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Inversion-like and Major-like Statistics of an Ordered Partition of a Multiset

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Abstract. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of a positive integer n, let $\mathrm{Tab}(\lambda, k)$ be the set of all tabloids of shape λ whose weights range over the set of all k-compositions of n and $\mathcal{OP}^k_{\lambda^{\mathrm{rev}}}$ the set of all ordered partitions into k blocks of the multiset $\{1^{\lambda_l} 2^{\lambda_{l-1}} \cdots l^{\lambda_1}\}$. In [2], Butler introduced an inversion-like statistic on $\mathrm{Tab}(\lambda, k)$ to show that the rank-selected Möbius invariant arising from the subgroup lattice of a finite abelian p-group of type λ has nonnegative coefficients as a polynomial in p. In this paper, we introduce an inversion-like statistic on the set of ordered partitions of a multiset and construct an inversion-preserving bijection between $\mathrm{Tab}(\lambda, k)$ and $\mathcal{OP}^k_{\hat{\lambda}}$. When $k = 2$, we also introduce a major-like statistic on $\mathrm{Tab}(\lambda, 2)$ and study its connection to the inversion statistic due to Butler.

1. Ordered Partitions of a Multiset

Let n be a positive integer. An ordered partition of $[n] := \{1, 2, \ldots, n\}$ is a disjoint union of nonempty subsets of $[n]$, and its nonempty subsets are called *blocks*. Conventionally we denote by $\pi = B_1/B_2/\cdots/B_k$ an ordered partition of $[n]$ into k blocks, where the elements in each block are arranged in the increasing order. The set of all ordered partitions of $[n]$ into k blocks will be denoted by $\mathcal{OP}^k_n$.

In the exactly same manner, one can define an ordered partition of a finite multiset. The set of all ordered partitions of a multiset S will be denoted by $\mathcal{OP}^k_S$. In particular, in case where S is a multiset given by

$$\{\underbrace{1, \cdots, 1}_{c_1-\text{times}}, \underbrace{2, \cdots, 2}_{c_2-\text{times}}, \cdots\cdots, \underbrace{l, \cdots, l}_{c_l-\text{times}}\}, \quad (\text{simply denoted by } \{1^{c_1} 2^{c_2} \cdots l^{c_l}\}),$$

we write $\mathcal{OP}^k_{(c_1, \cdots, c_l)}$ for $\mathcal{OP}^k_S$. For each $\pi = B_1/B_2/\cdots/B_k \in \mathcal{OP}^k_S$, the *type of* π is defined by a sequence $(b_1(\pi), b_2(\pi), \cdots, b_k(\pi))$, where $b_i(\pi)$ is the cardinality of

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B_i, $(1 \le i \le k)$. For simplicity, we write $\mathbf{type}(\pi)$ for the type of π. For instance, if $\pi = 1124/13/5/1/134/2$, then $\mathbf{type}(\pi) = (4,2,1,1,3,1)$.

Given a permutation ξ on a finite multiset S, define $\mathbf{inv}(\xi)$ by the total number of inversions of ξ, that is, the number of (i,j)'s such that $1 \le i < j \le |S|$, but $\xi_i > \xi_j$. On the other hand, for an ordered partition

$$\pi = \pi_{11}\pi_{12}\cdots\pi_{1b_1}/\cdots/\pi_{k1}\pi_{k2}\cdots\pi_{kb_k} \in \mathcal{OP}_n^k,$$

we define $\mathbf{inv}(\pi)$ by the number of all pairs $((i,j),(i',j'))$'s with $\pi_{ij} > \pi_{i'j'}$, where $1 \le i < i' \le k$, $1 \le j \le b_i$ and $1 \le j' \le b_{i'}$.

For each positive integer n, a *k-composition of n* is usually defined by a k-tuple, say $(c_1, c_2, \cdots, c_k)$, of nonnegative integers whose sum equals n. However, in the present paper, a k-composition of n will be used to mean by a k-tuple of positive integers and will be denoted by $(c_1, c_2, \cdots, c_k) \vDash n$. For positive integers n and k with $k \le n$ and for a k-composition $c = (c_1, c_2, \cdots, c_k)$ of n, we simply denote by $\mathbf{Perm}(n,k;c)$ the set of all permutations on $\{1^{c_1}2^{c_2}\cdots k^{c_k}\}$ of type c. We also let $\mathbf{Perm}(n,k)$ be the set of all permutations on $\{1^{c_1}2^{c_2}\cdots k^{c_k}\}$, where $(c_1, c_2, \cdots, c_k)$ ranges over the set of all k-compositions of n. Moreover, we simply denote by $\mathcal{OP}_n^k(c)$ the set of ordered partitions of $[n]$ into k-blocks of type c. With this notation, we can establish the following lemma.

Lemma 1.1. *Let n, k be positive integers with $k \le n$.*

(a) *Let $c = (c_1, c_2, \cdots, c_k)$ be a k-composition of n. Then there exists a bijection, say $\phi : \mathbf{Perm}(n,k;c) \to \mathcal{OP}_n^k(c)$, such that $\mathbf{inv}(\xi) = \mathbf{inv}(\phi(\xi))$ for all $\xi \in \mathbf{Perm}(n,k;c)$. Therefore*

$$\sum_{\xi \in \mathbf{Perm}(n,k;c)} q^{\mathbf{inv}(\xi)} = \sum_{\pi \in \mathcal{OP}_n^k(c)} q^{\mathbf{inv}(\pi)}.$$

(b) *There exists a bijection $\phi : \mathbf{Perm}(n,k) \to \mathcal{OP}_n^k$ such that $\mathbf{inv}(\xi) = \mathbf{inv}(\phi(\xi))$ for all $\xi \in \mathbf{Perm}(n,k)$. Therefore*

$$\sum_{\xi \in \mathbf{Perm}(n,k)} q^{\mathbf{inv}(\xi)} = \sum_{\pi \in \mathcal{OP}_n^k} q^{\mathbf{inv}(\pi)}.$$

Proof. (a) For a permutation $\xi = \xi_1\xi_2\cdots\xi_n$ on $\{1^{c_1}2^{c_2}\cdots k^{c_k}\}$, let $\phi(\xi)$ be the unique ordered partition whose r-th block consists of i's with $\xi_i = r$ for all $1 \le r \le k$. For instance, if $\xi = 1343222$, then $\phi(\xi) = 1/567/24/3$. Obviously this correspondence is bijective. One can easily verify that ϕ preserves the number of total inversions since (i,j) is an inversion if and only if the block containing j precedes the block containing i in $\phi(\xi)$.

(b) The desired assertion is straightforward from (a). □

In the following, we introduce a generalization of Lemma 1.1. For this purpose, the notion of tabloids is required since they can be viewed as a generalization of multiset permutations. Given a partition λ of n, a tabloid T of shape λ is a filling of the squares of λ with positive integers such that the entries in each row weakly increase from left to right. The weight of a tabloid is defined by its multiset of entries.

Definition 1.2.(see [2, Definition 1.3.1])

(a) If T is a tabloid of any shape, then the value of an entry x in T is the number of smaller entries in the same column and above x or in the next column to the right and below x. The value of T, denoted by $v(T)$, is the sum of the values of the entries in T.

(b) For a k-composition c of n, let $\mathrm{Tab}(\lambda, k; c)$ be the set of all tabloids of shape λ and weight $1^{c_1}2^{c_2}\cdots k^{c_k}$. And we let $\mathrm{Tab}(\lambda, k)$ be the set of all tabloids of shape λ whose weights range over the set of all k-compositions of n.

We first deal with the case where $\lambda = 1^n$. For each k-composition c of n, consider the function

$$\psi : \mathrm{Tab}(1^n, k; c) \to \mathrm{Perm}(n, k; c), \quad T \mapsto \psi(T),$$

where $\psi(T)$ is the permutation obtained by reading entries of T from bottom to top. Alternatively, $\psi(T)$ is obtained by rotating T by 90° clockwise. Obviously ψ is a bijection. Moreover, from Definition it follows that ψ is an inversion-preserving map, that is, $v(T) = \mathbf{inv}(\psi(T))$ for all $T \in \mathrm{Tab}(1^n, k; c)$. In view of Lemma , one can deduce that

$$\sum_{T\in \mathrm{Tab}(1^n,k;c)} q^{v(T)} = \sum_{\pi\in \mathcal{OP}_n^k(c)} q^{\mathbf{inv}(\pi)}. \tag{1.1}$$

by composing ϕ with ψ. In fact, $\phi \circ \psi(T)$ is the ordered partition into k blocks whose r-th block consists of i's with $T(i, 1) = r$ for each $1 \le r \le k$. Since c ranges over the set of all compositions of n, one can finally obtain the following identity:

$$\sum_{T\in \mathrm{Tab}(1^n,k)} q^{v(T)} = \sum_{\pi\in \mathcal{OP}_n^k} q^{\mathbf{inv}(\pi)}. \tag{1.2}$$

Before introducing a generalization of Eq.(1.1) and Eq.(1.2), let us explain the motivation briefly. Recall that $|\mathcal{OP}_n^k| = k!S(n, k)$, where $S(n, k)$ is a Stirling number of the second kind. There is a natural one to one correspondence between $\mathcal{OP}_n^k$ and the set of k-chains in the Boolean algebra, $B_n = [0, 1]^n$. (For the precise information, see [8].) On the other hand, it was shown in [2] that there exists close relationship between the subgroup lattice of a finite abelian p-group G of type $\lambda = (\lambda_1, \dots, \lambda_l)$ and the lattice $[0, \lambda_1] \times \cdots \times [0, \lambda_l]$. Here the *type of* G denotes the partition $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_l)$, $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l \ge 1$ when G is isomorphic to $\mathbb{Z}/p^{\lambda_1}\mathbb{Z} \times \mathbb{Z}/p^{\lambda_2}\mathbb{Z} \times \cdots \times \mathbb{Z}/p^{\lambda_l}\mathbb{Z}$. For instance, under this relationship, the case $\lambda = 1^n$ corresponds to $B_n = [0, 1]^n$. When G is a finite abelian p-group of type λ, define $\alpha_\lambda(S; p)$ to be the total number of chains with rank set S in the subgroup lattice of G.

Lemma 1.3.(see [2, Proposition 1.3.2]) *If* $S = \{a_1, a_2, \cdots, a_{k-1}\} \subseteq [|\lambda| - 1]$, *then*

$$\alpha_\lambda(S; p) = \sum_{T\in \mathrm{Tab}(\lambda,k;c)} p^{v(T)}, \tag{1.3}$$

where $c = (a_1, a_2 - a_1, \ldots, |\lambda| - a_{k-1})$.

Recall that, given a tabloid T, the value $v(T)$ of T is defined by

$$\sum_{(i,j)\in T} v(T;(i,j)),$$

where $v(T;(i,j))$ is the sum of

$$\sharp\{(i',j) \in T : 1 \le i' < i \text{ and } T(i',j) < T(i,j)\}$$

and

$$\sharp\{(i',j+1) \in T : i' > i \text{ and } T(i',j+1) < T(i,j)\}.$$

Here the notation $\sharp$ is used to denote the cardinality of a set. Now we define the inversion of an ordered partition of n into k-blocks as follows.

Definition 1.4. Given a k-composition c of n, let

$$\pi = \pi_{11}\pi_{12}\cdots\pi_{1c_1}/\cdots/\pi_{k1}\pi_{k2}\cdots\pi_{kc_k}$$

be an ordered partition into k blocks of type c.

(a) Given (i,j), we define $\mathbf{rep}(\pi:(i,j))$ by

$$\sharp\{(i',j') : \text{(i) } 1 \le i' < i \text{ and } \pi_{i'j'} = \pi_{ij}, \text{ or (ii) } i' = i \text{ and } \pi_{ij'} = \pi_{ij}\}.$$

(b) Given (i,j), $\mathbf{inv}(\pi;(i,j))$ is defined by the sum of

$$\sharp\{(i',j') : \text{(i) } 1 \le i' < i, \text{(ii) } \pi_{ij} < \pi_{i'j'}, \text{ and (iii) } \mathbf{rep}(\pi;(i',j')) = \mathbf{rep}(\pi;(i,j))\}$$

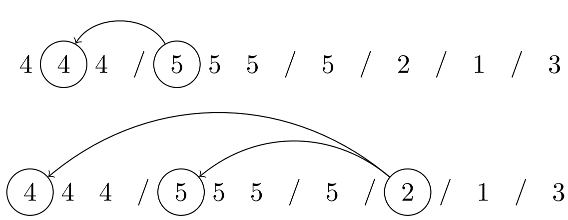
and

$$\sharp\{(i'',j'') : \text{(i) } 1 \le i'' < i, \text{(ii) } \pi_{i''j''} < \pi_{ij}, \text{ and (iii) } \mathbf{rep}(\pi;(i'',j'')) = \mathbf{rep}(\pi;(i,j)){+}1\}.$$

(c) The inversion of π, denoted by $\mathbf{inv}(\pi)$, is defined by

$$\sum_{\pi_{ij}\in\pi} \mathbf{inv}(\pi;(i,j)).$$

Example 1.5. Let $\pi = 444/555/5/2/1/3$. It is easy to show that the inversion table $(\mathbf{inv}(\pi;(i,j)))_{i,j}$ is given by $000/110/0/2/3/2$. For example, $\mathbf{inv}(\pi;(2,1))$ and $\mathbf{inv}(\pi;(4,1))$ can be illustrated well in the following diagram:



Let $C_\lambda(k;p)$ denote the total number of k-chains in the subgroup lattice of the finite abelian p-group G of type λ. The following theorem is the main result of this section.

Theorem 1.6. *For positive integers n, k with $k \le n$, let λ be a partition of n with $l(\lambda) = l$.*

(a) *Given a k-composition c of n, there exists a bijection*

$$\eta : \mathrm{Tab}(\lambda, k; c) \longrightarrow \mathcal{OP}^k_{\lambda^{\mathrm{rev}}}(c)$$

such that $v(T) = \mathbf{inv}(\phi(T))$ for all $T \in \mathrm{Tab}(\lambda, k; c)$. Here λ^{rev} denotes the partition $(\lambda_l, \cdots, \lambda_1)$. Therefore

$$\sum_{T\in \mathrm{Tab}(\lambda,k;c)} q^{v(T)} = \sum_{\pi \in \mathcal{OP}^k_{\lambda^{\mathrm{rev}}}(c)} q^{\mathbf{inv}(\pi)}.$$

(b) *For a prime p, we have*

$$C_\lambda(k;p) = \sum_{T\in \mathrm{Tab}(\lambda,k)} p^{v(T)} = \sum_{\pi \in \mathcal{OP}^k_{\lambda^{\mathrm{rev}}}} p^{\mathbf{inv}(\pi)}.$$

Proof. (a) Given $T \in \mathrm{Tab}(\lambda, k)$, let us obtain a word, say $w(T) = w_1 w_2 \cdots w_n$, by the reading which proceeds up columns from bottom to top and from left to right. Now define $\eta(T)$ in the following steps:

step 1: Each $T(i, j)$ is placed in the $T(i, j)$-th block as the value $l - i + 1$.

step 2: Assume that $T(i, j)$ appears in the k-th place among w_l's with the same value with $T(i, j)$. Then $l - i + 1$ appears in the k-th place of the $T(i, j)$-th block. Since $\eta(T)$ is an ordered partition with length k and weight $sh(T)$, ϕ is well-defined. Furthermore, it is a bijection since there is the inverse map which can be defined by reversing the above process.

In the following, let us show that η is weight preserving. Fix $(i, j) \in T$. Note that, if $\eta(T)_{ij}$ corresponds to the $T(a, b)$ under η, then $\mathbf{rep}(\eta(T); (i, j))$ is nothing other than b, the column of $T(a, b)$. Now we claim that there is a bijection between

$$\{(i', j) \in T : 1 \le i' < a \text{ and } T(i', b) < T(a, b)\}$$

and

$$\begin{aligned}\{(i', j') :\quad &\text{(i)}\quad 1 \le i' < i,\ \text{(ii)}\ \eta(T)_{ij} < \eta(T)_{i'j'},\\ &\text{(iii)}\quad \mathbf{rep}(\eta(T); (i', j')) = \mathbf{rep}(\eta(T); (i, j))\}.\end{aligned}$$

To see this, let $\eta(T)_{i'j'}$ correspond to the $T(a', b')$ under η. Then the third condition (iii) $\mathbf{rep}(\eta(T); (i', j')) = \mathbf{rep}(\eta(T); (i, j))$ implies $b' = b$. The first condition (i) $1 \le i' < i$ implies that $a' < a$. Finally, the second condition (ii) $\eta(T)_{ij} < \eta(T)_{i'j'}$ is equivalent to the condition $T(a', b') < T(a, b)$. In the same fashion, we can show that there is a bijection between

$$\{(i', j + 1) \in T : i' > i \text{ and } T(i', j + 1) < T(i, j)\}$$

and

$$\{(i'',j''):\quad \text{(i)}\quad 1\le i''<i,\ \text{(ii)}\ \pi_{i''j''}<\pi_{ij},\quad \text{(iii)}\quad \mathbf{rep}(\pi;(i'',j''))=\mathbf{rep}(\pi;(i,j))+1\}.$$

(b) First, note that there is a one to one correspondence between $\{S\subset[n-1]:|S|=k-1\}$ and the set all k-compositions of n. Thus, in view of Lemma 1.3, (b) can be derived from (a) by specializing q into p. $\square$

Remark 1.7. In the proof of Theorem 1.6, we can derive a bijection

$$\underline{\textbf{reading}}:\mathrm{Tab}(\lambda,k;c)\to \mathcal{OP}^l_c(\lambda^{\mathrm{rev}})$$

by reading the entries of a tabloid. So, by composing η^{-1} with $\underline{\textbf{reading}}$, we obtain a bijection

$$\theta:\mathcal{OP}^l_c(\lambda^{\mathrm{rev}})\to\mathcal{OP}^k_{\lambda^{\mathrm{rev}}}(c).$$

For instance, let T be the tabloid given by

2	2	2	3
1	1	1	
6			
4			
5			

then $\underline{\textbf{reading}}(T)=5/4/6/111/2223$ and $\eta(T)=444/555/5/2/1/3$. Thus we have

$$\theta(5/4/6/111/2223)=444/555/5/2/1/3.$$

2. A Major-like Statistics on the Set of Tabloids

In 1913, P. MacMahon introduced the *major* index statistic on the set of multiset permutations, which is defined by the sum of the descents, and this statistic is the same as the distribution of inversions. That is,

$$\sum_{\sigma\in\mathbf{Perm}(n,k;c)} q^{\mathbf{maj}(\sigma)}=\sum_{\sigma\in\mathbf{Perm}(n,k;c)} q^{\mathbf{inv}(\sigma)}=\begin{bmatrix} n\\ c_1,\cdots,c_k\end{bmatrix}_q. \tag{2.1}$$

In 1968, Foata constructed a map which assigns a permutation with a given major index to another with the same inversion index (see [4]).

After that, there were many results on these two statistics. An extension of MacMahon's result has been given by Kasraoui in [5] and many major-like statistics on the ordered partitions have been found by Kasraoui and Zeng in [6].

The purpose of this section is to introduce a major-like statistic, m, on the set $\mathrm{Tab}(\lambda,2)$ of tabloids such that

$$\sum_{\mathrm{T}\in\mathrm{Tab}(\lambda,2)} p^{v(T)}=\sum_{\mathrm{T}\in\mathrm{Tab}(\lambda,2)} p^{m(T)}.$$

Definition 2.1. If T is a tabloid with shape λ and weight $\{1^{c_1}, 2^{c_2}\}$, then the array of T, denoted $\mathcal{A}_T$, is obtained from T as follows.

(i) The last column of $\mathcal{A}_T$ is equal to that of T.

(ii) For every $1 \leq j \leq \lambda_1 - 1$, the jth column of $\mathcal{A}_T$ is defined by the following conditions:
- If there is an entry smaller than $T(i,j)$ in the $(j+1)$th column and below the ith row in T, then switch $T(i,j)$ with $T(i+1,j)$. Denote the result by $\hat{T}$.
- Continue the above process until no entries in $\hat{T}$ satisfy the conditions in the above step.

It should be noted that, from Definition 2.1, T can be recovered from $\mathcal{A}_T$ in a unique way. For all $1 \leq j \leq \lambda_1$, its jth column of $\mathcal{A}_T$ is contained in $\mathbf{Perm}(\lambda'_j, 2; c)$, where $c = (\lambda'_j - \nu'_j, \nu'_j)$. Define the inversion, **inv**, of $\mathcal{A}_T$ by

$$\mathbf{inv}(\mathcal{A}_T) = \sum_{1 \leq i \leq \lambda_1} \mathbf{inv}(\text{the } i\text{th column of } \mathcal{A}_T).$$

Then, by Definition 2.1, one has that $\mathbf{inv}(\mathcal{A}_T) = v(T)$.

Example 2.2. Let T be the tabloid as below. Then we have

T

2	2	2
1	1	1
1	2	
2		
1		

$v(T)$

1	1	0
·	·	·
·	1	
2		
·		

$\rightsquigarrow$

$\mathcal{A}_T$

1	1	2
2	2	1
1	2	
2		
1		

$\mathbf{inv}(\mathcal{A}_T)$

·	·	0
1	1	·
·	1	
2		
·		

Given a tabloid T with weight (c_1, c_2), we denote by $[k]T^{(j)}$ the number of occurrences of k, $(k = 1, 2)$ in the jth column of T. Similarly, we denote by $[k]\mathcal{A}_T^{(j)}$ the number of occurrences of $k\,(k = 1, 2)$ in the jth column of $\mathcal{A}_T$. Note that, for all j, we have

$$[k]T^{(j)} = [k]\mathcal{A}_T^{(j)}.$$

Before introducing a major-like statistic on the tabloid T, let us define the set of *descents* of $\mathcal{A}_T$, denoted $DES(\mathcal{A}_T)$, by

$$\{(i,j) \in \mathcal{A}_T : 1 \leq j \leq \lambda_1 - 1, [1]T^{(j+1)} + 1 < i \leq \lambda'_j, \mathcal{A}_T(i,j) > \mathcal{A}_T(i-1,j)\}$$
$$\cup \{(i,\lambda_1) \in \mathcal{A}_T : 2 \leq i \leq \lambda'_{\lambda_1}, \mathcal{A}_T(i,\lambda_1) > \mathcal{A}_T(i-1,\lambda_1)\}.$$

Now, define a major-like statistic on $\mathcal{A}_T$, denoted **maj**, by

$$\mathbf{maj}(\mathcal{A}_T) = \sum_{(i,j) \in DES(\mathcal{A}_T)} \lambda'_j - i + 1.$$

Definition 2.3. With the above notation, for a tabloid T of weight (c_1, c_2), the value of T, denoted by $m(T)$, is defined to be

$$\mathbf{maj}(\mathcal{A}_T) + \sum_{1 \le j \le \lambda_1 - 1} [2]T^{(j)} \cdot [1]T^{(j+1)}.$$

Example 2.4. Let T be the tabloid as below. Then we have

$$T = \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline 1 & 2 & \\ \cline{1-2} 2 & & \\ \cline{1-1} 1 & & \\ \cline{1-1} \end{array} \quad \rightsquigarrow \quad \mathcal{A}_T = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 & 1 \\ \hline 1 & 2 & \\ \cline{1-2} 2 & & \\ \cline{1-1} 1 & & \\ \cline{1-1} \end{array} \qquad \mathbf{maj}(\mathcal{A}_T) = \begin{array}{|c|c|c|}\hline \cdot & \cdot & 2 \\ \hline 2 & 2 & 1 \\ \hline 1 & 2 & \\ \cline{1-2} \mathbf{2} & & \\ \cline{1-1} 1 & & \\ \cline{1-1} \end{array}$$

In this case, $m(T)$ equals 4. Let $\alpha_\lambda(\mu; p)$ denote the number of subgroups of type μ in the finite abelian p-group of type λ.

Lemma 2.5.([2, Lemma 1.4.1], [3]) *For any partition μ with $\emptyset \neq \nu \subseteq \lambda$,*

$$\alpha_\lambda(\nu; p) = \prod_{j \ge 1} p^{\nu'_{j+1}(\lambda'_j - \nu'_j)} \binom{\lambda'_j - \nu'_{j+1}}{\nu'_j - \nu'_{j+1}}_p, \tag{2.2}$$

where λ' is the conjugate of λ and

$$\binom{n}{k}_p = \frac{(1-p^n)(1-p^{n-1})\cdots(1-p^{n-k+1})}{(1-p)(1-p^2)\cdots(1-p^k)}.$$

Lemma 2.6. *Let λ, ν be partitions with $\nu \subseteq \lambda$. Then we have*

$$\alpha_\lambda(\nu; p) = \sum_{T \in \mathrm{Tab}(\lambda, 2; (1^{|\nu|}, 2^{|\lambda| - |\nu|}))} p^{m(T)}.$$

Proof. In view of Eq. (2.2) and Definition 2.3, it suffices to show the equality

$$\prod_{j \ge 1} p^{\nu'_{j+1}(\lambda'_j - \nu'_j)} \binom{\lambda'_j - \nu'_{j+1}}{\nu'_j - \nu'_{j+1}}_p = \sum_{T \in \mathrm{Tab}(\lambda, 2; (1^{|\nu|}, 2^{|\lambda| - |\nu|}))} p^{[2]T^{(j)} \cdot [1]T^{(j+1)}} p^{\mathbf{maj}(\mathcal{A}_T)}. \tag{2.3}$$

The left hand side of the above equation can be rewritten as

$$\prod_{j \ge 1} p^{\nu'_{j+1}(\lambda'_j - \nu'_j)} \left(\sum_{\xi \in \mathbf{Perm}(\lambda'_j - \nu'_{j+1}, 2; (1^{\lambda'_j - \nu'_j}, 2^{\nu'_j - \nu'_{j+1}}))} p^{\mathbf{maj}(\xi)} \right) \tag{2.4}$$

$$= \sum_{\overline{\xi} = (\xi^{(1)}, \ldots, \xi^{(\lambda_1)})} p^{\mathbf{maj}(\overline{\xi})} p^{\nu'_{j+1}(\lambda'_j - \nu'_j)}, \tag{2.5}$$

where the sum ranges over the set

$$\mathbf{P} := \prod_{j\geq 1} \mathbf{Perm}(\lambda'_j - \nu'_{j+1}, 2; (1^{\lambda'_j-\nu'_j}, 2^{\nu'_j-\nu'_{j+1}}))$$

and $\mathbf{maj}(\overline{\xi}) = \sum_{1\leq i\leq \lambda_1} \mathbf{maj}(\xi^{(i)})$.

Given a sequence of permutations $\overline{\xi} = (\xi^{(1)}, \ldots, \xi^{(\lambda_1)})$ in $\mathbf{P}$, we let $\mathcal{A}_{\overline{\xi}}$ be the array with shape λ defined by

$$\mathcal{A}_{\overline{\xi}}(i,j) = \begin{cases} 1 & \text{if } i \leq \lambda'_j - \nu'_j, \\ {\xi^{(j)}}_{\lambda'_j - i+1} & \text{if } \lambda'_j - \nu'_j < i \leq \lambda'_j. \end{cases} \tag{2.6}$$

If we denote by $\mathcal{A}^j_{\overline{\xi}}$ the jth column of $\mathcal{A}_{\overline{\xi}}$, then Eq.(2.5) can be rewritten as

$$\sum_{\mathcal{A}_{\overline{\xi}}} p^{[2]\mathcal{A}^{(j)}_{\overline{\xi}} \cdot [1]\mathcal{A}^{(j+1)}_{\overline{\xi}}} p^{\mathbf{maj}(\mathcal{A}_{\overline{\xi}})},$$

where the sum runs over all $\mathcal{A}_{\overline{\xi}}$'s contained in

$$\{sh(\mathcal{A}_{\overline{\xi}}) = \lambda : [1]\mathcal{A}_{\overline{\xi}} = |\nu|,\ [2]\mathcal{A}_{\overline{\xi}} = |\lambda| - |\nu| \text{ and } \mathcal{A}_{\overline{\xi}} \text{ satisfies Eq.(2.6)}\}.$$

Also, recall that $\mathbf{maj}(\mathcal{A}_{\overline{\xi}}) = \sum_{1\leq j\leq \lambda_1} \mathbf{maj}(\xi^{(j)})$. Finally, by Definition , we can derive the equality

$$\sum_{T\in \mathrm{Tab}(\lambda,2;(1^{|\nu|},2^{|\lambda|-|\nu|}))} p^{[2]T^{(j)}\cdot[1]T^{(j+1)}} p^{\mathbf{maj}(\mathcal{A}_T)} = \sum_{T\in \mathrm{Tab}(\lambda,2;(1^{|\nu|},2^{|\lambda|-|\nu|}))} p^{m(T)}. \tag{2.7}$$

□

The main theorem of this section is stated as follows.

Theorem 2.7. *Let λ be a nonempty partition. Then*

$$C_\lambda(2;p) = \sum_{T\in \mathrm{Tab}(\lambda,2)} p^{m(T)}.$$

Proof. The desired result is almost straightforward since

$$\begin{aligned} C_\lambda(2;p) &= \sum_{\emptyset\subset\nu\subset\lambda} \alpha_\lambda(\nu;p) \\ &= \sum_{\emptyset\subset\nu\subset\lambda} \left(\sum_{T\in \mathrm{Tab}(\lambda,2;(1^{|\nu|},2^{|\lambda|-|\nu|}))} p^{m(T)} \right) \\ &= \sum_{T\in \mathrm{Tab}(\lambda,2)} p^{m(T)}. \end{aligned}$$

□

According to [7], let us define

$$\psi : \mathbf{Perm}(n,k;c) \to \mathbf{Perm}(n,k;c),\ \sigma \mapsto \beta^{(n-1)} \tag{2.8}$$

in the following steps.

1. Begin by setting $\beta^{(1)} = \sigma$, where $\sigma \in \mathbf{Perm}(n,k;c)$.
2. If ${\beta^{(1)}}_n \geq {\beta^{(1)}}_1$, draw a bar *before* each number in $\beta^{(1)}$ which is less than or equal to ${\beta^{(1)}}_n$, and also before ${\beta^{(1)}}_n$.
3. If ${\beta^{(1)}}_n < {\beta^{(1)}}_1$, draw a bar before each number in $\beta^{(1)}$ which is greater than ${\beta^{(1)}}_n$, and also before ${\beta^{(1)}}_n$.
4. Move each number at the beginning of a block to the end of the block. Then the last letter of $\beta^{(1)}$ is fixed.
5. Set $\beta^{(2)} = \beta^{(1)}$, and compare the $(n-1)$st letter with the first, creating blocks and draw an extra bar before the $(n-1)$st letter as above. Repeat the step (4).

It is well described in Chapter 10 of [7] that ψ is a bijective map such that $\mathbf{inv}(\sigma) = \mathbf{maj}(\psi(\sigma))$.

Theorem 2.8. *There is a bijective map from* $\mathrm{Tab}(\lambda, 2)$ *into itself such that*

$$\sum_{T \in \mathrm{Tab}(\lambda,2)} p^{v(T)} = \sum_{T \in \mathrm{Tab}(\lambda,2)} p^{m(T)},$$

where the sum runs over all tabloids in $\mathrm{Tab}(\lambda, 2)$*.*

Proof. More precisely, for any $\emptyset \subsetneq \nu \subsetneq \lambda$, we will construct a bijection Ψ from $\{T \in Tab(\lambda, 2; (1^{|\nu|}, 2^{|\lambda|-|\nu|}))\}$ into itself such that $v(T) = m(\Psi(T))$.

Given $T \in \{T \in Tab(\lambda, 2; (1^{|\nu|}, 2^{|\lambda|-|\nu|}))\}$, let us obtain $\mathcal{A}_T$ following Definition . For $1 \leq j \leq \lambda_1$, we let $\xi^{(j)}$ be the sub-column from (λ'_j, j) to $(\lambda'_j - \nu'_{j+1}, j)$ in $\mathcal{A}_T$. Then $\xi^{(j)}$ is obviously contained in $\mathbf{Perm}(\lambda'_j - \nu'_{j+1}, 2; (1^{\lambda'_j - \nu'_j}, 2^{\nu'_j - \nu'_{j+1}}))$. Applying the map ψ (in Eq.(2.8)) to $\xi^{(j)}$, one can obtain a unique $\tau^{(j)} := \psi(\xi^{(j)})$ satisfying that $\mathbf{inv}(\xi^{(j)}) = \mathbf{maj}(\tau^{(j)})$. Next, let us obtain $\mathcal{A}_{\hat{T}}$ from $\mathcal{A}_T$ by replacing $\xi^{(j)}$ by $\tau^{(j)}$ for all $1 \leq j \leq \lambda_1$. Finally, let us recover $\hat{T}$ from $\mathcal{A}_{\hat{T}}$ and then define $\Psi(T)$ by $\hat{T}$.

The bijectivity of Ψ follows from the uniqueness of ψ. Moreover, we have

$$\begin{aligned} v(T) &= \sum_{1 \leq j \leq \lambda_1} \mathbf{inv}({A_T}^{(j)}) \\ &= \sum_{1 \leq j \leq \lambda_1} [2]{A_T}^{(j)} \cdot [1]{A_T}^{(j+1)} + \mathbf{inv}(\xi^{(j)}) \\ &= \sum_{1 \leq j \leq \lambda_1} [2]{A_T}^{(j)} \cdot [1]{A_T}^{(j+1)} + \mathbf{maj}(\psi(\xi^{(j)})) \\ &= \sum_{1 \leq j \leq \lambda_1} [2]{A_T}^{(j)} \cdot [1]{A_T}^{(j+1)} + \mathbf{maj}(\tau^{(j)}) \\ &= m(\Psi(T)). \end{aligned}$$

This completes the proof. □

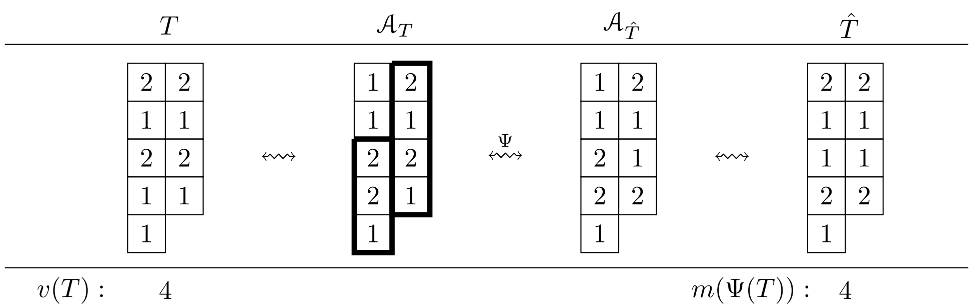
Remark 2.9. Specially, if $\lambda = 1^n$, then the set $\mathbf{Perm}(\lambda, 2)$ is related to ordered partitions of the set $\{1, \ldots, n\}$. In view of Eq. (1.2), we can easily define a major-like statistic on $\mathcal{OP}_n^k$, that is,

$$\sum_{T \in \mathrm{Tab}(1^n,k)} q^{m(T)} = \sum_{\pi \in \mathcal{OP}_n^k} q^{\mathbf{maj}(\pi)}.$$

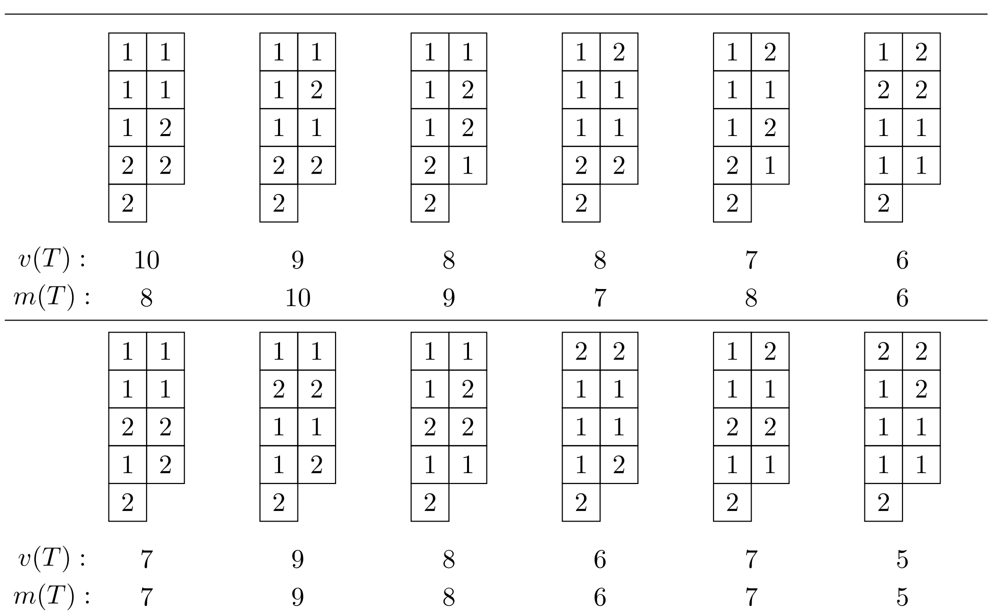
Example 2.10. Suppose that $\lambda = (2,2,2,2,1)$ and $\nu = (2,2,1)$. Then we get the following result.

$$\alpha_\lambda(\nu; p) = p^{2(5-3)} \binom{5-2}{3-2}_p \cdot p^0 \binom{4}{2}_p = p^4 \binom{3}{1}_p \cdot \binom{4}{2}_p$$

The next figure illustrates Ψ well.



To be precise, we have the following correspondence.



	1 1	1 1	1 1	2 2	2 2	2 2
	1 1	2 2	2 2	1 1	1 1	2 2
	2 2	1 1	2 2	1 1	2 2	1 1
	2 2	2 2	1 1	2 2	1 1	1 1
	1	1	1	1	1	1
$v(T)$:	8	7	6	6	5	4
$m(T)$:	6	8	7	5	6	4

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