2, we have

$$\begin{aligned}
(2.11)\ & \int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du+\int_{\frac{1}{2}}^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du \\
=\ & \int_0^1\frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}du+2\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du \\
=\ & \int_0^1\frac{u^{\alpha}}{(ub+(1-u)a)^2}du-\int_0^1\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}du+2\int_0^{\frac{1}{2}}\frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}du \\
\leq\ & \int_0^1\frac{u^{\alpha}}{(ub+(1-u)a)^2}du-\int_0^1\frac{(1-u)^{\alpha}}{(ub+(1-u)a)^2}du+2\int_0^{\frac{1}{2}}\frac{(1-2u)^{\alpha}}{(ub+(1-u)a)^2}du.
\end{aligned}$$

For the appearing integrals, we have

$$
\begin{aligned}
(2.12)\quad & \int_0^1 \frac{u^\alpha}{(ub+(1-u)a)^2}du - \int_0^1 \frac{(1-u)^\alpha}{(ub+(1-u)a)^2}du + 2\int_0^{\frac{1}{2}} \frac{(1-2u)^\alpha}{(ub+(1-u)a)^2}du \\
= & \int_0^1 (1-u)^\alpha b^{-2}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2} du - \int_0^1 u^\alpha b^{-2}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2} du \\
& + \int_0^1 \frac{(1-u)^\alpha}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^2}du \\
= & \int_0^1 (1-u)^\alpha b^{-2}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2} du - \int_0^1 u^\alpha b^{-2}\left(1-u\left(1-\frac{a}{b}\right)\right)^{-2} du \\
& + \int_0^1 v^\alpha \left(\frac{a+b}{2}\right)^{-2}\left(1-v\left(\frac{b-a}{b+a}\right)\right)^{-2} dv \\
= & \frac{b^{-2}}{\alpha+1}\ {}_2F_1\left(2,1;\alpha+2;\tfrac{b-a}{b+a}\right) - \frac{b^{-2}}{\alpha+1}\ {}_2F_1\left(2,\alpha+1;\alpha+2;\tfrac{b-a}{b+a}\right) \\
& + \frac{4(a+b)^{-2}}{(\alpha+1)}\ {}_2F_1\left(2,\alpha+1;\alpha+2;\tfrac{b-a}{b+a}\right) \\
= & C_2(\alpha).
\end{aligned}
$$

If we use (2.11) and (2.12) in (2.10), we have (2.9). This completes the proof. □

Corollary 2.4. *In Theorem 2.3:*

(1) *If we take* $\alpha = 1$ *we have the following Hermite-Hadamard-Fejér inequality for harmonically quasi-convex functions which is related to the right-hand side of* (1.6)*:*

$$
\begin{aligned}
& \left| \frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx - \int_a^b \frac{f(x)g(x)}{x^2}dx \right| \\
\leq & \frac{\|g\|_\infty (b-a)^2}{2} C_2(1)\left[\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right]^{\frac{1}{q}},
\end{aligned}
$$

(2) *If we take* $g(x) = 1$ *we have following Hermite-Hadamard type inequality for harmonically quasi-convex functions in fractional integral forms which is related to the right-hand side of* (1.5)*:*

$$
\begin{aligned}
& \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^\alpha \left\{ \begin{array}{c} J^\alpha_{1/a-}(f\circ h)(1/b) \\ +J^\alpha_{1/b+}(f\circ h)(1/a) \end{array} \right\} \right| \\
\leq & \frac{ab(b-a)}{2} C_2(\alpha)\left[\sup\left\{|f'(a)|^q, |f'(b)|^q\right\}\right]^{\frac{1}{q}},
\end{aligned}
$$

(3) *If we take* $\alpha = 1$ *and* $g(x) = 1$ *we have the following Hermite-Hadamard type inequality for harmonically quasi-convex functions which is related to the right-hand side of* (1.4)*:*

$$\begin{aligned} &\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right| \\ \leq \ & \frac{ab(b-a)}{2} C_2(1) \left[\sup \left\{ |f'(a)|^q , |f'(b)|^q \right\} \right]^{\frac{1}{q}} . \end{aligned}$$

We can state another inequality for $q > 1$ as follows:

Theorem 2.5. *Let* $f : I \subset (0, \infty) \to \mathbb{R}$ *be a differentiable function on* I° *such that* $f' \in L[a, b]$*, where* $a, b \in I$ *and* $a < b$ *. If* $|f'|^q, q > 1$*, is harmonically quasi-convex on* $[a, b]$*,* $g : [a, b] \to \mathbb{R}$ *is continuous and harmonically symmetric with respect to* $2ab/a + b$ *, then the following inequality for fractional integrals holds:*

$$\begin{aligned} &\left| \begin{array}{c} \frac{f(a)+f(b)}{2} \left[J^\alpha_{1/b+} (g \circ h)(1/a) + J^\alpha_{1/a-} (g \circ h)(1/b) \right] \\ - \left[J^\alpha_{1/b+} (fg \circ h)(1/a) + J^\alpha_{1/a-} (fg \circ h)(1/b) \right] \end{array} \right| \\ \leq \ & \frac{\|g\|_\infty ab(b-a)}{2^{\frac{1}{q}} \Gamma(\alpha+1)} \left(\frac{b-a}{ab} \right)^\alpha \\ & \times \left(\sup \left\{ |f'(a)|^q , |f'(b)|^q \right\} \right)^{\frac{1}{q}} \left[C_3^{\frac{1}{p}}(\alpha) + C_4^{\frac{1}{p}}(\alpha) \right] \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} C_3(\alpha) &= \left(\frac{a+b}{2} \right)^{-2p} \frac{1}{2(\alpha p+1)} \ {}_2F_1 \left(2p, \alpha p+1; \alpha p+2; \tfrac{b-a}{b+a} \right) , \\ C_4(\alpha) &= b^{-2p} \frac{1}{2(\alpha p+1)} \ {}_2F_1 \left(2p, 1; \alpha p+2; \tfrac{1}{2} \left(1 - \tfrac{a}{b} \right) \right) , \end{aligned}$$

with $0 < \alpha \leq 1$*,* $h(x) = 1/x$*,* $x \in \left[\frac{1}{b}, \frac{1}{a} \right]$ *and* $1/p + 1/q = 1$.

Proof. Using (2.4), Hölder's inequality and the harmonically quasi-convexity of $|f'|^q$, it follows that

$$
\begin{aligned}
(2.14)\quad & \left| \begin{array}{l} \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}\left(g\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(g\circ h\right)(1/b)\right] \\ \quad -\left[J^{\alpha}_{1/b+}\left(fg\circ h\right)(1/a)+J^{\alpha}_{1/a-}\left(fg\circ h\right)(1/b)\right] \end{array} \right| \\
\leq\ & \frac{\|g\|_{\infty}ab\,(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{l} \int_0^{\frac{1}{2}} \frac{(1-u)^{\alpha}-u^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right| du \\ +\int_{\frac{1}{2}}^1 \frac{u^{\alpha}-(1-u)^{\alpha}}{(ub+(1-u)a)^2}\left|f'(\frac{ab}{ub+(1-u)a})\right| du \end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab\,(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[\begin{array}{l} \left(\int_0^{\frac{1}{2}} \frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \\ \times\left(\int_0^{\frac{1}{2}}\left|f'(\frac{ab}{ub+(1-u)a})\right|^q du\right)^{\frac{1}{q}} \end{array}\right. \\
& \left. \begin{array}{l} +\left(\int_{\frac{1}{2}}^1 \frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}} \\ \times\left(\int_{\frac{1}{2}}^1\left|f'(\frac{ab}{ub+(1-u)a})\right|^q du\right)^{\frac{1}{q}} \end{array}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab\,(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha} \\
& \times\left[\left(\int_0^{\frac{1}{2}} \frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} \sup\left\{|f'(a)|^q,|f'(b)|^q\right\}du\right)^{\frac{1}{q}}\right. \\
& \left.+\left(\int_{\frac{1}{2}}^1 \frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 \sup\left\{|f'(a)|^q,|f'(b)|^q\right\}du\right)^{\frac{1}{q}}\right] \\
\leq\ & \frac{\|g\|_{\infty}ab\,(b-a)}{2^{\frac{1}{q}}\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}} \\
& \times\left[\left(\int_0^{\frac{1}{2}} \frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}+\left(\int_{\frac{1}{2}}^1 \frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du\right)^{\frac{1}{p}}\right].
\end{aligned}
$$

Using Lemma 1.2, we have

$$
(2.15)\qquad \int_0^{\frac{1}{2}} \frac{[(1-u)^{\alpha}-u^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du \leq \int_0^{\frac{1}{2}} \frac{(1-2u)^{\alpha p}}{(ub+(1-u)a)^{2p}}du
$$

and

$$
(2.16)\qquad \int_{\frac{1}{2}}^1 \frac{[u^{\alpha}-(1-u)^{\alpha}]^p}{(ub+(1-u)a)^{2p}}du \leq \int_{\frac{1}{2}}^1 \frac{(2u-1)^{\alpha p}}{(ub+(1-u)a)^{2p}}du.
$$

For the appearing integrals, we have

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} \frac{(1-2u)^{\alpha p}}{(ub+(1-u)a)^{2p}}du \\
=\;& \frac{1}{2}\int_0^1 \frac{(1-u)^{\alpha p}}{\left(\frac{u}{2}b+(1-\frac{u}{2})a\right)^{2p}}du \\
=\;& \frac{1}{2}\int_0^1 v^{\alpha p}\left(\frac{a+b}{2}\right)^{-2p}\left[1-v\left(\frac{b-a}{b+a}\right)\right]^{-2p}dv \\
=\;& \left(\frac{a+b}{2}\right)^{-2p}\frac{1}{2(\alpha p+1)}\;{}_2F_1\left(2p,\alpha p+1;\alpha p+2;\tfrac{b-a}{b+a}\right) \\
=\;& C_3(\alpha)
\end{aligned}
\tag{2.17}
$$

and

$$
\begin{aligned}
&\int_{\frac{1}{2}}^1 \frac{(2u-1)^{\alpha p}}{(ub+(1-u)a)^{2p}}du \\
=\;& \int_0^{\frac{1}{2}} \frac{(1-2u)^{\alpha p}}{(ua+(1-u)b)^{2p}}du \\
=\;& \frac{1}{2}\int_0^1 \frac{(1-v)^{\alpha p}}{\left(\frac{v}{2}a+(1-\frac{v}{2})b\right)^{2p}}dv \\
=\;& \frac{1}{2}\int_0^1 (1-v)^{\alpha p}\,b^{-2p}\left(1-\frac{v}{2}\left(1-\frac{a}{b}\right)\right)^{-2p}dv \\
=\;& b^{-2p}\frac{1}{2(\alpha p+1)}\;{}_2F_1\left(2p,1;\alpha p+2;\tfrac{1}{2}\left(1-\tfrac{a}{b}\right)\right) \\
=\;& C_4(\alpha).
\end{aligned}
\tag{2.18}
$$

If we use (2.15), (2.16), (2.17) and (2.18) in (2.14), we have (2.13). This completes the proof. □

Corollary 2.6. *In Theorem 2.5:*

(1) *If we take* $\alpha=1$ *we have the following Hermite-Hadamard-Fejér inequality for harmonically quasi-convex functions which is related to the right-hand side of* (1.6)*:*

$$
\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}\int_a^b \frac{g(x)}{x^2}dx-\int_a^b \frac{f(x)g(x)}{x^2}dx\right| \\
\leq\;& \frac{\|g\|_\infty (b-a)^2}{2^{\frac{1}{q}+1}}\left(\sup\left\{|f'(a)|^q,|f'(b)|^q\right\}\right)^{\frac{1}{q}}\left[C_3^{\frac{1}{p}}(1)+C_4^{\frac{1}{p}}(1)\right],
\end{aligned}
$$

(2) *If we take* $g(x) = 1$ *we have following Hermite-Hadamard type inequality for harmonically quasi-convex functions in fractional integral forms which is related to the right-hand side of* (1.5)*:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2} \left(\frac{ab}{b-a} \right)^{\alpha} \left\{ \begin{array}{c} J^{\alpha}_{1/a-} (f \circ h)(1/b) \\ +J^{\alpha}_{1/b+} (f \circ h)(1/a) \end{array} \right\} \right|$$
$$\leq \frac{ab(b-a)}{2^{\frac{1}{q}+1}} \left(\sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \left[C_3^{\frac{1}{p}}(\alpha) + C_4^{\frac{1}{p}}(\alpha) \right],$$

(3) *If we take* $\alpha = 1$ *and* $g(x) = 1$ *we have the following Hermite-Hadamard type inequality for harmonically quasi-convex functions which is related to the right-hand side of* (1.4)*:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{ab}{b-a} \int_a^b \frac{f(x)}{x^2} dx \right|$$
$$\leq \frac{ab(b-a)}{2^{\frac{1}{q}+1}} \left(\sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right)^{\frac{1}{q}} \left[C_3^{\frac{1}{p}}(1) + C_4^{\frac{1}{p}}(1) \right].$$

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