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Complex Dynamic Behaviors of an Impulsively Controlled Predator-prey System with Watt-type Functional Response

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Abstract. In this paper, we consider a discrete predator-prey system with Watt-type functional response and impulsive controls. First, we find sufficient conditions for stability of a prey-free positive periodic solution of the system by using the Floquet theory and then prove the boundedness of the system. In addition, a condition for the permanence of the system is also obtained. Finally, we illustrate some numerical examples to substantiate our theoretical results, and display bifurcation diagrams and trajectories of some solutions of the system via numerical simulations, which show that impulsive controls can give rise to various kinds of dynamic behaviors.

1. Introduction

Impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, biotechnology, pharmacokinetics, industrial robotics, and so forth. Recently, impulsive differential equations are significantly used to study the mathematical properties of an impulsive predator-prey system in population dynamics. Especially, controlling the population of insect pest(prey) has become an increasingly complex issue([1, 2, 3, 13, 19, 26, 27, 38]).

There are many methods that can be used to help manage insect pests. One of important methods for pest control is chemical control. Pesticides are useful because they quickly kill a significant portion of a pest population. However, there are many deleterious effects associated with the use of chemicals that need to be reduced or eliminated. These include human illness associated with pesticide applications, insect resistance to insecticides, contamination of soil and water, and diminution of

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biodiversity. As a result, it is required that we should combine pesticide efficacy tests with other ways of control. For the reason, biological control is presented as one of important alternatives. It is defined as the reduction in pest populations from the actions of other living organisms, often called natural enemies or beneficial species. Virtually all pests have some natural enemies, and the key to successful pest control is to identify the pest and its natural enemies and releasing them at fixed times for pest control. Spraying pesticide can affect natural enemies. But, in some cases, pesticides can be successfully integrated into a biological control strategy with little harming natural enemies([4, 6, 7, 8, 9, 23, 29, 30, 31, 32, 33, 35, 36, 37, 38]).

On the other hand, the relationship between pest and natural enemy can be expressed as a predator(natural enemy)-prey(pest) system mathematically as follows;

$$\begin{cases} x'(t) = ax(t)(1 - \frac{x(t)}{K}) - yP(x, y), \\ y'(t) = -dy(t) + eyP(x, y), \\ x(0) = x_0, y(0) = y_0, \end{cases} \tag{1.1}$$

where $x(t), y(t)$ represent the population density of the prey and the predator at time t, respectively. Usually, K is called the carrying capacity of the prey. The constant a is called intrinsic growth rate of the prey. The constants e, d are the conversion rate and the death rate of the predator, respectively. The function P is the functional response of the predator which means prey eaten per predator per unit of time.

Many scholars have studied such predator-prey systems with a functional response, such as Holling-type [20, 21, 25], Monod-type [20, 21, 28] and Beddington-type [15, 16, 18], etc. One of well-known function response is of Watt-type, proposed by [34]. The predator-prey system with Watt-type is described as the following differential equation:

$$\begin{cases} x'(t) = ax(t)(1 - \frac{x(t)}{K}) - b(1 - \exp(\frac{-cx(t)}{y(t)^\gamma}))y(t), \\ y'(t) = -dy(t) + e(1 - \exp(\frac{-cx(t)}{y(t)^\gamma}))y(t)), \end{cases} \tag{1.2}$$

where b is the maximum number of prey that can be eaten by a predator per unit time. The constant c is the constant for the decrease in motivation to hunt and γ is a nonnegative constant.

In order to accomplish the aims discussed above, we need to consider the following impulsive differential equation:

$$\begin{cases} x'(t) = ax(t)(1 - \frac{x(t)}{K}) - b(1 - \exp(\frac{-cx(t)}{y(t)^\gamma}))y(t), \ t \neq nT, \\ y'(t) = -dy(t) + e(1 - \exp(\frac{-cx(t)}{y(t)^\gamma}))y(t), \ t \neq nT, \\ x(t^+) = (1 - p)x(t), \ t = nT, \\ y(t^+) = y(t) + q, \ t = nT, \\ (x(0^+), y(0^+)) = (x_0, y_0), \end{cases} \tag{1.3}$$

where T is the period of the impulsive immigration or stock of the predator, $0 \leq p < 1$ presents the fraction of prey which die due to harvesting or pesticides etc and q is the size of immigration or stock of the predator.

In fact, impulsive control methods can be found in almost every field of applied sciences. Theoretical investigations and its application analysis can be found in Bainov and Simeonov[10, 11, 12], Lakshmikantham et al.[22]. Moreover, impulsive differential equations dealing with biological population dynamics are literate in [14, 4, 5, 17, 23, 24, 31, 32, 37]. Especially, the authors in [32] have studied Watt-type predator-prey systems with impulsive perturbations, considering only the impulsive control parameter q in system (1.3) with $p = 0$.

The main purpose of this paper is to investigate the dynamics of system (1.3). In the next section, we introduce some notations which are used in this paper. We study qualitative properties of system (1.3) in Section 3. In fact, we find conditions for the stability of a prey-free periodic solution and for the permanence of system (1.3) by using the Floquet theory. In Section 4, we numerically investigate the effects of impulsive perturbations on inherent oscillation by illustrating bifurcation diagrams and trajectories of solutions of the system.

2. Definitions and Basic Lemmas

In the section, we give some notations, definitions and Lemmas which will be useful for our main results.

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^2_+ = \{\mathbf{x} = (x(t), y(t)) \in \mathbb{R}^2 : x(t), y(t) \geq 0\}$. Denote $\mathbb{N}$ the set of all of nonnegative integers and $f = (f_1, f_2)^T$ the right hand side of system (1.3).

Let $V : \mathbb{R}_+ \times \mathbb{R}^2_+ \to \mathbb{R}_+$, then V is said to be in *a class* V_0 if

$$\begin{aligned} &(1)\ V \text{ is continuous on } (nT, (n+1)T] \times \mathbb{R}^2_+, \\ &\qquad \text{and } \lim_{\substack{(t,\mathbf{y}) \to (nT,\mathbf{x}) \\ t > nT}} V(t, \mathbf{y}) = V(nT^+, \mathbf{x}) \text{ exists.} \\ &(2)\ V \text{ is locally Lipschitzian in } \mathbf{x}. \end{aligned}$$

Definition 2.1. Let $V \in V_0, (t, \mathbf{x}) \in (nT, (n+1)T] \times \mathbb{R}^2_+$. The upper right derivatives of $V(t, \mathbf{x})$ with respect to the impulsive differential system (1.3) is defined as

$$D^+V(t, \mathbf{x}) = \limsup_{h \to 0+} \frac{1}{h}[V(t + h, \mathbf{x} + hf(t, \mathbf{x})) - V(t, \mathbf{x})].$$

Definition 2.2. System (1.3) is permanent if there exist $M \geq m > 0$ such that, for any solution $(x(t), y(t))$ of system (1.3) with $\mathbf{x}_0 = (x_0, y_0) > 0$,

$$m \leq \lim_{t\to\infty} \inf x(t) \leq \lim_{t\to\infty} \sup x(t) \leq M \text{ and } m \leq \lim_{t\to\infty} \inf y(t) \leq \lim_{t\to\infty} \sup y(t) \leq M.$$

Remark (1) A solution of system (1.3) is a piecewise continuous function $\mathbf{x} : \mathbb{R}_+ \to \mathbb{R}^2_+$, $\mathbf{x}(t)$ is continuous on $(nT, (n+1)T], n \in \mathbb{N}$ and $\mathbf{x}(nT^+) = \lim_{t \to nT^+} \mathbf{x}(t)$ exists.
(2) The smoothness properties of f guarantees the global existence and uniqueness of solution of system (1.3). (See [22] for the details).

The following lemma is obvious.

Lemma 2.3. *Let $\mathbf{x}(t) = (x(t), y(t))$ be a solution of system (1.3).*
(1) *If $\mathbf{x}(0^+) \geq 0$ then $\mathbf{x}(t) \geq 0$ for all $t \geq 0$.*
(2) *If $\mathbf{x}(0^+) > 0$ then $\mathbf{x}(t) > 0$ for all $t \geq 0$.*

We will use the following important comparison theorem on impulsive differential equations [22].

