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On the Order of Growth of Solutions to Complex Non-homogeneous Linear Differential Equations

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Abstract. In this paper, we study the order of growth of solutions to the non-homogeneous linear differential equation

$$f^{(k)}+A_{k-1}e^{az}f^{(k-1)}+\cdots+A_1e^{az}f'+A_0e^{az}f=F_1e^{az}+F_2e^{bz},$$

where $A_j\left(z\right)\ (\not\equiv 0)\ (j=0,1,\cdots,k-1)$, $F_j\left(z\right)\ (\not\equiv 0)\ (j=1,2)$ are entire functions and a, b are complex numbers such that $ab\left(a-b\right)\neq 0$.

1. Introduction and Statement of Results

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions [6, 12]. In what follows, we give the necessary notations and basic definitions.

Definition 1.1([6, 12]). Let f be a meromorphic function. Then the order $\sigma\left(f\right)$ of $f\left(z\right)$ is defined by

$$\sigma\left(f\right)=\limsup_{r\to+\infty}\frac{\log T\left(r,f\right)}{\log r},$$

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where $T(r,f)$ is the Nevanlinna characteristic function of f. If f is an entire function, then the order $\sigma(f)$ of $f(z)$ is defined by

$$\sigma(f)=\limsup_{r\to+\infty}\frac{\log T(r,f)}{\log r}=\limsup_{r\to+\infty}\frac{\log\log M(r,f)}{\log r},$$

where $M(r,f)=\max_{|z|=r}|f(z)|$.

Definition 1.2([6, 12]). Let f be a meromorphic function. Then the exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$\lambda(f)=\limsup_{r\to+\infty}\frac{\log N\left(r,\frac{1}{f}\right)}{\log r},$$

where $N\left(r,\frac{1}{f}\right)$ is the integrated counting function of zeros of $f(z)$ in $\{z:|z|\leq r\}$. Similarly, the exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}(f)=\limsup_{r\to+\infty}\frac{\log\overline{N}\left(r,\frac{1}{f}\right)}{\log r},$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z:|z|\leq r\}$.

In [10], Wang and Laine investigated the growth of solutions of some second order nonhomogeneous linear differential equation and obtained.

Theorem A([10]). *Let $A_j(z)(\not\equiv 0)$ $(j=0,1)$ and $F(z)$ be entire functions with* $\max\{\sigma(A_j)\ (j=0,1),\sigma(F)\}<1$, *and let a,b be complex constants that satisfy $ab\neq 0$ and $a\neq b$. Then every nontrivial solution f of the differential equation*

$$f''+A_1(z)e^{az}f'+A_0(z)e^{bz}f=F,$$

is of infinite order.

In this paper, we offer a higher-order result related to Theorem A. In fact we will prove the following results.

Theorem 1.1. *Let $A_j(z)$ $(\not\equiv 0)$ $(j=0,1,\cdots,k-1)$, $F_j(z)(\not\equiv 0)$ $(j=1,2)$ be entire functions with $\sigma(A_j)<1$ $(j=0,1,\cdots,k-1)$ and $\sigma(F_j)<1$ $(j=1,2)$, a and b be non-zero complex numbers such that $b=ca$ $(0<c<1)$. Suppose the following:*

(1) *there is exactly one s $(0\leq s\leq k-1)$ such that*

$$\sigma(A_s)>\max\{\sigma(A_j):j=0,1,\cdots,k-1 \text{ and } j\neq s\},$$

(2) *for any* τ *satisfying* $0 < \tau < \sigma(A_s)$, *there exists a subset* $H \subset (1, +\infty)$ *with infinite logarithmic measure, such that when* $|z| = r \in H$,

$$\log|A_s(z)| > r^{\tau}.$$

Then every solution f *of the differential equation*

$$f^{(k)} + A_{k-1}e^{az}f^{(k-1)} + \cdots + A_s e^{az} f^{(s)} + \cdots + A_1 e^{az} f' + A_0 e^{az} f = F_1 e^{az} + F_2 e^{bz} \tag{1.1}$$

has infinite order.

Corollary 1.1. *Let* $A_j(z)$ $(\not\equiv 0)$ $(j = 0, 1, \cdots, k-1)$, $F_j(z) (\not\equiv 0)$ $(j = 1, 2)$ *be entire functions with* $\sigma(A_j) < \frac{1}{2}$ $(j = 0, 1, \cdots, k-1)$ *and* $\sigma(F_j) < 1$ $(j = 1, 2)$, a *and* b *be non-zero complex numbers such that* $b = ca$ $(0 < c < 1)$. *Suppose that there is exactly one* s $(0 \le s \le k-1)$ *such that*

$$\sigma(A_s) > \max\{\sigma(A_j) : j = 0, 1, \cdots, k-1 \text{ and } j \ne s\}.$$

Then every solution f *of the differential equation* (1.1) *has infinite order.*

Remark 1.1. By the hypothesis of Corollary 1.1, we see that $0 < \sigma(A_s) < \frac{1}{2}$.

Theorem 1.2. *Under the hypotheses of Theorem 1.1, suppose further that* $\varphi(z) \not\equiv 0$ *is an entire function with finite order. Then every solution* f *of* (1.1) *satisfies*

$$\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = +\infty.$$

2. Preliminary Lemmas

Lemma 2.1([4, 9]). *Let* $P_1, P_2, \cdots, P_n$ $(n \ge 1)$ *be non-constant polynomials with degree* $d_1, d_2, \cdots, d_n$, *respectively, such that* $\deg(P_i - P_j) = \max\{d_i, d_j\}$ *for* $i \ne j$. *Let* $A(z) = \sum_{j=1}^{n} B_j(z) e^{P_j(z)}$ *where* $B_j(z) (\not\equiv 0)$ *are entire functions with* $\sigma(B_j) < d_j$. *Then* $\sigma(A) = \max_{1 \le j \le n}\{d_j\}$.

Lemma 2.2([8]). *Suppose that* $P(z) = (\alpha + i\beta) z^n + \cdots$ (α, β *are real numbers,* $|\alpha| + |\beta| \ne 0$) *is a polynomial with degree* $n \ge 1$, *that* $A(z)$ $(\not\equiv 0)$ *is an entire function with* $\sigma(A) < n$. *Set* $g(z) = A(z) e^{P(z)}$, $z = re^{i\theta}$, $\delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. *Then for any given* $\varepsilon > 0$, *there is a set* $E_1 \subset [0, 2\pi)$ *that has linear measure zero, such that for any* $\theta \in [0, 2\pi) \setminus (E_1 \cup E_2)$, *there is* $R > 0$, *such that for* $|z| = r > R$, *we have*

(i) *if* $\delta(P, \theta) > 0$, *then*

$$\exp\{(1 - \varepsilon)\delta(P, \theta) r^n\} \le \left|g\left(re^{i\theta}\right)\right| \le \exp\{(1 + \varepsilon)\delta(P, \theta) r^n\},$$

(ii) *if* $\delta(P,\theta)<0$, *then*

$$\exp\{(1+\varepsilon)\delta(P,\theta)r^n\}\le\left|g\left(re^{i\theta}\right)\right|\le\exp\{(1-\varepsilon)\delta(P,\theta)r^n\},$$

where $E_2=\{\theta\in[0,2\pi):\delta(P,\theta)=0\}$ *is a finite set.*

Lemma 2.3([5]). *Let* f *be a transcendental meromorphic function of finite order* σ. *Let* $\varepsilon>0$ *be a constant,* k *and* j *be integers satisfying* $k>j\ge 0$. *Then the following two statements hold:*

(i) *There exists a set* $E_3\subset(1,+\infty)$ *which has finite logarithmic measure, such that for all* z *satisfying* $|z|\notin E_3\cup[0,1]$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\le|z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.1}$$

(ii) *There exists a set* $E_4\subset[0,2\pi)$ *which has linear measure zero, such that if* $\theta\in[0,2\pi)\setminus E_4$, *then there is a constant* $R=R(\theta)>0$ *such that* (2.1) *holds for all* z *satisfying* $\arg z=\theta$ *and* $|z|\ge R$.

Lemma 2.4([11]). *Let* $f(z)$ *be an entire function and suppose that*

$$G(z):=\frac{\log^+\left|f^{(k)}(z)\right|}{|z|^\rho}$$

is unbounded on some ray $\arg z=\theta$ *with constant* $\rho>0$. *Then there exists an infinite sequence of points* $z_n=r_ne^{i\theta}$ $(n=1,2,\cdots)$, *where* $r_n\to+\infty$, *such that* $G(z_n)\to\infty$ *and*

$$\left|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\right|\le\frac{1}{(k-j)!}(1+o(1))\,r_n^{k-j},\ j<k$$

as $n\to+\infty$.

Lemma 2.5([11]). *Let* $f(z)$ *be an entire function with* $\sigma(f)=\sigma<+\infty$. *Suppose that there exists a set* $E_5\subset[0,2\pi)$ *which has linear measure zero, such that* $\log^+\left|f\left(re^{i\theta}\right)\right|\le Mr^\rho$ *for any ray* $\arg z=\theta\in[0,2\pi)\setminus E_5$, *where* M *is a positive constant depending on* θ, *while* ρ *is a positive constant independent of* θ. *Then* $\sigma(f)\le\rho$.

Lemma 2.6([3]). *Let* A_j $(j=0,1,\cdots,k-1)$, $F\not\equiv 0$ *be finite order meromorphic functions. If* $f(z)$ *is an infinite order meromorphic solution of the differential equation*

$$f^{(k)}+A_{k-1}f^{(k-1)}+\cdots+A_1f'+A_0f=F,$$

then f *satisfies*

$$\overline{\lambda}(f)=\lambda(f)=\sigma(f)=+\infty.$$

Remark 2.1([1, 2]). Let $h(z)$ be a transcendental entire function with order $\sigma(h) = \sigma < \frac{1}{2}$. Then there exists a subset $H \subset (1, +\infty)$ having infinite logarithmic measure, such that if $\sigma = 0$, then

$$\frac{\min\{\log|h(z)| : |z| = r\}}{\log r} \to +\infty \ \ (|z| = r \in H, \ r \to +\infty),$$

if $\sigma > 0$, then for any α $(0 < \alpha < \sigma)$,

$$\log|h(z)| > r^{\alpha} \ \ (|z| = r \in H, \ r \to +\infty).$$

3. Proof of the Theorems and Corollary

Proof of Theorem 1.1. We know that $b = ca$ $(0 < c < 1)$, then by (1.1) we get

$$e^{-az}f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_s f^{(s)} + \cdots + A_1 f' + A_0 f = F_1 + F_2 e^{(c-1)az}. \tag{3.1}$$

First, we prove that every solution f of (1.1) satisfies $\sigma(f) \geq 1$. We assume that $\sigma(f) < 1$. Obviously $\sigma\left(A_j f^{(j)}\right) < 1$ $(j = 0, 1, \cdots, k-1)$. Rewrite (3.1) as

$$F_2 e^{(c-1)az} - f^{(k)}e^{-az} = A_{k-1}f^{(k-1)} + \cdots + A_s f^{(s)} + \cdots + A_1 f' + A_0 f - F_1. \tag{3.2}$$

By (3.2) and the Lemma 2.1, we have

$$\begin{aligned} 1 &= \sigma\left\{F_2 e^{(c-1)az} - f^{(k)}e^{-az}\right\} \\ &= \sigma\left\{A_{k-1}f^{(k-1)} + \cdots + A_s f^{(s)} + \cdots + A_1 f' + A_0 f - F_1\right\} < 1. \end{aligned}$$

This is a contradiction. Hence, $\sigma(f) \geq 1$. Therefore f is a transcendental solution of equation (1.1).

Now, we prove that $\sigma(f) = +\infty$. Suppose that $\sigma(f) = \sigma < +\infty$. Set

$$\begin{aligned} &\alpha = \sigma(A_s), \\ &\beta = \max\{\sigma(A_j) : j = 0, 1, \cdots, k-1 \text{ and } j \neq s\}, \\ &\gamma = \max\{\sigma(F_j) : j = 1, 2\}. \end{aligned}$$

It is clear that, $0 \leq \beta < \alpha < 1$ and $0 \leq \gamma < 1$. Then for any given ε with $0 < \varepsilon < \min\{1-\alpha, 1-\beta, 1-\gamma\}$ and for sufficiently large r, we have

$$|A_s(z)| \leq \exp\left\{r^{\alpha+\varepsilon}\right\}, \tag{3.3}$$

$$|A_j(z)| \leq \exp\left\{r^{\beta+\varepsilon}\right\}, \ \ (j = 0, 1, \cdots, k-1 \text{ and } j \neq s), \tag{3.4}$$

$$|F_j(z)| \le \exp\left\{r^{\gamma+\varepsilon}\right\},\ \ (j=1,2). \tag{3.5}$$

By Lemma 2.2, there exists a set $E \subset [0,2\pi)$ of linear measure zero, such that whenever $\theta \in [0,2\pi)\setminus E$, then $\delta(az,\theta) \ne 0$. By Lemma 2.3, there exists a set $E_4 \subset [0,2\pi)$ which has linear measure zero, such that if $\theta \in [0,2\pi)\setminus E_4$, then there is a constant $R = R(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \ge R$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \le |z|^{k\sigma},\ 0 \le i < j \le k. \tag{3.6}$$

By the hypothesis of Theorem 1.1, we see that there exists a subset $H \subset (1,+\infty)$ having infinite logarithmic measure and $\tau > 0$ such that $0 \le \beta < \tau < \sigma(A_s)$,

$$|A_s(z)| > \exp\{r^{\tau}\},\ \ |z| = r \in H. \tag{3.7}$$

For any fixed $\theta \in [0,2\pi)\setminus(E \cup E_4)$, set

$$\delta_1 = \delta(-az,\theta),\ \ \delta_2 = \delta((c-1)az,\theta).$$

We can obtain

$$\delta_2 = (c-1)\delta(az,\theta) = (1-c)\delta(-az,\theta) = (1-c)\delta_1,$$

then $\delta_1 \ne 0$, $\delta_2 \ne 0$. We now discuss two cases separately.

Case 1. Suppose that $\delta_1 > 0$, then $\delta_2 > 0$. We can get

$$0 < \delta_2 = (1-c)\delta_1 < \delta_1.$$

By Lemma 2.2, for any given ε with $0 < 2\varepsilon < \min\left\{\frac{\delta_1-\delta_2}{\delta_1}, 1-\alpha, 1-\beta, 1-\gamma\right\}$, we obtain

$$\left|e^{-az}\right| \ge \exp\{(1-\varepsilon)\delta_1 r\}, \tag{3.8}$$

$$\left|e^{(c-1)az}\right| \le \exp\{(1+\varepsilon)\delta_2 r\} \tag{3.9}$$

for sufficiently large r. We now prove that $\log^+\left|f^{(k)}(z)\right|/|z|^{\gamma+\varepsilon}$ is bounded on the ray $\arg z = \theta$. We assume that $\log^+\left|f^{(k)}(z)\right|/|z|^{\gamma+\varepsilon}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 2.4, there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to +\infty$, and that

$$\frac{\log^+\left|f^{(k)}(z_m)\right|}{r_m^{\gamma+\varepsilon}} \to +\infty, \tag{3.10}$$

$$\left|\frac{f^{(j)}(z_m)}{f^{(k)}(z_m)}\right| \le \frac{1}{(k-j)!}(1+o(1))\,r_m^{k-j} \le 2r_m^{k-j},\ \ (j=0,1,\cdots,k-1), \tag{3.11}$$

for m is large enough. From (3.5) and (3.10), we get

$$\left|\frac{F_j(z_m)}{f^{(k)}(z_m)}\right| \to 0,\ (j=1,2), \tag{3.12}$$

for m is large enough. From (3.1), we obtain

$$\begin{aligned}\left|e^{-az}\right| \le{}& |A_{k-1}|\left|\frac{f^{(k-1)}}{f^{(k)}}\right| + \cdots + |A_s|\left|\frac{f^{(s)}}{f^{(k)}}\right| + \cdots + |A_1|\left|\frac{f'}{f^{(k)}}\right| \\ &+ |A_0|\left|\frac{f}{f^{(k)}}\right| + \left|\frac{F_1}{f^{(k)}}\right| + \left|\frac{F_2}{f^{(k)}}\right|\left|e^{(c-1)az}\right|.\end{aligned} \tag{3.13}$$

Substituting (3.3), (3.4), (3.8), (3.9), (3.11) and (3.12) into (3.13), we have

$$\begin{aligned}\exp\{(1-\varepsilon)\delta_1 r_m\} &\le \left|e^{-az_m}\right| \\ &\le |A_{k-1}(z_m)|\left|\frac{f^{(k-1)}(z_m)}{f^{(k)}(z_m)}\right| + \cdots + |A_s(z_m)|\left|\frac{f^{(s)}(z_m)}{f^{(k)}(z_m)}\right| \\ &\quad + \cdots + |A_0(z_m)|\left|\frac{f(z_m)}{f^{(k)}(z_m)}\right| + \left|\frac{F_1(z_m)}{f^{(k)}(z_m)}\right| + \left|\frac{F_2(z_m)}{f^{(k)}(z_m)}\right|\left|e^{(c-1)az_m}\right| \\ &\le M_0 r_m^{M_1}\exp\left\{r_m^{\alpha+\varepsilon}\right\}\exp\{(1+\varepsilon)\delta_2 r_m\},\end{aligned} \tag{3.14}$$

where $M_0>0$ and $M_1>0$ are some constants. By $0<\varepsilon<\frac{\delta_1-\delta_2}{2\delta_1}$ and (3.14), we can get

$$\exp\left\{\frac{(\delta_1-\delta_2)^2}{2\delta_1}r_m\right\} \le M_0 r_m^{M_1}\exp\left\{r_m^{\alpha+\varepsilon}\right\},$$

which is a contradiction because $\alpha+\varepsilon<1$. Therefore, $\log^+\left|f^{(k)}(z)\right|/|z|^{\gamma+\varepsilon}$ is bounded and we have

$$\left|f^{(k)}(z)\right| \le M\exp\left\{r^{\gamma+\varepsilon}\right\}$$

on the ray $\arg z=\theta$. By the same reasoning as in the proof of Lemma 3.1 in [7], we immediately conclude that

$$\begin{aligned}|f(z)| &\le (1+o(1))\,r^k\left|f^{(k)}(z)\right| \le (1+o(1))\,Mr^k\exp\left\{r^{\gamma+\varepsilon}\right\} \\ &\le M\exp\left\{r^{\gamma+2\varepsilon}\right\}\end{aligned}$$

on the ray $\arg z=\theta$.

Case 2. Suppose that $\delta_1<0$, then $\delta_2<0$. By Lemma 2.2, for any given ε with $0<2\varepsilon<\min\{1-\alpha,1-\beta,1-\gamma,\tau-\beta\}$, we obtain

$$\left|e^{-az}\right| \le \exp\{(1-\varepsilon)\delta_1 r\} < 1, \tag{3.15}$$

$$\left|e^{(c-1)az}\right| \leq \exp\left\{(1-\varepsilon)\,\delta_2 r\right\} < 1, \tag{3.16}$$

for sufficiently large r. We now prove that $\log^+\left|f^{(s)}(z)\right| / |z|^{\gamma+\varepsilon}$ is bounded on the ray $\arg z = \theta$. We assume that $\log^+\left|f^{(s)}(z)\right| / |z|^{\gamma+\varepsilon}$ is unbounded on the ray $\arg z = \theta$. Then by Lemma 2.4, there is a sequence of points $z_m = r_m e^{i\theta}$, such that $r_m \to +\infty$, and that

$$\frac{\log^+\left|f^{(s)}(z_m)\right|}{r_m^{\gamma+\varepsilon}} \to +\infty, \tag{3.17}$$

$$\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right| \leq \frac{1}{(s-j)!}\left(1+o(1)\right) r_m^{s-j} \leq 2r_m^{s-j},\ \ (j=0,1,\cdots,s-1), \tag{3.18}$$

for m is large enough. From (3.5) and (3.17), we get

$$\left|\frac{F_j(z_m)}{f^{(s)}(z_m)}\right| \to 0,\ \ (j=1,2) \tag{3.19}$$

for m is large enough. From (3.1), we obtain

$$-1 = \frac{1}{A_s}e^{-az}\frac{f^{(k)}}{f^{(s)}} + \sum_{j=s+1}^{k-1}\left(\frac{A_j}{A_s}\frac{f^{(j)}}{f^{(s)}}\right) + \sum_{j=0}^{s-1}\left(\frac{A_j}{A_s}\frac{f^{(j)}}{f^{(s)}}\right) - \frac{1}{A_s}\frac{F_1}{f^{(s)}} - \frac{1}{A_s}\frac{F_2}{f^{(s)}}e^{(c-1)az}. \tag{3.20}$$

By (3.7), we obtain

$$\left|\frac{1}{A_s(z_m)}\right| < \exp\left\{-r_m^{\tau}\right\}, \tag{3.21}$$

$$\left|\frac{A_j(z_m)}{A_s(z_m)}\right| < \exp\left\{r_m^{\beta+\varepsilon} - r_m^{\tau}\right\},\ \ (j=0,1,\cdots,k-1 \text{ and } j\neq s), \tag{3.22}$$

for m is large enough. Substituting (3.6), (3.15), (3.16), (3.18), (3.19), (3.21) and (3.22) into (3.20), we have

$$\begin{aligned} 1 \leq &\left|\frac{1}{A_s(z_m)}\right|\left|e^{-az_m}\right|\left|\frac{f^{(k)}(z_m)}{f^{(s)}(z_m)}\right| \\ &+ \sum_{j=s+1}^{k-1}\left(\left|\frac{A_j(z_m)}{A_s(z_m)}\right|\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\right) + \sum_{j=0}^{s-1}\left(\left|\frac{A_j(z_m)}{A_s(z_m)}\right|\left|\frac{f^{(j)}(z_m)}{f^{(s)}(z_m)}\right|\right) \\ &+ \left|\frac{1}{A_s(z_m)}\right|\left|\frac{F_1(z_m)}{f^{(s)}(z_m)}\right| + \left|\frac{1}{A_s(z_m)}\right|\left|\frac{F_2(z_m)}{f^{(s)}(z_m)}\right|\left|e^{(c-1)az_m}\right| \end{aligned}$$

$$\leq r_m^{k\sigma} \exp\{-r_m^\tau\} \exp\{(1-\varepsilon)\delta_1 r_m\} + (k-s-1) r_m^{k\sigma} \exp\left\{r_m^{\beta+\varepsilon} - r_m^\tau\right\}$$
$$+2sr_m^s \exp\left\{r_m^{\beta+\varepsilon} - r_m^\tau\right\} + o(1)\exp\{-r_m^\tau\}$$
$$+o(1)\exp\{-r_m^\tau\}\exp\{(1-\varepsilon)\delta_2 r_m\}. \tag{3.23}$$

Obviously,

$$\exp\{-r_m^\tau\} \to 0, \tag{3.24}$$

$$r_m^{k\sigma}\exp\{-r_m^\tau\}\exp\{(1-\varepsilon)\delta_1 r_m\} \to 0, \tag{3.25}$$

$$\exp\{-r_m^\tau\}\exp\{(1-\varepsilon)\delta_2 r_m\} \to 0, \tag{3.26}$$

$$r_m^{k\sigma}\exp\left\{r_m^{\beta+\varepsilon} - r_m^\tau\right\} \to 0, \tag{3.27}$$

$$r_m^s\exp\left\{r_m^{\beta+\varepsilon} - r_m^\tau\right\} \to 0 \tag{3.28}$$

as $r_m \to +\infty$ because $\beta+\varepsilon < \tau$. From (3.23)–(3.28), we obtain $1 \leq 0$ as $r_m \to +\infty$, which is a contradiction. Therefore, $\log^+ \left|f^{(s)}(z)\right| / |z|^{\gamma+\varepsilon}$ is bounded and we have

$$\left|f^{(s)}(z)\right| \leq M\exp\left\{r^{\gamma+\varepsilon}\right\}$$

on the ray $\arg z = \theta$. This implies, as in Case 1, that

$$|f(z)| \leq M\exp\left\{r^{\gamma+2\varepsilon}\right\}. \tag{3.29}$$

Therefore, for any given $\theta \in [0,2\pi)\setminus(E\cup E_4)$, we have got (3.29) on the ray $\arg z = \theta$, provided that r is large enough. Then by Lemma 2.5, we have $\sigma(f) \leq \gamma + 2\varepsilon < 1$, which is a contradiction. Hence every transcendental solution f of (1.1) must be of infinite order. □

Proof of Corollary 1.1. By the hypothesis of Corollary 1.1, we see that $0 < \sigma(A_s) < \frac{1}{2}$. Using Remark 2.1 and using the same reasoning as above, we can get $\sigma(f) = +\infty$. □

Proof of Theorem 1.2. Suppose that f is a solution of equation (1.1). Then, by Theorem 1.1 we have $\sigma(f) = +\infty$. Set $g(z) = f(z) - \varphi(z)$, $g(z)$ is an entire function and $\sigma(g) = \sigma(f) = +\infty$. Substituting $f = g + \varphi$ into (1.1), we obtain

$$g^{(k)} + A_{k-1}e^{az}g^{(k-1)} + \cdots + A_1e^{az}g' + A_0e^{az}g = D, \tag{3.30}$$

where

$$D = F_1e^{az} + F_2e^{bz} - \left[\varphi^{(k)} + A_{k-1}e^{az}\varphi^{(k-1)} + \cdots + A_1e^{az}\varphi' + A_0e^{az}\varphi\right].$$

We prove that $D \not\equiv 0$. In fact, if $D \equiv 0$, then

$$\varphi^{(k)} + A_{k-1}e^{az}\varphi^{(k-1)} + \cdots + A_1 e^{az}\varphi' + A_0 e^{az}\varphi = F_1 e^{az} + F_2 e^{bz}.$$

Hence $\sigma(\varphi) = +\infty$, which is a contradiction. Therefore $D \not\equiv 0$. We know that the functions A_j $(j = 0, 1, \cdots, k-1)$, D are of finite order. By Lemma 2.6 and (3.30) we have

$$\overline{\lambda}(g) = \lambda(g) = \sigma(g) = \sigma(f) = +\infty.$$

Therefore

$$\overline{\lambda}(f - \varphi) = \lambda(f - \varphi) = \sigma(f) = +\infty,$$

which completes the proof. □

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