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Reconfiguring k-colourings of Complete Bipartite Graphs

Marcel Celaya
School of Mathematics, Georgia Tech, Atlanta, GA, USA
e-mail : mcelaya6@math.gatech.edu

Kelly Choo and Gary MacGillivray*
Mathematics and Statistics, University of Victoria, Victoria, BC, Canada
e-mail : chook@uvic.ca *and* gmacgill@uvic.ca

Karen Seyffarth
Mathematics and Statistics, University of Calgary, Calgary, AB, Canada
e-mail : kseyffar@ucalgary.ca

Abstract. Let H be a graph, and $k \geq \chi(H)$ an integer. We say that H has a *cyclic Gray code of k-colourings* if and only if it is possible to list all its k-colourings in such a way that consecutive colourings, including the last and the first, agree on all vertices of H except one. The Gray code number of H is the least integer $k_0(H)$ such that H has a cyclic Gray code of its k-colourings for all $k \geq k_0(H)$. For complete bipartite graphs, we prove that $k_0(K_{\ell,r}) = 3$ when both ℓ and r are odd, and $k_0(K_{\ell,r}) = 4$ otherwise.

1. Introduction

Let H be a graph and k a positive integer. The *k-colouring graph of H, $G_k(H)$*, has as its vertices the proper k-colourings of H, any two of which are joined by an edge if and only if they agree on all but one vertex of H. When this graph is connected, any given k-colouring can be reconfigured into any other via a sequence of recolourings which each change the colour of exactly one vertex. When it is hamiltonian, there is a cyclic list that contains all of the k-colourings of H and consecutive elements of the list differ in the colour of exactly one vertex.

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* Corresponding Author.
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The *Gray code number of* H, denoted $k_0(H)$, is defined to be the smallest integer k such that $G_k(H)$ has a Hamilton cycle for all $k \geq k_0(H)$; that is, $k_0(H)$ is the least integer such that there exists a *cyclic Gray code of* k*-colourings* of H. It is shown in [7] that for any simple graph H, $k_0(H)$ is well-defined; i.e., for $k \geq col(G)+2$, where $col(G)$ denotes the colouring number of G, it is always possible to enumerate all proper k-colourings of H in such a way that any two successive colourings, including the first and the last, differ on only one vertex. A discussion of the origins of the Gray code number can be found in [7].

For our purposes, a *proper* k*-colouring* of a graph H is a function $f : V(H) \longrightarrow \{1, 2, \ldots, k\}$ such that if $xy \in E(H)$, $f(x) \neq f(y)$. We refer to the function values as the *colours* of the vertices, and for convenience use the term k*-colouring* (since we only consider proper k-colourings). This terminology is consistent with Bondy and Murty [2], and we refer the reader to that text for notation and terminology not defined here.

Choo and MacGillivray [7] establish Gray code numbers for various classes of graphs. For complete graphs, $k_0(K_1) = 3$ and $k_0(K_n) = n + 1$ when $n \geq 2$. For cycles, $k_0(C_n) = 4$ for $n \geq 3$. Any tree T satisfies $k_0(T) = 3$, except if T is a star with an odd number (at least three) of vertices, in which case $k_0(T) = 4$. The results here extend the work presented in [7] in that we determine the Gray code numbers of complete bipartite graphs, of which stars are a special case. The general case of bipartite graphs that are not complete remains largely unexplored. Connectivity and hamiltonicity of the k-colouring graphs of complete multipartite graphs is addressed in [1].

Connectivity of k-colouring graphs arises in random sampling of k-colourings, and approximating the number of k-colourings (see [8, 12, 13]). Neither the 2-colouring graph of a bipartite graph nor the 3-colouring graph of a 3-chromatic graph is ever connected, but for each $k \geq 4$ there exist k-chromatic graphs for which the k-colouring graph is connected, and others for which it is disconnected [4, 5]. On the other hand, for any graph H, the k-colouring graph is connected for all $k \geq col(H)+1$ [8]. While it is Polynomial to decide if the 3-colouring graph of a bipartite graph is connected [3], it is NP-complete to decide if two given colourings belong to the same component of such a graph [6]. In [3] it is shown that the diameter of any component of the 3-colouring graph of a bipartite graph is bounded by a quadratic function of the number of vertices, but for each $k \geq 4$ there exist bipartite graphs on n vertices for which the diameter of some component of the k-colouring graph is exponential in n; for each $k \geq 4$ it is PSPACE complete to decide if two given k-colourings belong to the same component of the k-colouring graph.