Lemma 2.4.(Comparison theorem) *Suppose $V \in V_0$ and*

$$\begin{cases} D^+V(t, \mathbf{x}) \leq g(t, V(t, \mathbf{x})), \ t \neq nT, \\ V(t, \mathbf{x}(t^+)) \leq \psi_n(V(t, \mathbf{x})), \ t = nT, \end{cases} \tag{2.1}$$

$g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is continuous on $(nT, (n+1)T] \times \mathbb{R}_+$ and for $u(t) \in \mathbb{R}_+, n \in \mathbb{N}$, $\lim_{(t,y) \to (nT^+, u)} g(t, y) = g(nT^+, u)$ exists, $\psi_n : \mathbb{R}_+ \to \mathbb{R}_+$ is non-decreasing. Let $r(t)$ be the maximal solution of the scalar impulsive differential equation

$$\begin{cases} u'(t) = g(t, u(t)), \ t \neq nT, \\ u(t^+) = \psi_n(u(t)), \ t = nT, \\ u(0^+) = u_0, \end{cases} \tag{2.2}$$

existing on $[0, \infty)$. Then $V(0^+, \mathbf{x}_0) \leq u_0$ implies that $V(t, \mathbf{x}(t)) \leq r(t), t \geq 0$, where $\mathbf{x}(t)$ is any solution of equation (2.1).

Similar result can be obtained when all conditions of the inequalities in Lemma 2.4 are reversed. Note that if we have some smoothness conditions of $g(t, u(t))$ to guarantee the existence and uniqueness of the solutions for equation (2.2), then $r(t)$ is exactly the unique solution of equation (2.2).

Now, we give the basic properties of the following impulsive differential equation.

$$\begin{cases} y'(t) = -dy(t), \ t \neq nT, \\ y(t^+) = y(t) + q, \ t = nT, \\ y(0^+) = y_0. \end{cases} \tag{2.3}$$

Then we can easily obtain the following results.

Lemma 2.5. (1) *$y^*(t) = \dfrac{q \exp(-d(t - nT))}{1 - \exp(-dT)}$, $t \in (nT, (n+1)T]$, $n \in \mathbb{N}$ and $y^*(0^+) = \dfrac{q}{1 - \exp(-dT)}$ is a positive periodic solution of system (2.3).*

(2) $y(t) = \left(y(0^+) - \dfrac{q}{1-\exp(-dT)}\right)\exp(-dt) + y^*(t)$ *is the solution of system (2.3) with* $y_0 \geq 0$, $t \in (nT, (n+1)T]$ *and* $n \in \mathbb{N}$.
(3) *All solutions* $y(t)$ *of system (1.3) with* $y_0 \geq 0$ *tend to* $y^*(t)$. *i.e.,* $|y(t) - y^*(t)| \to 0$ *as* $t \to \infty$.

It is from Lemma 2.4 that the general solution $y(t)$ of equation (2.3) can be synchronized with the positive periodic solution $y^*(t)$ of equation (2.3) for sufficiently large t and we can obtain the complete expression for the prey-free periodic solution of system (1.3)

$$(0, y^*(t)) = \left(0, \frac{q\exp(-d(t-nT))}{1-\exp(-dT)}\right) \text{ for } t \in (nT, (n+1)T].$$

To study the stability of the prey-free periodic solution as a solution of system (1.3) we present the Floquet theory for the linear T-periodic impulsive equation:

$$\begin{cases} x'(t) & = A(t)x(t), t \neq \tau_k, t \in \mathbb{R}, \\ x(t^+) & = x(t) + B_k x(t), t = \tau_k, k \in \mathbb{Z}. \end{cases} \tag{2.4}$$

We introduce the following conditions:

(H_1) $A(\cdot) \in PC(\mathbb{R}, C^{n\times n})$ and $A(t+T) = A(t)(t \in \mathbb{R})$, where $PC(\mathbb{R}, C^{n\times n})$ is the set of all piecewise continuous matrix functions which is left continuous at $t = \tau_k$, and $C^{n\times n}$ is the set of all $n \times n$ matrices.

(H_2) $B_k \in C^{n\times n}$, $\det(E + B_k) \neq 0, \tau_k < \tau_{k+1} (k \in \mathbb{Z})$.

(H_3) There exists a $q \in \mathbb{N}$ such that $B_{k+q} = B_k, \tau_{k+q} = \tau_k + T(k \in \mathbb{Z})$.

Let $\Phi(t)$ be a fundamental matrix of equation (2.4), then there exists unique nonsingular matrix $M \in C^{n\times n}$ such that

$$\Phi(t+T) = \Phi(t)M(t \in \mathbb{R}). \tag{2.5}$$

By equality (2.5) there corresponds to the fundamental matrix $\Phi(t)$ and the constant matrix M which we call the monodromy matrix of equation (2.4) (corresponding to the fundamental matrix of $\Phi(t)$).

All monodromy matrices of equation (2.4) are similar and have the same eigenvalues. The eigenvalues $\mu_1, \cdots, \mu_n$ of the monodromy matrices are called the Floquet multipliers of equation (2.4).

Lemma 2.6.([10])(Floquet theory) *Let conditions* $(H_1)-(H_3)$ *hold. Then the linear* T*-periodic impulsive equation (2.4) is*
(1) *stale if and only if all multipliers* $\mu_j (j = 1, \cdots, n)$ *of equation (2.4) satisfy the inequality* $|\mu_j| \leq 1$, *and moreover, to those* μ_j *for which* $|\mu_j| = 1$, *there correspond simple elementary divisors;*
(2) *asymptotically stable if and only if all multipliers* $\mu_j (j = 1, \cdots, n)$ *of equation (2.4) satisfy the inequality* $|\mu_j| < 1$;
(3) *unstable if* $|\mu_j| > 1$ *for some* $j = 1, \cdots, n$.

3. Main Results

Now, we present a condition which guarantees locally asymptotical stability of the prey-free periodic solution $(0, y^*(t))$.

Theorem 3.1. *If*

$$aT - bc\left(\frac{q}{1-\exp(-dT)}\right)^{1-\gamma}(1-\exp(-d(1-\gamma)T))\frac{1}{d(1-\gamma)} < \ln\frac{1}{1-p}$$

then $(0, y^*(t))$ *is locally asymptotically stable.*

Proof. The local stability of the periodic solution $(0, y^*(t))$ of system (1.3) may be determined by considering the behavior of small amplitude perturbations of the solution. Let $(x(t), y(t))$ be any solution of system (1.3). Define $x(t) = u(t), y(t) = y^*(t) + v(t)$. Then they may be written as

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = \Phi(t)\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}, 0 \le t \le T, \tag{3.1}$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi}{dt} = \begin{pmatrix} a - bcy^*(t) & 0 \\ ecy^*(t) & -d \end{pmatrix}\Phi(t) \tag{3.2}$$

and $\Phi(0) = I$, the identity matrix. The linearization of the third and fourth equation of system (1.3) becomes

$$\begin{pmatrix} u(nT^+) \\ v(nT^+) \end{pmatrix} = \begin{pmatrix} 1-p & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u(nT) \\ v(nT) \end{pmatrix}. \tag{3.3}$$

Note that all eigenvalues of $S = \begin{pmatrix} 1-p & 0 \\ 0 & 1 \end{pmatrix}\Phi(T)$ are $\mu_1 = \exp(-dT) < 1$ and $\mu_2 = (1-p)\exp(\int_0^T a - bcy^*(t)^{1-\gamma}dt)$. Since

$$\int_0^T y^*(t)^{1-\gamma}dt = \left(\frac{q}{1-\exp(-dT)}\right)^{1-\gamma}(\exp(d(1-\gamma)T) - 1)\frac{1}{d(1-\gamma)}$$

we have

$$\mu_2 = (1-p)\exp\left(aT - bc\left(\frac{q}{1-\exp(-dT)}\right)^{1-\gamma}(1-\exp(-d(1-\gamma)T))\frac{1}{d(1-\gamma)}\right)$$

The condition $|\mu_2| < 1$ is equivalent to the equation

$$aT - bc\left(\frac{q}{1-\exp(-dT)}\right)^{1-\gamma}(1-\exp(-d(1-\gamma)T))\frac{1}{d(1-\gamma)} < \ln\frac{1}{1-p}.$$

According to Lemma , $(0, y^*(t))$ is locally asymptotically stable. $\square$

We show that all solutions of system (1.3) are uniformly ultimately bounded.

Proposition 3.2. *There is an $M > 0$ such that $x(t), y(t) \leq M$ for all t large enough, where $(x(t), y(t))$ is a solution of system (1.3).*

Proof. Let $\mathbf{x}(t) = (x(t), y(t))$ be a solution of system (1.3) and let $V(t, \mathbf{x}) = ex(t) + by(t)$. Then $V \in V_0$, if $t \neq n\tau$

$$D^+V + \beta V = -\frac{ea}{K}x(t)^2 + e(a+\beta)x(t) + b(\beta - d)y(t), \tag{3.4}$$

and $V(n\tau^+) = V(n\tau) + q$. Clearly, the right hand of (3.4), is bounded when $0 < \beta < d$. Thus we can choose $0 < \beta_0 < d$ and $M_0 > 0$ such that

$$\begin{cases} D^+V \leq -\beta_0 V + M_0, t \neq n\tau, \\ V(n\tau^+) = V(n\tau) + q. \end{cases} \tag{3.5}$$

From Lemma 2.4, we can obtain that

$V(t) \leq (V(0^+) - \frac{M_0}{\beta_0})\exp(-\beta_0 t) + \frac{p(1-\exp(-(n+1)\beta_0\tau))}{1-\exp(-\beta_0\tau)}\exp(-\beta_0(t - n\tau)) + \frac{M_0}{\beta_0}$ for $t \in (n\tau, (n+1)\tau]$.

Therefore, $V(t)$ is bounded by a constant for sufficiently large t. Hence there is an $M > 0$ such that $x(t) \leq M, y(t) \leq M$ for a solution $(x(t), y(t))$ with all t large enough. $\square$

Theorem 3.3. *System (1.3) is permanent if*

$$aT - bc\Big(\frac{q}{1-\exp(-dT)}\Big)^{1-\gamma}(1-\exp(-d(1-\gamma)T))\frac{1}{d(1-\gamma)} > \ln\frac{1}{1-p}.$$

Proof. Let $(x(t), y(t))$ be any solution of system (1.3) with $(x_0, y_0) > 0$. From Proposition 3.2, we may assume that $x(t) \leq M$, $y(t) \leq M$, $t \geq 0$ and $M > (\frac{a}{bc})^{\frac{1}{1-\gamma}}$. Let $m_2 = \frac{p\exp(-dT)}{1-\exp(-dT)} - \epsilon_2$, $\epsilon_2 > 0$. From Lemma 2.4, clearly we have $y(t) \geq m_2$ for all t large enough. Now we shall find an $m_1 > 0$ such that $x(t) \geq m_1$ for all t large enough. We will do this in the following two steps.

(Step 1) Since

$$aT - bc\Big(\frac{q}{1-\exp(-dT)}\Big)^{1-\gamma}(1-\exp(-d(1-\gamma)T))\frac{1}{d(1-\gamma)} > \ln\frac{1}{1-p},$$

we can choose $m_3 > 0$, $\epsilon_1 > 0$ small enough such that $\delta \equiv \dfrac{qm_3}{m_3 + bm_2^\gamma} < d$ and $R = (1-p)\exp\Big(\int_0^T a - \frac{a}{K}m_3 - bc(u^*(t) + \epsilon_1)^{1-\gamma}dt\Big) > 1$, where $u^*(t) =$

$\dfrac{q\exp((-d+\delta)(t-nT))}{1-\exp((-d+\delta)T)}$, $t \in (nT,(n+1)T]$ and $n \in \mathbb{N}$. Now we can prove that $x(t) < m_3$ cannot hold for all t. Otherwise, we can get $y'(t) \le y(t)(-d+\delta)$. By Lemma 2.4, we have $y(t) \le u(t)$ and $u(t) \to u^*(t)$, $t \to \infty$, where $u(t)$ is the solution of

$$\begin{cases} u'(t) = (-d+\delta)u(t), \ t \neq nT, \\ u(t^+) = u(t) + q, \ t = nT \\ u(0^+) = y_0. \end{cases} \tag{3.6}$$

Then there exists $T_1 > 0$ such that $y(t) \le u(t) \le u^*(t) + \epsilon_1$. Since $1 - \exp(\frac{-cx(t)}{y(t)^\gamma}) \le \frac{cx(t)}{y(t)^\gamma}$ we obtain that

$$\begin{aligned} x'(t) &= x(t)(a - \frac{a}{K}x(t)) - b\Big(1 - \exp(\frac{-cx(t)}{y(t)^\gamma})\Big)y(t) \\ &\ge x(t)\Big(a - \frac{a}{K}m_3 - bcy(t)^{1-\gamma}\Big) \\ &\ge x(t)\Big(a - \frac{a}{K}m_3 - bc(u^*(t)+\epsilon_1)^{1-\gamma}\Big) \text{ for } t \ge T_1 \text{ and } t \neq nT. \end{aligned}$$

Let $N_1 \in \mathbb{N}$ and $N_1 T \ge T_1$. We have, for $n \ge N_1$

$$\begin{cases} x'(t) & \ge x(t)\Big(a - \frac{a}{K}m_3 - bc(u^*(t)+\epsilon_1)^{1-\gamma}\Big), t \neq nT, \\ x(t^+) & = (1-p)x(t), t = nT. \end{cases} \tag{3.7}$$

Integrating (3.7) on $(nT,(n+1)T] (n \ge N_1)$, we obtain

$$x((n+1)T) \ge x(nT^+)\exp\Big(\int_{nT}^{(n+1)T} a - \frac{a}{K}m_3 - bc(u^*(t)+\epsilon_1)^{1-\gamma}dt\Big) = x(nT)R.$$

Then $x((N_1+k)T) \ge x(N_1T)R^k \to \infty$ as $k \to \infty$ which is a contradiction. Hence there exists a $t_1 > 0$ such that $x(t_1) \ge m_3$.

(Step 2) If $x(t) \ge m_3$ for all $t \ge t_1$, then we are done. If not, we may let $t^* = \inf_{t>t_1}\{x(t) < m_3\}$. Then $x(t) \ge m_3$ for $t \in [t_1, t^*]$. If $t^* \neq nT$ for all $n \in \mathbb{N}$ and $x(t)$ is continuous then $x(t^*) = m_3$. If $t^* = n_0T$ for some $n_0 \in \mathbb{N}$, let $t^{**} = t^* - \epsilon_0$, where ϵ_0 is small enough, then $x(t^{**}) \ge m_3$. Without loss of generality, we may assume that $t^* \neq nT$ for all $n \in \mathbb{N}$. Suppose that $t^* \in [n_1T,(n_1+1)T)$ for some $n_1 \in \mathbb{N}$. Select $n_2, n_3 \in \mathbb{N}$ such that $n_2T > \dfrac{\log(\frac{\epsilon_1}{M+p})}{-d+\delta}$ and $(1-p)^{n_2+1}\exp((n_2+1)\sigma T)R^{n_3} > 1$, where $\sigma = a - \frac{a}{K}m_3 - bcM^{1-\gamma} < 0$. Let $T' = n_2T + n_3T$. There are two possible cases for $t \in (t^*, (n_1+1)T']$.

Case 1) $x(t) < m_3$ for all $t \in (t^*, (n_1+1)T]$.

In this case we will show that there exists $t_2 \in [(n_1+1)T, (n_1+1)T+T']$ such that $x_2(t_2) \ge m_3$. Suppose not. i.e., $x(t) < m_3$, for all $t \in [(n_1+1)T, (n_1+1+n_2+n_3)T]$.

Then $x(t) < m_3$ for all $t \in (t^*, (n_1+1+n_2+n_3)T]$. By (3.6) with $u((n_1+1)T^+) = y((n_1+1)T^+)$, we have

$$u(t) = \left(u((n_1+1)T^+) - \frac{q}{1-\exp(-d+\delta)}\right)\exp((-d+\delta)(t-(n_1+1)T)) + u^*(t)$$

for $t \in (nT, (n+1)T], n_1+1 \le n \le n_1+n_2+n_3$. So we get $|u(t) - u^*(t)| \le (M + q)\exp((-d+\delta)n_2T) < \epsilon_1$ and $y(t) \le u(t) \le u^*(t) + \epsilon_1$ for $t \in [(n_1+1+n_2)T, (n_1+1+n_2+n_3)T]$, which implies (3.7) holds on $[(n_1+1+n_2)T, (n_1+1+n_2+n_3)T]$. As in step 1, we have

$$x((n_1+1+n_2+n_3)T) \ge x_2((n_1+1+n_2)T)R^{n_3}.$$

Since $y(t) \le M$, for $t \in (t^*, (n_1+1+n_2)T]$, we obtain

$$\begin{cases} x'(t) & \ge x(t)\Big(a - \dfrac{a}{K}m_3 - bcM^{1-\gamma}\Big), t \ne nT, \\ x(t^+) & = (1-p)x(t), t = nT. \end{cases} \tag{3.8}$$

Integrating it on $[t^*, (n_1+1+n_2)T]$ we get

$$x((n_1+1+n_2)T) \ge m_3(1-p)^{n_2+1}\exp(\sigma(n_2+1)T).$$

Thus $x((n_1+1+n_2+n_3)T) \ge m_3(1-p)^{n_2+1}\exp(\sigma(n_2+1)T)R^{n_3} > m_3$ which is a contradiction. Now, let $\bar{t} = \inf_{t>t^*}\{x(t) \ge m_3\}$. Then $x(t) \le m_3$ for $t^* \le t < \bar{t}$ and $x(\bar{t}) = m_3$. Thus (3.8) holds for $t \in [t^*, \bar{t})$. By the integration of it on $[t^*, t)(t^* \le t \le \bar{t})$, we can get that $x(t) \ge x(t^*)\exp(\sigma(t-t^*)) \ge m_3(1-p)^{1+n_2+n_3}\exp(\sigma(1+n_2+n_3)T) \equiv m_1$.

Case 2) There is a $t' \in (t^*, (n_1+1)T]$ such that $x_2(t') \ge m_3$.
Let $\hat{t} = \inf_{t>t^*}\{x(t) \ge m_3\}$. Then $x(t) \le m_3$ for $t \in [t^*, \hat{t})$ and $x(\hat{t}) = m_3$. Also, (3.8) holds for $t \in [t^*, \hat{t})$. Integrating the equation on $[t^*, t)(t^* \le t \le \hat{t})$, we can get that $x(t) \ge x(t^*)\exp(\sigma(t-t^*)) \ge m_3\exp(\sigma T) \ge m_1$.

Thus in both case the similar argument can be continued since $x(t) \ge m_3$ for some $t > t_1$. This completes the proof. $\square$

Remark Let $q_{\max} = (1-\exp(-dT))\big(\frac{d(1-\gamma)(aT+\ln(1-p))}{bc(1-\exp(-d(1-\gamma)T))}\big)^{\frac{1}{1-\gamma}}$. From Theorem 3.1 and Theorem 3.3, we know that the prey-free periodic solution is locally asymptotically stable if $q > q_{\max}$ and otherwise, the prey and predator can coexist. Thus $q_{\max}$ plays a role of a critical value that discriminates between stability and permanence.

4. Numerical Simulation

In this section, we will study dynamic behaviors of system (1.3) by means of numerical simulation because the continuous system (1.3) cannot be solved explicitly. Especially, we investigate the influence of impulsive perturbations numerically.

For this, we fix the parameters as follows:

$$a = 4.0, K = 2.0, b = 1.0, c = 5.5, d = 0.2, e = 9.0, \gamma = 0.4, p = 0.2. \tag{4.1}$$

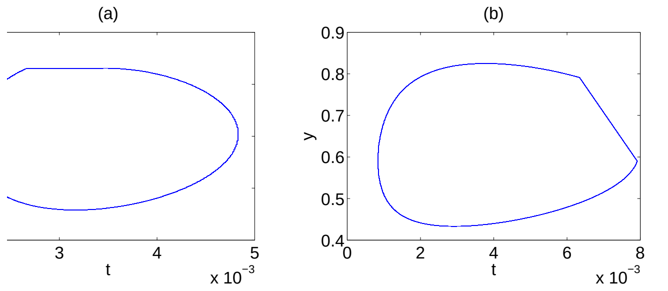

Figure 1: Dynamical behavior of system (1.3). (a) Phase portrait of a T-period solution for $q = 0$. (b) Phase portrait of a T-period solution for $q = 0.2$.

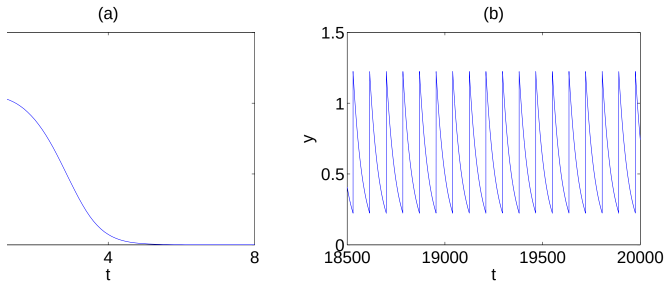

Figure 2: Dynamical behavior of system (1.3) with $q = 1.0$. (a) The trajectory of x is plotted. (b) The trajectory of y is plotted.

From Figure 1(a), we can figure out that there exists a limit cycle of system (1.3) when $q = 0$. It follows from Theorem 3.1 that the prey-free periodic solution $(0, y^*(t))$ is locally asymptotically stable provided that $q > q_{\max} = 0.8517$. A typical prey-free periodic solution of system (1.3) is exhibited in the Figure 2(a) and (b), where we observe how the variable $y(t)$ oscillates in a stable cycle. In contrast, the prey $x(t)$ rapidly decreases to zero. On the other hand, if the amount q of releasing species is smaller than $q_{\max}$, then prey and predator can coexist on a stable positive periodic solution (Figure 1(b)) and system (1.3) can be permanent which follows from Theorem 3.1. In Figure 3, we display a bifurcation diagram for prey and predator populations as q increasing from 0 to 1 with initial value $\mathbf{x_0} = (1.0, 1.0)$. The resulting bifurcation diagram clearly shows that system (1.3) has rich dynamics

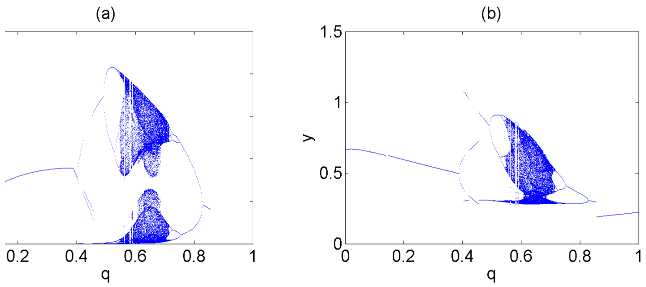

Figure 3: Bifurcation diagrams of system (1.3) for q when $0 < q < 1.0$. (a) x is plotted. (b) y is plotted.

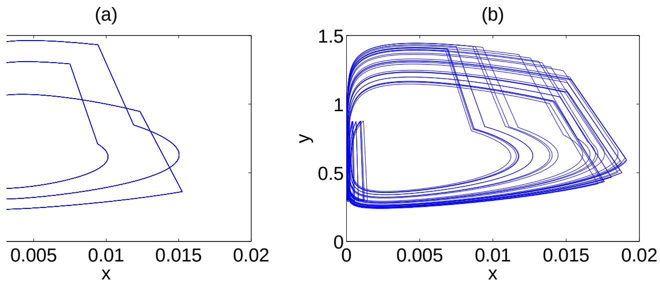

Figure 4: Coexistence of prey and predator when $q = 0.58$. (a) A solution with initial value $(1, 1)$. (b) A solution with initial value $(0.04, 0.6)$.

including cycles, periodic doubling bifurcation, chaotic bands, periodic window, period-halving bifurcation, etc. In Figure 3, solutions with period T are still stable for $q < 0.3885$. When $q > 0.3885$, they become unstable and solutions with period 3 begin to appear. Figure 3 illustrates an evidence for cascade of period doubling bifurcations leading to chaos when $0.4915 < q < 0.53$. We can capture a typical chaotic attractor when $q = 0.7$. (Figure 5(a)). We can also find that there exist sudden changes in Figure 3 when $q \approx 0.407, 0.4529, 0.58$ and 0.6071. Furthermore, they can lead to non-unique attractors. Specially, there exist two attractors when $q = 0.58$, shown in Figure 4. These results show that just one parameter can give rise to multiple attractors. Narrow periodic windows and wide periodic windows are intermittently scattered. At the end of the chaotic region, there is a cascade of period-halving bifurcation from chaos to one cycle. (see Figure 5). Periodic halving is the flip bifurcation in the opposite direction.

The results we obtain in this paper show that the impulsive perturbations have significant effects on the stable limit cycle of system (1.2) and make the dynamics of system (1.3) more complicated.

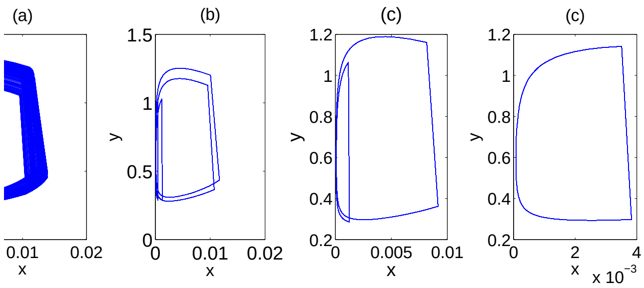

Figure 5: Period-halving bifurcation from chaos to cycle. (a) Chaotic attractor for $q = 0.7$. (b) Phase portrait of a $4T$-period solution for $q = 0.74$. (c) Phase portrait of a $2T$-period solution for $q = 0.78$. (d) Phase portrait of a T-period solution for $q = 0.84$.

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