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G-vector-valued Sequence Space Frames

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Abstract. G-vector-valued sequence space frames and g-Banach frames for Banach spaces are introduced and studied in this paper. Also, the concepts of duality mapping and β-dual of a BK-space are used to define frame mapping and synthesis operator of these frames, respectively. Finally, some results regarding the existence of g-vector-valued sequence space frames and g-Banach frames are obtained. In particular, it is proved that if X is a separable Banach space and Y is a Banach space with a Schauder basis, then there exist a Y-valued sequence space Y_v and a g-Banach frame for X with respect to Y and Y_v.

1. Introduction

Various generalizations of frames have been proposed recently. For example in [12], Sun introduced the concept of g-frames and showed that many basic properties of g-frames are similar to frames. Frames on Hilbert spaces were extended to Banach spaces by Gröchenig [8], and studied in [2, 3]. Motivated by the results of p-frames in [4, 7] and Banach frames in [1, 2, 3, 10], we define g-vector-valued sequence space frames and g-Banach frames on Banach spaces and show that many properties of these frames can be shared with X_d-frames and Banach frames by considering some extra conditions.

In this article X, Y and Z are Banach spaces and X^* is the dual space of X. If $x \in X$ and $x^* \in X^*$, we denote $x^*(x)$ by $\langle x, x^* \rangle$. Let $W(X)$ and $\phi(X)$ signify the space of all sequences in X and the space of all finite sequences in X, respectively. A sequence space in X is a linear subspace of $W(X)$. Let E be an X-valued sequence space. For $x \in E$ and $k \in \mathbb{N}$, we write x_k stands for the k-th term of x. For $x_0 \in X$ and $k \in \mathbb{N}$, we let $e^k(x_0)$ be the sequence $(0, 0, 0, ..., 0, x_0, 0, ...)$ with x_0 in the k-th position and let $e(x_0)$ be the sequence $(x_0, x_0, x_0, ...)$. An X-valued sequence space E is said to be normal if $\{y_k\} \in E$ whenever $\|y_k\|_X \leq \|x_k\|_X$ for all $k \in \mathbb{N}$ and

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$\{x_k\} \in E$. Suppose that the X-valued sequence space E is endowed with some linear topology τ. Then E is called a K-space if, for each $k \in \mathbb{N}$, the k-th coordinate mapping $p_k : E \to X$, $p_k(x) = x_k$, is continuous on E. In addition if (E, τ) is a Frechet (Banach) space, then E is called a FK (BK)-space. Now, suppose that E contains $\phi(X)$, then E is said to have property AK if $\sum_{k=1}^{n} e^k(x_k) \to x$ in E as $n \to \infty$ for every $x = \{x_k\} \in E$.
For an X-valued sequence space E, the β-dual of E is defined by

$$E^\beta = \{\{f_k\} : \quad f_k \in X^*, \quad \sum_{k=1}^{\infty} f_k(x_k) \ converges, \quad \forall \{x_k\} \in E\}.$$

If E is a BK-space, we define a norm on E^β by the formula

$$\|\{f_k\}\|_{E^\beta} = sup_{\|\{x_k\}\|_E \leq 1} |\sum_{k=1}^{\infty} f_k(x_k)|.$$

Lemma 1.1.([11]) *Let E be an X-valued sequence space which is a BK-space containing $\phi(X)$. Then for each $k \in \mathbb{N}$, the mapping $T_k : X \to E$, defined by $T_k(x) = e^k(x)$, is continuous.*

Theorem 1.2.([11]) *If E is a BK-space having property AK, then E^β and E^* are isometrically isomorphic.*

Note that if E has property AK, then for $x = \{x_k\} \in E$ and $f \in E^*$, we have

$$f(x) = \sum_{k=1}^{\infty} f(e^k(x_k)) = \sum_{k=1}^{\infty} (foT_k)(x_k). \tag{1.1}$$

Definition 1.3.([2]) Let X_d be a scalar-valued sequence space which is a BK-space. A countable family $\{g_i\} \subseteq X^*$ is called an X_d-frame for X if
(i) $\{g_i(x)\} \in X_d$, $x \in X$,
(ii) there exist constants $A, B > 0$ such that

$$A\|x\|_X \leq \|\{g_i(x)\}\|_{X_d} \leq B\|x\|_X, \quad x \in X. \tag{1.2}$$

The constants A and B are called X_d-frame bounds. If (i) and the second inequality in (1.2) are satisfied, then $\{g_i\}$ is called an X_d-Bessel sequence for X.

Definition 1.4.[2] Let X_d be a scalar-valued sequence space which is a BK-space.

Given a bounded operator $K : X_d \to X$, and an X_d-frame $\{g_i\} \subseteq X^*$, we say that $(\{g_i\}, K)$ is a Banach frame for X with respect to X_d if

$$K(\{g_i(x)\}) = x, \quad x \in X.$$

2. G-vector-valued Sequence Space Frames

Definition 2.1. Let Y_v be a Y-valued sequence space which is a BK-space. We call a sequence $\{\Lambda_i \in B(X,Y),\ i \in \mathbb{N}\}$ a g-Y_v-frame for X with respect to Y if
(i) $\{\Lambda_i(x)\} \in Y_v, \quad x \in X,$
(ii) there exist constants $A, B > 0$ such that

$$A\|x\|_X \leq \|\{\Lambda_i(x)\}\|_{Y_v} \leq B\|x\|_X, \quad x \in X. \tag{2.1}$$

The constants A and B are called the lower and upper frame bounds, respectively. If (i) and the second inequality in (2.1) are satisfied, then $\{\Lambda_i\}$ is called a g-Y_v-Bessel sequence for X with respect to Y. We call $\{\Lambda_i\}$ a tight g-Y_v-frame if A and B can be chosen such that $A = B$ and a Parseval g-Y_v-frame if A and B can be chosen such that $A = B = 1$.

Let $1 < p < \infty$. If $Y_v = (\sum_{i\in\mathbb{N}} \oplus Y_i)_{\ell^p}$, where for each $i \in \mathbb{N}$, $Y_i = Y$, then (2.1) means that $\{\Lambda_i\}$ is a pg-frame for X with respect to Y [1].

Example 2.2. Let Y_v be a Y-valued sequence space which is a BK-space. The k-th coordinate mapping $p_k : Y_v \to Y$ is a Parseval g-Y_v-frame for Y_v with respect to Y.

Let $(Y, \|.\|)$ be a Banach space and $b = (b_k)$ be a bounded sequence of positive real numbers. The Y-valued sequence space of Maddox is defined as

$$\ell^\infty(Y, b) = \{y = \{y_k\} : \sup_k \|y_k\|^{b_k} < \infty\}.$$

Example 2.3. Let Y_v be a $C[0,1]$-valued sequence space, $\ell^\infty(C[0,1], (b_k))$, where for each $k \in \mathbb{N}$, $b_k = 1$. We define $\Lambda_i : C[0,1] \to C[0,1]$ by

$$\Lambda_i(f) = f \circ g_i,$$

where $g_i : C[0,1] \to C[0,1]$ is defined by $g_i(x) = x^i$. Then $\{\Lambda_i\}$ is a Parseval g-$\ell^\infty(C[0,1], (b_k))$-frame for $C[0,1]$ with respect to $C[0,1]$.

Suppose that Y_v is a Y-valued sequence space which is a BK-space and has the property AK. If $\{\Lambda_i\}$ is a g-Y_v-Bessel sequence for X with respect to Y, then we define two operators

$$U : X \to Y_v, \quad U(x) = \{\Lambda_i x\},$$

and

$$T : Y_v^* \to X^*, \quad T(f) = \sum_{i=1}^{\infty} (foT_i)\Lambda_i,$$

where T_i is the mapping mentioned in Lemma 1.1. The operator U is called the analysis operator and T is called the synthesis operator of $\{\Lambda_i\}$. The new idea that occurs here is that we use the isomorphism of Y_v^* and Y_v^β in order to define analysis and synthesis operators of g-Y_v-Bessel sequences and show that they are bounded.

Theorem 2.4. *Let Y_v be a Y-valued sequence space which is a BK-space and has the property AK. Let $\{\Lambda_i\} \subseteq B(X, Y)$ and for each $x \in X - \{0\}$, $\{\Lambda_i x\} \neq 0$ and $\{\Lambda_i x\} \in Y_v$. Then $\{\Lambda_i\}$ is a g-Y_v-Bessel sequence for X with respect to Y with upper bound B if and only if T is a well-defined, bounded operator and $\|T\| \leq B$.*

Proof. Suppose that $\{\Lambda_i\}$ is a g-Y_v-Bessel sequence for X with respect to Y with upper bound B. Then for each $f \in Y_v^*$ and $x \in X$, since Y_v has the property AK, by (1.1) we have

$$\begin{aligned} |T(f)(x)| &= |\sum_{i=1}^{\infty} (foT_i)\Lambda_i x| = |f(\{\Lambda_i x\})| \\ &\leq \|f\|_{Y_v^*} \|\{\Lambda_i x\}\|_{Y_v} \leq B\|f\|_{Y_v^*} \|x\|_X. \end{aligned}$$

So T is well-defined and bounded. Conversely, assume that T is well-defined and $\|T\| \leq B$. Since Y_v has the property AK, by Theorem 1.2, for each $x \in X$, $F_x : Y_v^* \simeq Y_v^\beta \to \mathbb{C}$, can be defined by

$$F_x(f) = T(f)(x) = \sum_{i=1}^{\infty} (foT_i)\Lambda_i x, \quad f \in Y_v^*.$$

Then

$$|F_x(f)| = |T(f)(x)| \leq \|Tf\|_{X^*} \|x\|_X \leq \|T\| \|f\|_{Y_v^*} \|x\|_X. \tag{2.2}$$

Clearly for each $x \in X$, $F_x \in Y_v^{**}$.
On the other hand by the Hahn-Banach theorem, there is $f \in Y_v^*$ with $\|f\|_{Y_v^*} \leq 1$ such that

$$\|\{\Lambda_i x\}\|_{Y_v} = |f(\{\Lambda_i x\})|. \tag{2.3}$$

By (1.1), (2.2) and (2.3), we have

$$\begin{aligned} \|\{\Lambda_i x\}\|_{Y_v} &= |f(\{\Lambda_i x\})| = |\sum_{i=1}^{\infty} (foT_i)\Lambda_i x| \\ &= |F_x(f)| \leq \|T\| \|f\|_{Y_v^*} \|x\|_X \\ &\leq \|T\| \|x\|_X \leq B\|x\|_X. \end{aligned}$$

□

Lemma 2.5. *Let X be a Banach space and Y_v be a Y-valued sequence space which is a BK-space and has the property AK. Suppose that $\{\Lambda_i\}$ is a g-Y_v-Bessel sequence for X with respect to Y. Then*
(i) $U^* = T$.
(ii) *If X and Y_v are reflexive, then $T^* = U$.*

Proof. (i) By (1.1), for each $f \in Y_v^*$ and $x \in X$, we have

$$U^*f(x) = \langle x, U^*f\rangle = \langle Ux, f\rangle = f(\{\Lambda_i x\}) = \sum_{i=1}^{\infty}(foT_i)\Lambda_i x = T(f)(x).$$

(ii) Since X and Y_v are reflexive, we deduce the proof. □

Lemma 2.6.([9]) *Given a bounded operator $U : X \to Y$, the adjoint operator $U^* : Y^* \to X^*$ is surjective if and only if U has a bounded inverse on its range $R(U)$.*

Theorem 2.7. *Suppose that X is a reflexive Banach space and Y_v is a reflexive Y-valued sequence space which is a BK-space and has the property AK. Let $\{\Lambda_i\} \subseteq B(X,Y)$ and for each $x \in X - \{0\}$, $\{\Lambda_i x\} \neq 0$ and $\{\Lambda_i x\} \in Y_v$. Then $\{\Lambda_i\}$ is a g-Y_v-frame for X with respect to Y if and only if T is a well-defined and bounded operator of Y_v^* onto X^*. In this case, the frame bounds are $\|(T^*)^{-1}\|^{-1}$ and $\|T\|$.*

Proof. By Theorem 2.4, the upper g-Y_v-frame condition satisfies if and only if T is a well-defined and bounded operator of Y_v^* into X^*. Now suppose that $\{\Lambda_i\}$ is a g-Y_v-frame for X with respect to Y. Then U has a bounded inverse on its range $R(U)$. By Lemma 2.6, U^* is surjective and so T is surjective by Lemma 2.5.
Conversely, suppose that T is a well-defined and bounded operator of Y_v^* onto X^*. By Lemma 2.5, for each $x \in X$, we have

$$\|Ux\|_{Y_v} = \|T^*x\|_{Y_v} \leq \|T\|\|x\|_X.$$

Since T is bounded and surjective, T^* is one to one. Hence T^* has a bounded inverse on $R(T^*)$. So for each $x \in X$ we have

$$\|x\|_X = \|(T^*)^{-1}T^*x\|_X \leq \|(T^*)^{-1}\|\|Ux\|_{Y_v}.$$

□

After this in this section, X is a reflexive Banach space and Y_v is a reflexive Y-valued sequence space which is a BK-space and has the property AK.

Definition 2.8. A family $\{\Lambda_i \in B(X,Y) : i \in \mathbb{N}\}$ is called a g-Y_v^*-Riesz basis for X^* with respect to Y if
(i) $\{x : \Lambda_i x = 0,\ i \in \mathbb{N}\} = \{0\}$ (i.e. $\{\Lambda_i\}$ is g-complete);

(ii) for each $x \in X$, $\{\Lambda_i x\} \in Y_v$ and the operator $T : Y_v^* \to X^*$ is well-defined and there are constants $A, B > 0$ such that

$$A\|f\|_{Y_v^*} \leq \|Tf\|_{X^*} \leq B\|f\|_{Y_v^*}, \; f \in Y_v^*. \tag{2.4}$$

The constants A and B are called the lower and upper Riesz basis bounds, respectively.

The following theorem shows that every g-Y_v^*-Riesz basis with bounds A and B is a g-Y_v-frame with the same bounds.

Theorem 2.9. *Let $\{\Lambda_i\}$ be a g-Y_v^*-Riesz basis for X^* with respect to Y with lower and upper Riesz basis bounds A and B, respectively. Suppose that for each $x \in X - \{0\}$, $\{\Lambda_i x\} \neq 0$. Then $\{\Lambda_i\}$ is a g-Y_v-frame for X with respect to Y with frame bounds A and B.*

Proof. Since $\{\Lambda_i\}$ is a g-Y_v^*-Riesz basis for X^* with respect to Y, the operator T is well-defined, bounded and surjective. By Theorem 2.7, $\{\Lambda_i\}$ is a g-Y_v-frame for X with respect to Y. The upper Riesz basis bound coincides with the upper frame bound by Theorem 2.4. Now we show that A is a lower frame bound for $\{\Lambda_i\}$. Since $T : Y_v^* \to X^*$ is invertible, for each $g \in X^*$, there exists a unique $f \in Y_v^*$, such that $Tf = g$. So by (2.4) we have

$$\|T^{-1}g\| = \|f\| \leq \frac{1}{A}\|Tf\| = \frac{1}{A}\|g\|.$$

This implies that $A \leq \|(T^{-1})^*\|^{-1}$. Now we conclude the proof by Theorem 2.7. □

Definition 2.10. Let $\{\Lambda_i\}$ be a g-Y_v-frame for X with respect to Y. We say that $\{\Lambda_i^*(Y^*)\}_{i\in\mathbb{N}}$ is a Riesz decomposition of X^*, if for each $g \in X^*$, there is a unique choice of $f \in Y_v^*$ such that $\sum_{i=1}^{\infty}(foT_i)\Lambda_i = g$.

Theorem 2.11. *Let $\{\Lambda_i\}$ be a g-Y_v-frame for X with respect to Y. Then the following conditions are equivalent:*
(i) *$\{\Lambda_i\}$ is a g-Y_v^*-Riesz basis for X^*.*
(ii) *T is injective.*
(iii) *$R(U) = Y_v$.*
(iv) *$\{\Lambda_i^*(Y^*)\}_{i\in\mathbb{N}}$ is a Riesz decomposition of X^*.*

Proof. $(i) \to (ii)$ It is evident by the definition of g-Y_v^*-Riesz basis.
$(ii) \to (i)$ The operator T is well-defined, bounded and surjective by Theorem 2.7, and is injective by (ii), so it has a bounded inverse and therefore $\{\Lambda_i\}$ is a g-Y_v^*-Riesz basis for X^*.
$(i) \to (iii)$ By assumption, T has a bounded inverse on $R(T) = X^*$. By Lemma 2.6, T^* is surjective and Lemma 2.5, implies that $R(U) = Y_v$.
$(iii) \to (i)$ Since U is surjective, by Lemma 2.5, T is injective. So $\{\Lambda_i\}$ is a g-Y_v^*-Riesz basis for X^*.
$(ii) \to (iv)$ Suppose that there exist $f, h \in Y_v^*$ such that $f \neq h$ and

$$\sum_{i=1}^{\infty}(foT_i)\Lambda_i = \sum_{i=1}^{\infty}(hoT_i)\Lambda_i.$$

Then for each $x \in X$,

$$\sum_{i=1}^{\infty}(foT_i)\Lambda_i x = \sum_{i=1}^{\infty}(hoT_i)\Lambda_i x.$$

This implies that $f(\{\Lambda_i x\}) = h(\{\Lambda_i x\})$. Since T is injective, $f = h$.
$(iv) \to (ii)$ Suppose that for some $f \in Y_v^*$, $T(f) = 0$. Then for each $x \in X$, $\sum_{i=1}^{\infty}(foT_i)\Lambda_i x = 0$. Since $\{\Lambda_i^*(Y^*)\}_{i\in\mathbb{N}}$ is a Riesz decomposition of X, $f = 0$. $\square$

3. G-Y_v-Frame Mapping

In this section X is a reflexive Banach space and Y_v is a reflexive Y-valued sequence space which is a BK-space and has the property AK. Here we extend the results of [10] to define g-Y_v-frame mapping and investigate its invertibility.
We recall that when the family $\{g_i\}$ is a frame for a Hilbert space H, the frame mapping $S : H \to H$, $S(f) = TU(f)$ is the composition of the synthesis operator $T : \ell^2 \to H$ and the analysis operator $U : H \to \ell^2$. Now suppose that the family $\{\Lambda_i\} \subseteq B(X,Y)$ is a g-Y_v-frame for X with respect to Y. In order to compose the analysis operator $U : X \to Y_v$ and the synthesis operator $T : Y_v^* \to X^*$, we need a mapping $J : Y_v \to Y_v^*$, to this aim we use the definition of duality mapping.
First we recall that the Banach space X is strictly convex if whenever $x, y \in X$ with $x \neq y$, $\|x\| = \|y\| = 1$ then $\|\lambda x + (1-\lambda)y\| < 1$ for all $\lambda \in (0,1)$.
The mapping ϕ_X from X into the set of subsets of X^*, defined by

$$\phi_X x = \{x^* \in X^* : x^*(x) = \|x^*\|\|x\|, \|x^*\| = \|x\|\},$$

is called the duality mapping on X. By the Hahn-Banach theorem, for each $x \in X$, $\phi_X x$ is nonempty. In general, the duality mapping is set valued, but for certain spaces it is single-valued and such spaces are called smooth.

Proposition 3.1.([6]) *The following statements are true:*
(i) *If X^* is strictly convex, then for each $x \in X$, $\phi_X x$ consists of unique element $x^* \in X^*$.*
(ii) *If X and X^* are strictly convex and X is reflexive, then ϕ_X is bijective.*
(iii) *If H is a Hilbert space, then for each $x \in H$, $\phi_H x = x$.*

Definition 3.2. Let Y_v^* be strictly convex and $\{\Lambda_i\}$ be a g-Y_v-Bessel sequence for X with respect to Y. The mapping $S : X \to X^*$, defined by

$$S = T\phi_{Y_v}U,$$

is called the g-Y_v-frame mapping of $\{\Lambda_i\}$.

We note that if $\{\Lambda_i\}$ is a g-Y_v-frame for X with respect to Y with frame bounds A and B, then by the definition of the duality mapping, for each $x \in X$, we have

$$A^2\|x\|^2 \leq Sx(x) \leq B^2\|x\|^2.$$

Let $\{g_i\}$ be a frame for H and T be the synthesis operator for $\{g_i\}$. Since T is surjective and bounded operator, it has a pseudo-inverse, the unique operator $T^{\dagger} : H \to \ell^2$, for which $TT^{\dagger} = I_{R(T)}$, $\ker(T^{\dagger}) = (R(T))^{\perp}$ and $R(T^{\dagger}) = (\ker(T))^{\perp}$. The pseudo-inverse $T^{\dagger}$ is actually the bounded operator $(T|_{(\ker(T))^{\perp}})^{-1}$, where $T|_{(\ker(T))^{\perp}}$ is the restriction of T to the orthogonal complement of $\ker(T)$.
The existence of a right inverse of the synthesis operator for a given Hilbert frame is due to the fact that a closed subspace of a Hilbert space H is always complemented in H. However, a closed subspace of a Banach space X is not necessarily complement in X and the existence of a right inverse in the Banach space needs some complementary conditions, to approach to this aim we have the following proposition.
The following proposition is the extension of [Prop. 5.2, [10]] to g-Y_v-frames.

Proposition 3.3. *Let $\{\Lambda_i\}$ be a g-Y_v-frame for X with respect to Y. Then the following assertions are equivalent:*
(i) *T has a bounded right inverse.*
(ii) *$\ker(T)$ is complemented in Y_v^*.*

Proof. $(i) \to (ii)$ Let $W : X^* \to Y_v^*$ be the right inverse of the operator T. Put $P = WT : Y_v^* \to Y_v^*$. Clearly $P^2 = P$ and we have $Y_v^* = \ker(P) \oplus R(P)$. Since $\ker(P) = \ker(T)$, $\ker(T)$ is complemented in Y_v^*.
$(ii) \to (i)$ Suppose that M is a complement of $\ker(T)$ in Y_v^*. By Theorem 2.7, the operator T is bounded and surjective. So $(T|_M)^{-1}$ is a bounded right inverse of T. □

Now we recall the concept of semi-inner product in Banach spaces in order to define the right inverse of T. A mapping $[.,.]$ from $X \times X$ into $\mathbb{R}$ is said to be a semi-inner product on X if it has the following properties:
(i) for all $x \in X$, $[x, x] \geq 0$ and $[x, x] = 0$ iff $x = 0$;
(ii) for all $\alpha, \beta \in \mathbb{R}$ and for all $x, y, z \in X$, $[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z]$;
(iii) for all $x, y \in X$, $|[x, y]|^2 \leq [x, x][y, y]$.
If X^* is strictly convex, then there is a unique semi-inner product on X such that for each $x \in X$, $\|x\|_X = [x, x]^{\frac{1}{2}}$ and for all $x, y \in X$, $\phi_X x(y) = [y, x]$, where ϕ_X is the duality mapping on X. In this case an operator $A : X \to X$ is said to be adjoint abelian if for all $x, y \in X$, $[Ax, y] = [x, Ay]$ or equivalently $A^*\phi_X = \phi_X A$. The element $x \in X$ is called (Giles) orthogonal to the element $y \in X$ (denoted by $x \perp y$), if $[y, x] = 0$. If M is a linear subspace of X, the notation $M^{\perp}$ is used to show the orthogonal complement of M in Giles sense, i.e. $M^{\perp} = \{x \in X : x \perp y, \forall y \in M\}$.

Proposition 3.4. *Suppose that X and Y are reflexive Banach spaces. Let X and X^* be strictly convex spaces. Suppose that $\overline{span}\{\Lambda_i^*(Y^*)\}_{i\in\mathbb{N}}$ and $(\overline{span}\{\Lambda_i^*(Y^*)\}_{i\in\mathbb{N}})^{\perp}$ (in Giles sense) are topologically complementary in X^*. Then*
(i) *If $\{\Lambda_i \in B(X, Y),\ i \in \mathbb{N}\}$ is a g-Y_v-frame for X with respect to Y, then $X^* = \overline{span}\{\Lambda_i^*(Y^*)\}_{i\in\mathbb{N}}$.*
(ii) *$\{\Lambda_i \in B(X, Y),\ i \in \mathbb{N}\}$ is a g-complete sequence on X if and only if $X^* = \overline{span}\{\Lambda_i^*(Y^*)\}_{i\in\mathbb{N}}$.*

Proof. (i) Suppose that $g \in (\overline{span}\{\Lambda_i^*(Y^*)\}_{i\in\mathbb{N}})^{\perp}$. Then for each $i \in \mathbb{N}$ and $y^* \in Y^*$, $g \perp \Lambda_i^*(y^*)$. Since X and Y are reflexive and X is strictly convex, we have

$$[\Lambda_i^*(y^*), g] = \langle \Lambda_i^*(y^*), \phi_{X^*} g\rangle = \langle y^*, \Lambda_i \phi_{X^*} g\rangle = 0,$$

so for each $i \in \mathbb{N}$, $\Lambda_i \phi_{X^*} g = 0$. Since $\{\Lambda_i\}$ is a g-Y_v-frame and X is a reflexive space and X, X^* are strictly convex spaces, ϕ_{X^*} is bijective by Proposition 3.1. So we deduce that $g = 0$.

(ii) Let $\{\Lambda_i\}$ be a g-complete sequence on X. Similar to the proof of (i), we can show that $X^* = \overline{span}\{\Lambda_i^*(Y^*)\}_{i\in\mathbb{N}}$. Conversely suppose that for each $i \in \mathbb{N}$, $\Lambda_i x = 0$. Then

$$\langle \Lambda_i x, y^*\rangle = \langle x, \Lambda_i^* y^*\rangle = 0, \quad y^* \in Y^*.$$

Since $X^* = \overline{span}\{\Lambda_i^*(Y^*)\}_{i\in\mathbb{N}}$, for each $g \in X^*$, $\langle x, g\rangle = 0$ and so $x = 0$. □

Remark 3.5. Let Y_v be a strictly convex space and $\{\Lambda_i\}$ be a g-Y_v-frame for X with respect to Y. Suppose that $\ker(T)$ and $(\ker(T))^{\perp}$ are topologically complementary in Y_v^*. Then by Proposition 3.3, the operator $T|_{(\ker(T))^{\perp}}$ is invertible and $T^{\perp} = (T|_{(\ker(T))^{\perp}})^{-1}$ is a bounded right inverse of T.

Definition 3.6. Let Y_v be a strictly convex space and $\{\Lambda_i\}$ be a g-Y_v-frame for X with respect to Y. Suppose that $\ker(T)$ and $(\ker(T))^{\perp}$ are topologically complementary in Y_v^*. We define the mapping $F : X^* \to X^{**} \simeq X$ by

$$F = (T^{\perp})^* \phi_{Y_v^*} T^{\perp}.$$

Lemma 3.7. *Let Y_v be a strictly convex space and $\{\Lambda_i\}$ be a g-Y_v-frame for X with respect to Y. Suppose that $\ker(T)$ and $(\ker(T))^{\perp}$ are topologically complementary in Y_v^*. Then for each $g \in X^*$*

$$Fg(g) \geq \frac{1}{B^2}\|g\|_{X^*}^2,$$

where B is the upper frame bound for $\{\Lambda_i\}$.

Proof. For each $g \in X^*$, we have

$$\begin{aligned} Fg(g) &= \langle g, Fg\rangle = \langle g, (T^{\perp})^* \phi_{Y_v^*} T^{\perp} g\rangle \\ &= \langle T^{\perp} g, \phi_{Y_v^*} T^{\perp} g\rangle = \|T^{\perp} g\|_{Y_v^*}^2. \end{aligned} \tag{3.1}$$

By Theorem 2.4, $\|T\| \leq B$. Hence by (3.1)

$$\|g\|_{X^*}^2 = \|TT^{\perp} g\|_{X^*}^2 \leq B^2 \|T^{\perp} g\|_{Y_v^*}^2 = Fg(g)B^2.$$

□

Theorem 3.8. *Let Y_v and Y_v^* be strictly convex spaces and $\{\Lambda_i\}$ be a g-Y_v-frame for X with respect to Y. Suppose that $\ker(T)$ and $(\ker(T))^{\perp}$ are topologically complementary in Y_v^* and the operator $T^{\perp}T$ is adjoint abelian. Then the following*

assertions hold:
(i) S *is invertible and* $S^{-1} = F$.
(ii) $S^{-1} = U^{-1}\phi_{Y_v^*}T^{\perp}$.

Proof. (i) By Proposition 3.1, $\phi_{Y_v^*}\phi_{Y_v} = I_{Y_v}$. Since $T^{\perp}T$ is adjoint abelian, we have

$$\begin{aligned} FS &= (T^{\perp})^*\phi_{Y_v^*}T^{\perp}T\phi_{Y_v}T^* \\ &= (T^{\perp})^*(T^{\perp}T)^*\phi_{Y_v^*}\phi_{Y_v}T^* \\ &= (TT^{\perp}TT^{\perp})^* = I_X. \end{aligned}$$

Similarly, we can show that $SF = I_{X^*}$.
(ii) Since $U = T^*$ and $T^{\perp}T$ is adjoint abelian, for each $g \in X^*$, we have

$$\begin{aligned} \phi_{Y_v^*}T^{\perp}g &= \phi_{Y_v^*}T^{\perp}TT^{\perp}g \\ &= (T^{\perp}T)^*\phi_{Y_v^*}T^{\perp}g \\ &= U(T^{\perp})^*\phi_{Y_v^*}T^{\perp}g. \end{aligned}$$

Therefore, $\phi_{Y_v^*}T^{\perp}g \in R(U)$, and we have

$$U^{-1}\phi_{Y_v^*}T^{\perp}g = (T^{\perp})^*\phi_{Y_v^*}T^{\perp}g = S^{-1}g.$$

□

4. G-Banach Frames

We recall that a sequence $\{x_i\}$ in Banach space X is called a Schauder basis for X if for each $x \in X$ there is a unique sequence of scalars $\{a_i\}$ such that $x = \sum a_i x_i$. The unique elements $x_i^* \in X^*$ satisfying

$$x = \sum \langle x, x_i^* \rangle x_i, \quad \forall x \in X,$$

are called the biorthogonal functionals for $\{x_i\}$. So if X is a Banach space with a Schauder basis $\{x_i\}$, we always have a retrieval formula, that we can get every $x \in X$ by the sequence $\{x_i\}$. But the question is how we can find a retrieval formula for the spaces with no Schauder basis. Gröchenig and Casazza answered this question by considering the concepts of atomic decompositions and Banach frames. By inspiration of their results, in this section we present retrieval formulas for these Banach spaces by the concept of g-Banach frames.

Definition 4.1. Let Y_v be a Y-valued sequence space which is a BK-space. Given a bounded operator $K : Y_v \to X$ and a g-Y_v-frame $\{\Lambda_i\} \subseteq B(X, Y)$, we say that $(\{\Lambda_i\}, K)$ is a g-Banach frame for X with respect to Y and Y_v, if

$$K(\{\Lambda_i x\}) = x, \quad x \in X. \tag{3.1}$$

Note that (3.1) can be considered as a kind of generalized retrieval formula, in the sense that it tells how to come back to $x \in X$, based on the coefficients $\{\Lambda_i(x)\}$. By extending [Prop. 3.4, [2]], the following proposition gives the equivalent conditions in which we can obtain g-Banach frames from g-Y_v-frames.

Proposition 4.2. *Suppose that $\{\Lambda_i\}$ is a g-Y_v-frame for X with respect to Y. Then the following conditions are equivalent:*
(i) *$R(U)$ is complemented in Y_v.*
(ii) *The operator $U^{-1} : R(U) \to X$ can be extended to a bounded linear operator $V : Y_v \to X$.*
(iii) *There exists a linear bounded operator K such that $(\{\Lambda_i\}, K)$ is a g-Banach frame for X with respect to Y and Y_v.*

Proof. $(i) \to (ii)$ Since $R(U)$ is complemented in Y_v, there exists a closed subspace N of Y_v such that $Y_v = R(U) \oplus N$. We define $V : Y_v \to X$, $V(g) = U^{-1}(g)$ if $g \in R(U)$ and $V(g) = 0$ if $g \in N$.
$(ii) \to (i)$ Let $V : Y_v \to X$ be a linear bounded extension of U^{-1}. Now consider the bounded operator $P : Y_v \to R(U)$ defined by $P = UV$. Using the fact that $VU = I$, we get $P^2 = P$. For each $x \in X$, we have

$$Ux = UVUx = P(Ux) \in R(P).$$

Thus $R(U)$ is complemented in Y_v.
$(ii) \to (iii)$ Since there exists a bounded operator $V : Y_v \to X$, such that for each $x \in X$, $V(\{\Lambda_i x\}) = VUx = x$ and $\{\Lambda_i\}$ is a g-Y_v-frame for X with respect to Y, we conclude the proof.
$(iii) \to (ii)$ Since for each $x \in X$, $K(\{\Lambda_i x\}) = x$, the operator K is a bounded extension of U^{-1}. □

In the rest of this section we investigate the conditions that help us construct g-Y_v-frames and g-Banach frames.

Proposition 4.3.([3]) Every separable Banach space has a Banach frame with frame bounds $A = B = 1$.

By inspiration of the result of Proposition 4.3, in the following theorem we show that if X is a separable Banach space and Y is a Banach space with a Schauder basis $\{e_i\}$, then we can find a Y-valued sequence space Y_v, and a g-Banach frame for X with respect to Y and Y_v.

Theorem 4.4. *Let X be a separable Banach space and Y be a Banach space with a Schauder basis $\{e_i\}$ such that for each $i \in \mathbb{N}$, $\|e_i\| = 1$. Then there exist a Y-valued sequence space Y_v, which is a BK-space and a g-Banach frame for X with respect to Y and Y_v with frame bounds $A = B = 1$.*

Proof. Since X is a separable Banach space, we can choose a sequence $g_i \in X^*$ with $\|g_i\|_{X^*} = 1$, such that for every $x \in X$, we have

$$\|x\|_X = sup_i |g_i(x)|. \tag{3.2}$$

We define the operator $\Lambda_i : X \to Y$ by $\Lambda_i(x) = g_i(x)e_i$. Let Y_v be the subspace of $\ell^\infty(Y, (b_k))$, where for each $k \in \mathbb{N}$, $b_k = 1$, given by

$$Y_v = \{\{\Lambda_i x\} : x \in X\}.$$

Let $K : Y_v \to X$, $K(\{\Lambda_i x\}) = x$. Now by (3.2), K is an isometrical isomorphism of Y_v onto X and therefore Y_v is a Banach space. Also we define

$$p_k : Y_v \to Y, \quad p_k(\{\Lambda_i x\}) = \Lambda_k(x).$$

Then

$$\|p_k(\{\Lambda_i x\})\| = \|\Lambda_k(x)\| = \|g_k(x)e_k\| \leq \|g_k\|\|x\| = \|x\| = \|\{\Lambda_i(x)\}\|_{\ell^\infty}.$$

Therefore Y_v is a BK-space and $(\{\Lambda_i\}, K)$ is a g-Banach frame for X with respect to Y and Y_v with frame bounds $A = B = 1$. □

Theorem 4.5. *Let X be a Banach space and Y_v be a Y-valued sequence space which is a BK-space. Then the following statements hold:*
(i) *There exists a g-Y_v-frame for X with respect to Y if and only if X is isomorphic to a subspace of Y_v.*
(ii) *There exists a g-Banach frame for X with respect to Y and Y_v if and only if X is isomorphic to a complemented subspace of Y_v.*

Proof. (i) Let $\{\Lambda_i\}$ be a g-Y_v-frame for X with respect to Y. Then the mapping $U : X \to Y_v$, $U(x) = \{\Lambda_i(x)\}$ is an isomorphism of X into Y_v.
Conversely, let X be a subspace of Y_v. By the definition of a BK-space, the i-th coordinate mapping $p_i : Y_v \to Y$, $p_i(\{y_k\}) = y_i$ is continuous. Let $\Lambda_i = p_i \mid_X$. Then for each $x \in X$, $\{\Lambda_i(x)\} = x \in Y_v$ and $\|x\|_X = \|\{\Lambda_i(x)\}\|_{Y_v}$.
(ii) Assume that X is isomorphic to a complemented subspace of Y_v. Suppose that M is a complemented subspace of Y_v. Then $F : X \to M$ is an isomorphism. Let $P : Y_v \to M$ be the projection of Y_v onto $R(F)$. Define $K : Y_v \to X$ by $Ky = F^{-1}Py$. Let p_i be the i-th coordinate mapping of Y_v onto Y. Then for each $x \in X$, we define

$$\Lambda_i(x) = p_i(Fx), \quad i \in \mathbb{N}.$$

Hence, $Fx = \{\Lambda_i(x)\}$. Since F is an isomorphism, it follows that $(\{\Lambda_i\}, K)$ is a g-Banach frame for X with respect to Y and Y_v.
Conversely, suppose that there exists a g-Banach frame for X with respect to Y and Y_v. Then by Proposition 4.2, $R(U)$ is complemented in Y_v and so X is isomorphic to a complement subspace of Y_v. □

Lemma 4.6. *Let $\{\Lambda_i \in B(X,Y): \ i \in \mathbb{N}\}$ be a g-complete sequence on X. Then there exist a BK-space Y_v and a bounded operator $K: Y_v \to X$ such that $(\{\Lambda_i\}, K)$ is a g-Banach frame for X with respect to Y and Y_v.*

Proof. Let define the normed linear space

$$Y_v = \{\{\Lambda_i(x)\} : x \in X\}, \quad \|\{\Lambda_i(x)\}\|_{Y_v} = \|x\|_X.$$

Since $\{\Lambda_i\}$ is a g-complete sequence on X, the norm on Y_v is well-defined and the operator $W: X \to Y_v$, $W(x) = \{\Lambda_i x\}$ is an isometrical isomorphism of X onto Y_v. Therefore Y_v is a BK-space and $(\{\Lambda_i\}, K = W^{-1})$ is a g-Banach frame for X with respect to Y and Y_v. □

Corollary 4.7. *Let $\{\Lambda_i\}$ be a g-Y_v-frame for X with respect to Y. If for $j_0 \in \mathbb{N}$, Λ_{j_0} is a surjective operator, then there exist an X^*-valued sequence space $(X^*)_v$, which is a BK-space and a bounded operator $W: (X^*)_v \to Y^*$ such that $(\{\Lambda_i^*\}, W)$ is a g-Banach frame for Y^* with respect to X^* and $(X^*)_v$.*

Proof. Let for each $i \in \mathbb{N}$, $\Lambda_i^* g = 0$. Then we have

$$\langle x, \Lambda_{j_0}^* g\rangle = \langle \Lambda_{j_0} x, g\rangle = 0, \quad x \in X.$$

Since Λ_{j_0} is surjective, $g = 0$. So $\{\Lambda_i^*\}$ is a g-complete sequence on Y^*. We conclude the proof by Lemma 4.6. □

In the next example, we show the existence of a g-Banach frame, which is not a g-frame.

Example 4.8. Let $g(x) = e^{\frac{-x^2}{2}}$ be the Gaussian and $\{\alpha_{m,n} : m, n \in \mathbb{Z}\}$ be an orthonormal basis for $\ell^2(\mathbb{Z}^2)$. Define

$$\Lambda_j f = \sum_{m,n \in \mathbb{Z}} \langle f(x), e^{i2\pi m x} g(x - 2n - j)\rangle \alpha_{m,n}, \quad j = 1, 2.$$

We deduce from [5] that, $\{\Lambda_i\}_{i=1,2}$ is not a g-frame but is a g-complete sequence on $L^2(\mathbb{R})$. However, by Lemma 4.6, $(\{\Lambda_i\}_{i=1,2}, K = W^{-1})$, for which $W: L^2(\mathbb{R}) \to Z_v$, $W(f) = \{\Lambda_i f\}_{i=1,2}$ is a g-Banach frame for $L^2(\mathbb{R})$ with respect to $\ell^2(\mathbb{Z}^2)$ and the Bk-space

$$Z_v = \{\{\Lambda_i f\} : f \in L^2(\mathbb{R})\},$$

with norm

$$\|\{\Lambda_i f\}\|_{Z_v} = \|f\|_{L^2(\mathbb{R})}.$$

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