Other k-colouring graphs have also been considered. Viewing a k-colouring of H as a partition of $V(H)$ with at most k cells leads to the k*-Bell colour graph*, while viewing it as a partition into exactly k parts leads to the k*-Stirling colour graph*. Every graph on n vertices has a hamiltonian n-Bell colour graph, and for each $k \geq 4$, the k-Stirling colour graph of a tree is hamiltonian [9]. The *canonical* k*-colouring graph* of H with respect to a fixed ordering Π of $V(H)$ is the subgraph of $G_k(H)$ obtained by first defining two k-colourings to be equivalent if they give rise to the same partition of $V(H)$, and then taking the subgraph induced by the set of equivalence class representatives which are lexicographically least with respect to Π. For every tree T there exists an ordering Π of the vertices such that the canonical k-colouring graph of T with respect to Π is Hamiltonian for all $k \geq 3$ [10]. For any graph H and any vertex ordering Π, the canonical k-colouring graph of H with respect to Π is a spanning subgraph of the k-Bell colour graph of H. Finally, connectivity of the graph of list-$L(2,1)$-labellings – proper colourings with some additional restrictions – has recently been studied in [11].

2. Gray Code Numbers of Complete Bipartite Graphs

Let $K_{\ell,r}$ be a complete bipartite graph with bipartition (L, R), where the sets L and R are $L = \{p_1, p_2, \ldots, p_\ell\}$ and $R = \{q_1, q_2, \ldots, q_r\}$, respectively. A colouring f of $K_{\ell,r}$ with $f(p_i) = a_i$, $1 \leq i \leq \ell$ and $f(q_i) = b_i$, $1 \leq i \leq r$ is denoted $\langle a_1 a_2 \ldots a_\ell | b_1 b_2 \ldots b_r \rangle$.

We begin by establishing a lower bound on $k_0(K_{\ell,r})$.

Theorem 2.1. *For positive integers ℓ and r, $G_2(K_{\ell,r})$ is not hamiltonian, and $G_3(K_{\ell,r})$ is hamiltonian if and only if ℓ, r are both odd.*

Proof. A 2-colouring of $K_{\ell,r}$ is completely determined by the colour of any one of its vertices, implying that $|V(G_2(K_{\ell,r}))| = 2$. Moreover, these two 2-colourings cannot be joined by an edge since the colours of all vertices of $K_{\ell,r}$ must be changed to obtain one 2-colouring from the other. Since $K_{\ell,r}$ has a least two vertices, $G_2(K_{\ell,r})$ is not connected and hence not hamiltonian.

Notice that every 3-colouring of $K_{\ell,r}$ leaves at least one of L, R monochromatic, so for each j, $1 \leq j \leq 3$, we define L_j to be the subgraph of $G_3(H)$ induced by 3-colourings f in which $f(p) = j$ for all $p \in L$; R_j is defined analogously. Thus every vertex of $G_3(H)$ belongs to (at least) one of $L_1, L_2, L_3, R_1, R_2, R_3$.

The colourings in L_1 have all vertices of L coloured with 1 and the vertices of R coloured with 2 and 3. Thus each colouring in L_1 can be thought of as binary string of length r over $\{2, 3\}$, implying that L_1 is isomorphic to the r-dimensional cube, Q_r. It is routine to prove (and also follows from a result in [14]) that Q_r has a Hamilton path between $\underbrace{00\ldots0}_{r}$ and $\underbrace{11\ldots1}_{r}$ if and only if r is odd. Thus if r is odd, there is a Hamilton path $P_{L,1}$ in L_1 between $\langle 11\ldots1|22\ldots2\rangle$ and $\langle 11\ldots1|33\ldots3\rangle$. If ℓ is also odd, then $R_3 \cong Q_\ell$, so R_3 has a Hamilton path $P_{R,3}$ between $\langle 11\ldots1|33\ldots3\rangle$ and $\langle 22\ldots2|33\ldots3\rangle$. Analogously,

- L_2 has a Hamilton path $P_{L,2}$ between $\langle 22\ldots2|33\ldots3\rangle$ and $\langle 22\ldots2|11\ldots1\rangle$;
- R_1 has a Hamilton path $P_{R,1}$ between $\langle 22\ldots2|11\ldots1\rangle$ and $\langle 33\ldots3|11\ldots1\rangle$;
- L_3 has a Hamilton path $P_{L,3}$ between $\langle 33\ldots3|11\ldots1\rangle$ and $\langle 33\ldots3|22\ldots2\rangle$;
- R_2 has a Hamilton path $P_{R,2}$ between $\langle 33\ldots3|22\ldots2\rangle$ and $\langle 11\ldots1|22\ldots2\rangle$.

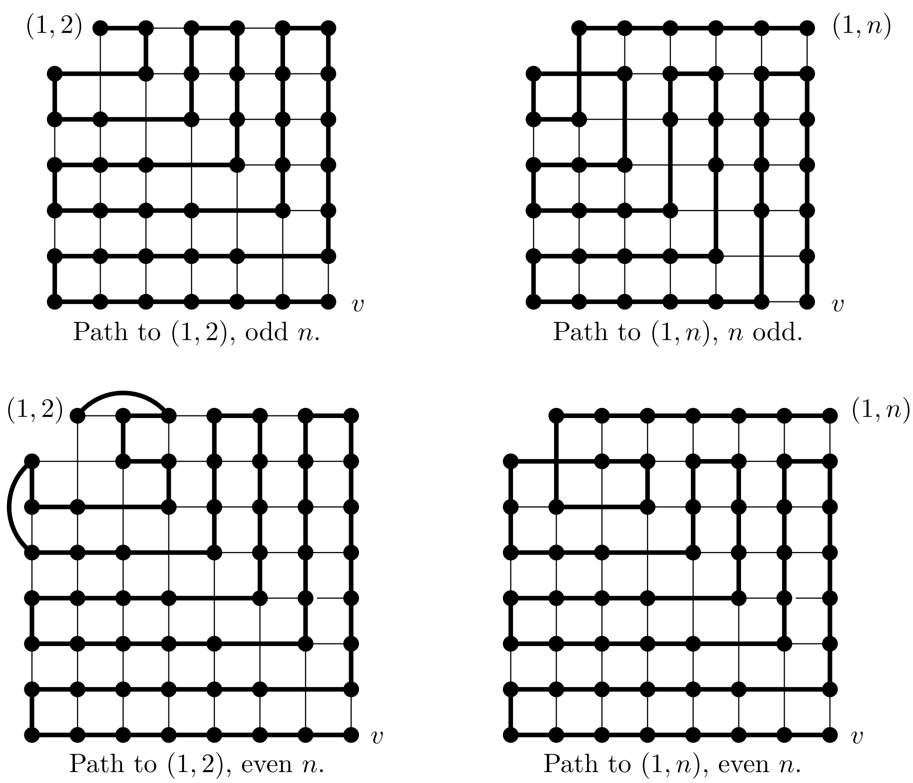

Figure 1: Hamilton paths in the graph J_n of Lemma 2.2 when $n = 7$ and $n = 8$. Not all edges are shown.

It follows that

$$P_{L,1} \cup P_{R,3} \cup P_{L,2} \cup P_{R,1} \cup P_{L,3} \cup P_{R,2}$$

is a Hamilton cycle of $G_3(K_{\ell,r})$.

Conversely, if r is even, then $G_3(K_{\ell,r})$ is not hamiltonian. The two-vertex set $\{\langle 11\dots1|22\dots2\rangle, \langle 11\dots1|33\dots3\rangle\}$ forms a cut of $G_3(K_{\ell,r})$, since one must encounter at least one of these two vertices before leaving or entering L_1. Therefore, a Hamilton cycle of $G_3(K_{\ell,r})$ must contain a Hamilton path of L_1 that starts and ends at these two vertices. Since r is even, $L_1 \cong Q_r$ contains no such Hamilton path, and thus $G_3(K_{\ell,r})$ is not hamiltonian. □

Theorem 2.1 implies that if $\ell, r \geq 1$ and at least one of these is even, then $k_0(K_{\ell,r}) \geq 4$. It remains to show that this inequality is an equality.

Consider the complete graph K_n with vertex set $\{1, 2, \dots, n\}$, and the cartesian product $K_n \square K_n$ with vertex set $\{(i, j) \mid 1 \leq i, j \leq n\}$. Denote by J_n the graph obtained from $K_n \square K_n$ by deleting the set of vertices $\{(i, i) \mid 1 \leq i \leq n - 1\}$.

Lemma 2.2. *For $n \geq 3$, J_n has a Hamilton path between (n,n) and any vertex of $J_n - (n,n)$.*

Proof. Let $v = (n,n)$. In Figure 1, we depict Hamilton paths between v and $(1,2)$ when n is odd and when n is even, and Hamilton paths between v and $(1,n)$ when n is even and when n is odd. The lemma is proved by showing that for every $w \in V(J_n)$, $w \neq v$, there is an automorphism of J_n that fixes v and maps w to either $(1,2)$ or $(1,n)$.

For any $\pi \in S_n$, define $\phi_\pi : V(J_n) \to V(J_n)$ by

$$\phi_\pi(a,b) = (\pi(a), \pi(b)).$$

If $\pi(n) = n$, then it is straightforward to see that ϕ_π is an automorphism of J_n.

Suppose $w = (w_1, w_2) \in V(J_n)$ is such that neither w_1 nor w_2 is equal to n. Choose $\pi = (1\ w_1)(2\ w_2)$, so that ϕ_π is an automorphism of J_n. Then

$$\phi_\pi(w) = (\pi(w_1), \pi(w_2)) = (1,2),$$

and hence J_n has a Hamilton path between v and w. If $w = (w_1, n)$, then choosing $\pi = (1\ w_1)$ again ensures that ϕ_π is an automorphism of J_n, and

$$\phi_\pi(w) = (\pi(w_1), \pi(n)) = (1,n);$$

i.e., J_n has a Hamilton path between v and w. Finally, suppose $w = (n, w_2)$, and let $\tau : V(J_n) \to V(J_n)$ be the automorphism of J_n in which

$$\tau(a,b) = (b,a).$$

Choosing $\pi = (1\ w_2)$ ensures that $\phi_\pi \circ \tau$ is an automorphism of J_n in which

$$\phi_\pi \circ \tau(n, w_2) = \phi_\pi(w_2, n) = (\pi(w_2), \pi(n)) = (1,n).$$

Again, there is a Hamilton path in J_n between v and w. $\square$

We now use Lemma 2.2 to prove our main theorem.

Theorem 2.3. *Let $1 \leq \ell \leq r$ and let $k \geq 4$. Then $G_k(K_{\ell,r})$ is hamiltonian.*

Proof. The proof is by induction on ℓ. When $\ell = 1$, the graph $K_{\ell,r}$ is a star, and it is known [7, Corollary 5.6] that $G_k(K_{1,r})$ is hamiltonian for $k \geq 4$.

For $\ell \geq 2$, let $K_{\ell,r}$ have bipartition (L,R) with $u \in L$ and $v \in R$, and let H denote the graph obtained from $K_{\ell,r}$ by deleting u and v. Then $H \cong K_{\ell-1,r-1}$, and has bipartition (L', R') where $L' = L \backslash \{u\}$ and $R' = R \backslash \{v\}$. Suppose $f_0, f_1, \ldots f_{N-1}, f_0$ is a Hamilton cycle in $G_k(H)$. For $0 \leq i \leq N-1$, define F_i to be the subgraph of $G_k(K_{\ell,r})$ induced by the colourings that agree with f_i on H. In what follows, the subscripts of f_i and F_i are taken modulo N. Let $[F_i, F_{i+1}]$ denote the set of edges that have one end in F_i and the other end in F_{i+1}.

Suppose $i \in \{0, 1, \ldots, N-1\}$. A colouring $t_i \in V(F_i)$ is called a *sink* if it is incident to an edge in $[F_i, F_{i+1}]$. If t_i is a sink, then it is adjacent to exactly one colouring in $V(F_{i+1})$.

Claim. For any $s_i \in V(F_i)$, there exists a sink $t_i \neq s_i$, and a Hamilton path in F_i between s_i and t_i.

Proof. Assume that the set of all colours is $C := \{1, 2, \ldots, k\}$. Let $U_\ell(i)$ and $U_r(i)$ be the sets of colours used in L' and R', respectively, under the colouring f_i. Then $A_\ell(i) := C \setminus U_r(i)$ and $A_r(i) := C \setminus U_\ell(i)$ are the sets of colours available for u and v, respectively, to extend f_i to a colouring in F_i.

Since only one vertex of H changes colour between f_i and f_{i+1}, at least one of the equalities $U_\ell(i+1) = U_\ell(i)$ or $U_r(i+1) = U_r(i)$ holds, implying that $A_r(i+1) = A_r(i)$ or $A_\ell(i+1) = A_\ell(i)$, respectively. Without loss of generality, assume that $A_r(i+1) = A_r(i)$.

Define $\alpha_i = |A_\ell(i)|$, $\beta_i = |A_\ell(i)|$, and let $A_\ell(i) = \{x_1, x_2, \ldots, x_{\alpha_i}\}$ and $A_r(i) = \{y_1, y_2, \ldots, y_{\beta_i}\}$. If $A_\ell(i+1) \not\supseteq A_\ell(i)$, then the colour change from f_i to f_{i+1} introduces a new colour to R', i.e., there exists a colour $x_j \in U_r(i+1) \setminus U_r(i)$. Since only one vertex of H changes colour between f_i and f_{i+1}, x_j is unique and we may assume, without loss of generality, that $A_\ell(i) \setminus A_\ell(i+1) = \{x_1\}$, and hence $x_1 \in U_r(i+1) \setminus U_r(i)$. It follows that if a colouring $t_i \in V(F_i)$ is not a sink, then $t_i(u) = x_1$.

Let $d_i := |A_\ell(i) \cap A_r(i)|$ be the number of colours available to both u and v when extending f_i to a colouring in F_i. Then $d_i < \min\{\alpha_i, \beta_i\}$ since $A_\ell(i)$, $A_r(i)$ each contains colours not found in the other, namely, the colours used in $U_r(i)$, $U_\ell(i)$, respectively. Assume $x_j = y_j$ for all j, $1 \leq j \leq d_i$.

If $d_i = 0$, then all colours of C are used in f_i and $\{U_\ell(i), U_r(i)\}$ is a partition of C. It follows that $U_r(i+1) \subseteq U_r(i)$, and hence $A_\ell(i) \subseteq A_\ell(i+1)$. Since $A_r(i) = A_r(i+1)$, every colouring in $V(F_i)$ is a sink. In this case, $F_i \cong K_{\alpha_i} \square K_{\beta_i}$; since $\alpha_i + \beta_i \geq 4$, F_i is hamiltonian. We obtain a Hamilton path with $s_i \in V(F_i)$ as one end by deleting an edge incident to s_i in an arbitrary Hamilton cycle of F_i.

Now suppose $d_i \geq 1$; then $\alpha_i \geq 2$ and $\beta_i \geq 2$. Let $s_i \in V(F_i)$. In what follows, we construct a Hamilton cycle in F_i so that on the Hamilton cycle, s_i is adjacent to a sink t_i. The subsequent deletion of the edge $s_i t_i$ results in the required Hamilton path.

First consider the case when $\alpha_i = 2$. Then $d_i = 1$, $x_1 = y_1$, and $\beta_i \geq 3$ (since $k \geq 4$ and $A_\ell(i) \cup A_r(i) = \{1, 2, \ldots, k\}$). If $s_i(v) \neq y_1$, then we may assume without loss of generality that $y_2 = s_i(v)$. Figure 2 shows a Hamilton cycle in F_i when $\alpha_i = 2$ and $\beta_i = 7$, where the hollow vertices represent sinks. This Hamilton cycle generalizes to arbitrary $\beta_i \geq 3$. Notice that if $s_i(v) = y_1$ (recall that $y_1 = x_1$), then $s_i(u) = x_2$; otherwise, $s_i(v) = y_2$. In either case, s_i is adjacent to a hollow vertex (sink) t_i on the Hamilton cycle.

Now suppose $\alpha_i \geq 3$. Figures 3 and 4 show Hamilton cycles in F_i when $\alpha_i = 4$ and $\beta_i = 7, 6$, respectively; again, the hollow vertices are sinks, and the Hamilton

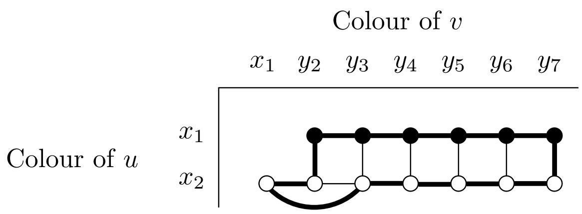

Figure 2: $d_i = 1$ and $\alpha_i = 2$.

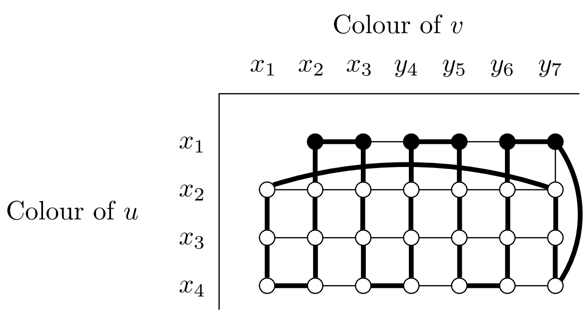

Figure 3: $d_i = 3$, $\alpha_i = 4$, and $\beta_i = 7$.

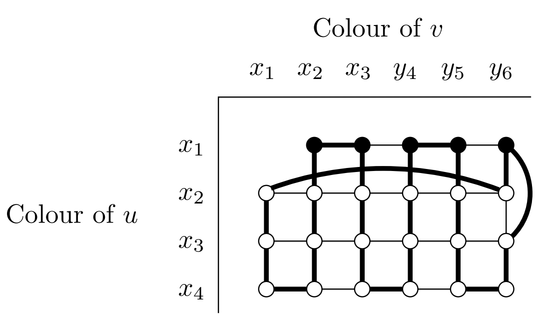

Figure 4: $d_i = 3$, $\alpha_i = 4$, and $\beta_i = 6$.

cycles generalize to arbitrary α_i and β_i odd/even, respectively. Notice that any $s_i \in V(F_i)$ is adjacent to a hollow vertex (sink) t_i on the Hamilton cycle. □

We now describe a Hamilton cycle of $G_k(K_{\ell,r})$ for $r \geq \ell \geq 2$. Choose $f_0 = \langle \underbrace{11\ldots1}_{\ell-1} | \underbrace{22\ldots2}_{r-1} \rangle$; since $r \geq \ell \geq 2$, $1 \notin U_r(1)$ and $2 \notin U_\ell(1)$. Thus $\langle \underbrace{11\ldots1}_{\ell} | \underbrace{22\ldots2}_{r} \rangle$ is a sink in $V(F_0)$, so we define $t_0 = \langle \underbrace{11\ldots1}_{\ell} | \underbrace{22\ldots2}_{r} \rangle$.

For $1 \leq i \leq N-2$, define $s_i \in V(F_i)$ to be the vertex adjacent to t_{i-1}. By our earlier claim, there is a Hamilton path in F_i between s_i and a sink t_i. Suppose s_{N-1} is the colouring in F_{N-1} adjacent to t_{N-2}. Observe that all vertices of F_{N-1} are sinks since the colours used in f_0 are used in f_{N-1}. Thus the Hamilton cycle in F_{N-1} (whose existence is guaranteed in the proof of the claim) offers two choices for t_{N-1}: the two colourings adjacent to s_{N-1} in the Hamilton cycle. Choose t_{N-1} so that it is not adjacent to t_0, and let s_0 be the colouring in F_0 adjacent to t_{N-1}. This choice guarantees that $s_0 \neq t_0$. Since F_0 is isomorphic to the graph J_n in Lemma 2.2 with $n = k-1$, it follows from that lemma that F_0 contains a Hamilton path between s_0 and t_0. The union of the Hamilton paths contained in the union of the F_i, $0 \leq i \leq n-1$, along with the edges $t_i s_{i+1}$, $0 \leq i \leq n-1$, yields the required Hamilton cycle. □

References

[1] S. Bard, *Colour graphs of complete multipartite graphs*, M.Sc. Thesis, University of Victoria, Victoria, BC, Canada, 2014.

[2] J. A. Bondy and U. S. R. Murty, *Graph Theory*, GTM 224, Springer, Berlin, 2008.

[3] P. Bonsma and L. Cereceda, *Finding paths between graph colourings: PSPACE-completeness and superpolynomial distances*, Theoretical Computer Science, **410**(2009), 5215–5226.

[4] L. Cereceda, J. van den Heuvel and M. Johnson, *Connectedness of the graph of vertex colourings*, Discrete Math., **308**(2008), 913–919.

[5] L. Cereceda, J. van den Heuvel and M. Johnson, *Mixing 3-colourings in bipartite graphs*, European J. Combin., **30**(2009), 1593–1606.

[6] L. Cereceda, J. van den Heuvel and M. Johnson, *Finding Paths Between 3-Colorings*, *J. Graph Theory*, **67**(2011), 69–82.

[7] K. Choo and G. MacGillivray, *Gray code numbers for graphs*, Ars Math. Contemp., **4**(2011), 125–139.

[8] M. Dyer, A. Flaxman, A. Frieze and E. Vigoda, *Randomly coloring sparse random graphs with fewer colors than the maximum degree*, Random Structures Algorithms, **29**(2006), 450–465.

[9] S. