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A Generalization of Formal Local Cohomology Modules

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Abstract. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a commutative Noetherian ring R, M a finitely generated R-module and i an integer. In this paper we study formal local cohomology modules with respect to a pair of ideals. We denote the i-th $\mathfrak{a}$-formal local cohomology module M with respect to $\mathfrak{b}$ by $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M)$. We show that if $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M)$ is artinian, then $\mathfrak{a} \subseteq \sqrt{(0 : \mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M))}$. Also, we show that $\mathfrak{F}^{\dim M}_{\mathfrak{a},\mathfrak{b}}(M)$ is artinian and we determine the set $\operatorname{Att}_R \mathfrak{F}^{\dim M}_{\mathfrak{a},\mathfrak{b}}(M)$.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring with identity, $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of R and M is a finitely generated R-module. Let $\underline{x} = x_1, \cdots, x_r$ denote a system of elements of R and $\mathfrak{b} := Rad(\underline{x}R)$. Let $C_{\underline{x}}$ denote the Čech complex of R with respect to $\underline{x}$. For an integer i, the cohomology module $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M) := \mathrm{H}^i(\varprojlim_n (C_{\underline{x}} \otimes (M/\mathfrak{a}^n M))$ is called the i-th $\mathfrak{a}$-formal local cohomology module M with respect to $\mathfrak{b}$. If $(R, \mathfrak{m})$ is a local ring and $\mathfrak{b} = \mathfrak{m}$, we speak simply about the i-th $\mathfrak{a}$-formal local cohomology and denote it by $\mathfrak{F}^i_{\mathfrak{a}}(M)$ (see [12, Definition 3.1]). Note that by [12, Proposition 3.2], $\mathfrak{F}^i_{\mathfrak{a}}(M) \simeq \varprojlim_n \mathrm{H}^i_{\mathfrak{m}}(M/\mathfrak{a}^n M)$ for all $i \in \mathbb{Z}$.

In the case of a regular local ring, formal local cohomology modules have been studied by Peskine and Szpiro (cf. [10]). Formal local cohomology modules have been studied by several authors; see for example [12], [1], [6], [11] and [2].

In this paper we study some properties of formal local cohomology modules with respect to a pair of ideals of R.

Recall that for an R-module M and an ideal $\mathfrak{a}$, the cohomological dimension of M with respect to $\mathfrak{a}$ defined as $\mathrm{cd}(\mathfrak{a}, M) := \max\{i \in \mathbb{Z} : \mathrm{H}^i_{\mathfrak{a}}(M) \neq 0\}$. For more details see [5]. In this paper we show that $\max\{i \in \mathbb{Z} : \mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M) \neq 0\} = \mathrm{cd}(\mathfrak{b}, M/\mathfrak{a}M)$ and $\mathfrak{F}^{\dim M}_{\mathfrak{a},\mathfrak{b}}(M)$ is artinian. Also we show that if $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M)$ is artinian

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then $\mathfrak{a} \subseteq \sqrt{(0 : \mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M))}$.

A non-zero R-module M is called secondary if its multiplication map by any element a of R is either surjective or nilpotent. A secondary representation for an R-module M is an expression for M as a finite sum of secondary submodules. If such a representation exists, we will say that M is representable. A prime ideal $\mathfrak{p}$ of R is said to be an attached prime of M if $\mathfrak{p} = (N :_R M)$ for some submodule N of M. If M admits a reduced secondary representation, $M = S_1 + S_2 + \ldots + S_n$, then the set of attached primes $\mathrm{Att}_R(M)$ of M is equal to $\{\sqrt{0 :_R S_i} : i = 1, \ldots, n\}$ (see [8]). In [2], the author and M. H. Bijan-Zadeh have investigated the attached primes of $\mathfrak{F}^{\dim M}_{\mathfrak{a}}(M)$. Here, we detemine the set of attached primes of formal local cohomology module $\mathfrak{F}^{\dim M}_{\mathfrak{a},\mathfrak{b}}(M)$.

2. Main Results

The following result is important in this paper.

Proposition 2.1 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and M an R-module. Then there is the following short exact sequence:*

$$0 \to \varprojlim_n{}^1 \mathrm{H}^{i+1}_{\mathfrak{b}}(M/\mathfrak{a}^n M) \to \mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M) \to \varprojlim_n \mathrm{H}^i_{\mathfrak{b}}(M/\mathfrak{a}^n M) \to 0.$$

for all $i \in \mathbb{Z}$.

Proof. See [12, Proposition 3.2]. □

We shall use the following lemma in the proof of some of our results.

Lemma 2.2. *Let M and N be two finitely generated R-modules with $\mathrm{Supp}_R(N) \subseteq \mathrm{Supp}_R(M)$. Then $\mathrm{cd}(\mathfrak{a}, N) \leq \mathrm{cd}(\mathfrak{a}, M)$. In particular, $\mathrm{cd}(\mathfrak{a}, N) = \mathrm{cd}(\mathfrak{a}, M)$ whenever $\mathrm{Supp}_R(N) = \mathrm{Supp}_R(M)$.*

Proof. [5, Theorem 1.2]. □

Theorem 2.3 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and M a finitely generated R-module. If $t := \mathrm{cd}(b, M/\mathfrak{a}M)$, then $\mathfrak{F}^t_{\mathfrak{a},\mathfrak{b}}(M) \simeq \varprojlim_n \mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^n M)$.*

Proof. Since $\mathrm{Supp}_R(M/\mathfrak{a}M) = \mathrm{Supp}_R(M/\mathfrak{a}^n M)$, by Lemma 2.2 we have $\mathrm{cd}(\mathfrak{b}, M/\mathfrak{a}M) = \mathrm{cd}(\mathfrak{b}, M/\mathfrak{a}^n M) = t$ for all $n \in \mathbb{N}$. Thus $\mathrm{H}^{t+1}_{\mathfrak{b}}(M/\mathfrak{a}^n M) = 0$ for all $n \in \mathbb{N}$ and so $\varprojlim_n{}^1 \mathrm{H}^{t+1}_{\mathfrak{b}}(M/\mathfrak{a}^n M) = 0$. Therefore the result follows by Proposition 2.1 □

Theorem 2.4 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of R and M a finitely generated R-module. Then $\max\{i \in \mathbb{Z} : \mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M) \neq 0\} = \mathrm{cd}(\mathfrak{b}, M/\mathfrak{a}M)$.*

Proof. Let $t := \mathrm{cd}(\mathfrak{b}, M/\mathfrak{a}M)$. By Lemma 2.2, $\mathrm{cd}(\mathfrak{b}, \mathfrak{a}^n M/\mathfrak{a}^{n+1} M) \leq t$ and

$\mathrm{cd}(\mathfrak{b}, M/\mathfrak{a}M) = \mathrm{cd}(\mathfrak{b}, M/\mathfrak{a}^nM)$ for all $n \in \mathbb{N}$. Thus the short exact sequence

$$0 \to \mathfrak{a}^nM/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^{n+1}M \to M/\mathfrak{a}^nM \to 0$$

induces an epimorphism $\mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^{n+1}M) \to \mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^nM) \to 0$ of non-zero R-modules for all $n \in \mathbb{N}$, and so $\varprojlim_n \mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^nM) \neq 0$. Hence by Theorem 2.3, $\mathfrak{F}^t_{\mathfrak{a},\mathfrak{b}}(M) \neq 0$. On the other hand $\mathrm{H}^i_{\mathfrak{b}}(M/\mathfrak{a}^nM) = 0$ for all $i > t$. Thus from Proposition 2.1 we have $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M) = 0$ for all $i > t$ and this completes the proof. □

Theorem 2.5 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of* R *and* M *a finitely generated* R*-module of finite dimension* d*. Then*

i) $\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M) \simeq \varprojlim_n \mathrm{H}^d_{\mathfrak{b}}(M/\mathfrak{a}^nM)$,

ii) $\mathfrak{F}^{d-1}_{\mathfrak{a},\mathfrak{b}}(M) \simeq \varprojlim_n \mathrm{H}^{d-1}_{\mathfrak{b}}(M/\mathfrak{a}^nM)$.

Proof. i) Since $\dim(M/\mathfrak{a}^nM) \leq d$ by Grothendieck's Vanishing Theorem, [3, 6.1.2], $\mathrm{H}^{d+1}_{\mathfrak{b}}(M/\mathfrak{a}^nM) = 0$ for all $n \in \mathbb{N}$, and so ${\varprojlim_n}^1 \mathrm{H}^{d+1}_{\mathfrak{b}}(M/\mathfrak{a}^nM) = 0$. Now the result follows by Proposition 2.1.

ii) By [3, 6.1.2] and [3, 7.1.7], $\mathrm{H}^d_{\mathfrak{b}}(M/\mathfrak{a}^nM)$ is an artinian module for all $n \in \mathbb{N}$. Since ${\varprojlim_n}^1$ vanishes on the projective system of artinian modules, we have ${\varprojlim_n}^1 \mathrm{H}^d_{\mathfrak{b}}(M/\mathfrak{a}^nM) = 0$. Therefore the result follows by Proposition 2.1. □

Theorem 2.6 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of a ring* R *and* M *a finitely generated* R*-module of finite dimension* d*. Then* $\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M)$ *is a homomorphic image of* $\mathrm{H}^d_{\mathfrak{b}}(M)$*, and so* $\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M)$ *is an artinian* R*-module.*

Proof. We have $\dim \mathfrak{a}^nM \leq \dim M$ for all $n \in \mathbb{N}$, so that, by Grothendieck's Vanishing Theorem, the short exact sequence

$$0 \longrightarrow \mathfrak{a}^nM \longrightarrow M \longrightarrow M/\mathfrak{a}^nM \longrightarrow 0$$

induces an exact sequence

$$\mathrm{H}^d_{\mathfrak{b}}(M) \xrightarrow{\phi_n} \mathrm{H}^d_{\mathfrak{b}}(M/\mathfrak{a}^nM) \longrightarrow 0.$$

The family of R-modules $\{\ker \phi_n\}_{n\in\mathbb{N}}$ is a family of Artinian R-modules. Thus, the above exact sequence induces an exact sequence $\varprojlim_n \mathrm{H}^d_{\mathfrak{b}}(M) \to \varprojlim_n \mathrm{H}^d_{\mathfrak{b}}(M/\mathfrak{a}^nM) \to 0$. Hence by Theorem 2.5(i), we obtain an exact sequence $\mathrm{H}^d_{\mathfrak{b}}(M) \to \mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M) \to 0$. But by [3, 7.1.7], $\mathrm{H}^d_{\mathfrak{b}}(M)$ is artinian, and so the proof is complete. □

Proposition 2.7 *Let* R *be a ring and* $(Q_n)_{n\geq 1}$ *be an inverse system of* R*-modules, with maps* $\varphi_{mn} : Q_m \to Q_n$ *for* $m \geq n$*. Let* $\mathfrak{a}$ *be an ideal of* R *such that* $u^kQ_k = 0$

for all $u \in \mathfrak{a}$ *and all* $k \in \mathbb{N}$. *If* N *is an arbitrary submodule of* $\varprojlim_n Q_n$ *such that* $\varprojlim_n Q_n/N$ *is non-zero and representable, then* $\mathfrak{a} \subseteq \mathfrak{p}$ *for all* $\mathfrak{p} \in \mathrm{Att}_R(\varprojlim_n Q_n/N)$.

Proof. Let $\varprojlim_n Q_n/N = S_1 + S_2 + ... + S_n$ be a minimal secondary representation of $\varprojlim_n Q_n/N$ where S_j is $\mathfrak{p}_j$-secondary for $j = 1, 2, ..., n$. Suppose that there exists an integer $j \in \{1, \dots, n\}$ such that $\mathfrak{a} \nsubseteq \mathfrak{p}_j$ and look for a contradiction. Then there exists $u \in \mathfrak{a} \setminus \mathfrak{p}_j$. Take $0 \neq g = N + (g_k) \in S_j \subseteq \varprojlim_n Q_n/N$. Let g_k be the first non-zero component of g. Since $u \notin \mathfrak{p}_j$, we have $uS_j = S_j$. But $u^k S_j \subseteq u^k(\varprojlim_n Q_n/N)$, and so $S_j \subseteq u^k(\varprojlim_n Q_n/N)$. As $u^k Q_k = 0$, it follows that the k-th component of each element of $u^k(\varprojlim_n Q_n/N)$ is zero. But $g \in u^k(\varprojlim_n Q_n/N)$ and the k-th component of g is non-zero, which is a contradiction. □

Proposition 2.8 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of* R, t *an integer and* M *a finitely generated* R*-module. If* $\varprojlim_n \mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^n M)$ *is representable, then* $\mathfrak{a} \subseteq \mathfrak{p}$ *for all* $\mathfrak{p} \in \mathrm{Att}_R(\varprojlim_n \mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^n M))$.

Proof. Since $u^k \mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^k M) = 0$ for all $u \in \mathfrak{a}$ and $k \in \mathbb{N}$, the result follows by Proposition 2.7. □

Proposition 2.9 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of a ring* R, t *an integer and* M *a finitely generated* R*-module. If* $\varprojlim_n^1 \mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^n M)$ *is representable, then* $\mathfrak{a} \subseteq \mathfrak{p}$ *for all* $\mathfrak{p} \in \mathrm{Att}_R(\varprojlim_n^1 \mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^n M))$.

Proof. Consider the inverse system $(\mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^n M))_n$ with maps φ_{mn} for $m \geq n$. Take $Q_n := \prod_{i \leq n}(\mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^i M)/\varphi_{ni}(\mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^n M)))$. By [7, Proposition 4.2] there exists a submodule N of $\varprojlim_n Q_n$ such that $\varprojlim_n^1 \mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^n M) \simeq \varprojlim_n Q_n/N$. Let $k \in \mathbb{N}$. Since $u^k \mathrm{H}^t_{\mathfrak{b}}(M/\mathfrak{a}^i M) = 0$ for all $u \in \mathfrak{a}$ and $i \leq k$, we deduce that $u^k Q_k = 0$. Now the result follows by Proposition 2.7. □

The following Theorem is one of our main results.

Theorem 2.10 *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two ideals of* R, i *an integer and* M *a finitely generated* R*-module. If* $\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M)$ *is artinian, then* $\mathfrak{a} \subseteq \mathfrak{p}$ *for all* $\mathfrak{p} \in \mathrm{Att}_R(\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M))$.

Proof. Proposition 2.1 implies that $\varprojlim_n^1 \mathrm{H}^{i+1}_{\mathfrak{b}}(M/\mathfrak{a}^n M)$ and $\varprojlim_n \mathrm{H}^i_{\mathfrak{b}}(M/\mathfrak{a}^n M)$ are artinian modules. Since artinian modules are representable, by Proposition 2.9, $\mathrm{Att}_R(\varprojlim_n^1 \mathrm{H}^{i+1}_{\mathfrak{b}}(M/\mathfrak{a}^n M)) \subseteq V(\mathfrak{a})$ and by Proposition 2.8, $\mathrm{Att}_R(\varprojlim_n \mathrm{H}^i_{\mathfrak{b}}(M/\mathfrak{a}^n M)) \subseteq$

$\mathrm{V}(\mathfrak{a})$. Therefore

$$\mathrm{Att}_R(\mathfrak{F}^i_{\mathfrak{a},\mathfrak{b}}(M)) \subseteq \mathrm{Att}_R(\varprojlim_n{}^1 \mathrm{H}^{i+1}_{\mathfrak{b}}(M/\mathfrak{a}^n M)) \cup \mathrm{Att}_R(\varprojlim_n \mathrm{H}^i_{\mathfrak{b}}(M/\mathfrak{a}^n M)) \subseteq \mathrm{V}(\mathfrak{a})$$

and the proof is complete. □

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a ring R and M a finitely generated R-module of finite dimension d. By Theorem 2.6, $\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M)$ is an artinian R-module. In the following result, we determine the set $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M))$. This is based on the proof of [2, Theorem 3.1].

Theorem 2.12 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of a Noetherian local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d. Then*

$$\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}\, M : \mathrm{cd}(\mathfrak{b}, R/\mathfrak{p}) = d, \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Proof. If $\mathrm{cd}(\mathfrak{b}, M/\mathfrak{a}M) < \dim M$, then $\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M) = 0$, and so

$$\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}\, M : \mathrm{cd}(\mathfrak{b}, R/\mathfrak{p}) = d, \mathfrak{p} \supseteq \mathfrak{a}\} = \phi.$$

Thus we assume that $\mathrm{cd}(\mathfrak{b}, M/\mathfrak{a}M) = \dim M$. By Theorem 2.6, $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M)) \subseteq \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{b}}(M))$ and by [4, Theorem A],

$$\mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{b}}(M)) = \{\mathfrak{p} \in \mathrm{Ass}\, M : \mathrm{cd}(\mathfrak{b}, R/\mathfrak{p}) = d\}.$$

On the other hand, by Theorem 2.10, $\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M)) \subseteq \mathrm{V}(\mathfrak{a})$. Therefore

$$\mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M)) \subseteq \{\mathfrak{p} \in \mathrm{Ass}\, M : \mathrm{cd}(\mathfrak{b}, R/\mathfrak{p}) = \dim M, \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Let $\mathfrak{p} \in \mathrm{Ass}\, M$ such that $\mathrm{cd}(\mathfrak{b}, R/\mathfrak{p}) = \dim M$ and $\mathfrak{p} \supseteq \mathfrak{a}$. By [9, 6.8], there exists a $\mathfrak{p}$-primary submodule N of M such that $\mathrm{Ass}_R(M/N) = \{\mathfrak{p}\}$ and $\mathfrak{p} = \sqrt{(0 : (M/N))}$. By the assumption $\mathfrak{a} \subseteq \mathfrak{p}$, and so $\sqrt{\mathfrak{a}} \subseteq \sqrt{(0 : (M/N))}$. Hence

$$\sqrt{(0 : (M/N)/\mathfrak{a}(M/N))} = \sqrt{\mathfrak{a} + (0 : (M/N))} = \sqrt{(0 : (M/N))}.$$

This implies that

$$\mathrm{Supp}_R((M/N)/\mathfrak{a}(M/N)) = \mathrm{Supp}_R(M/N) = \mathrm{Supp}_R(R/\mathfrak{p}),$$

and so by Lemma 2.2 we have

$$\mathrm{cd}(\mathfrak{b}, (M/N)/\mathfrak{a}(M/N)) = \mathrm{cd}(\mathfrak{b}, M/N) = \mathrm{cd}(\mathfrak{b}, R/\mathfrak{p}) = d.$$

Now by Theorem 2.4, $\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M/N) \neq 0$. Hence by Theorem 2.6,

$$\phi \neq \mathrm{Att}_R(\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M/N)) \subseteq \mathrm{Att}_R(\mathrm{H}^d_{\mathfrak{b}}(M/N)) \subseteq \mathrm{Ass}_R(M/N) = \{\mathfrak{p}\}.$$

Therefore we have $\operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M/N) = \{\mathfrak{p}\}$. On the other hand, the exact sequence

$$0 \to N \to M \to M/N \to 0$$

induces an exact sequence $M/\mathfrak{a}^n M \to (M/N)/\mathfrak{a}^n(M/N) \to 0$ for all $n \in \mathbb{N}$. Thus we obtain an exact sequence $\mathrm{H}^d_b(M/\mathfrak{a}^n M) \to \mathrm{H}^d_b((M/N)/\mathfrak{a}^n(M/N)) \to 0$ for all $n \in \mathbb{N}$. If $\dim(M/\mathfrak{a}^n M) \lneqq d$ then $\mathrm{H}^d_b(M/\mathfrak{a}^n M) = 0$ for all $n \in \mathbb{N}$. But by the assumption $\operatorname{cd}(\mathfrak{b}, M/\mathfrak{a}M) = d$, and so $\mathrm{H}^d_b(M/\mathfrak{a}^n M) \neq 0$ for all $n \in \mathbb{N}$. Thus we conclude that $\dim(M/\mathfrak{a}^n M) = d$ and by using [3, 7.1.7], $\mathrm{H}^d_b(M/\mathfrak{a}^n M)$ is artinian for all $n \in \mathbb{N}$. So $\varprojlim_n$ is exact on the above exact sequence and we get the following exact sequence:

$$\varprojlim_n \mathrm{H}^d_b(M/\mathfrak{a}^n M) \to \varprojlim_n \mathrm{H}^d_b((M/N)/\mathfrak{a}^n(M/N)) \to 0$$

Thus by Theorem 2.3, there is an exact sequence:

$$\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M) \to \mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M/N) \to 0.$$

Now $\mathfrak{p} \in \operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M/N))$ implies that $\mathfrak{p} \in \operatorname{Att}_R(\mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M))$. This completes the proof. □

The following corollary is due to Bijan-zadeh and the author [2].

Corollary 2.13 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and M a finitely generated R-module of dimension d. Then*

$$\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a}}(M) = \{\mathfrak{p} \in \operatorname{Ass} M : \dim(R/\mathfrak{p}) = \dim M, \mathfrak{p} \supseteq \mathfrak{a}\}.$$

Proof. Use the fact that $\operatorname{cd}(\mathfrak{m}, R/\mathfrak{p}) = \dim(R/\mathfrak{p})$ and $\mathfrak{F}^d_{\mathfrak{a},\mathfrak{m}}(M) \simeq \mathfrak{F}^d_{\mathfrak{a}}(M)$ and Theorem 2.12. □

corollary 2.14 *Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and let M and N be two finitely generated R-modules of dimension d such that $\operatorname{Supp}_R M = \operatorname{Supp}_R N$. Then $\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M) = \operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(N)$.*

Proof. Since $\operatorname{Supp}_R M = \operatorname{Supp}_R N$, by [4, Corollary 3] $\operatorname{Att}_R \mathrm{H}^d_{\mathfrak{b}}(M) = \operatorname{Att}_R \mathrm{H}^d_{\mathfrak{b}}(N)$ and so $\operatorname{Att}_R \mathrm{H}^d_{\mathfrak{b}}(M) \cap \mathrm{V}(\mathfrak{a}) = \operatorname{Att}_R \mathrm{H}^d_{\mathfrak{b}}(N) \cap \mathrm{V}(\mathfrak{a})$. Thus by Theorem 2.12 we have $\operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(M) = \operatorname{Att}_R \mathfrak{F}^d_{\mathfrak{a},\mathfrak{b}}(N)$. □

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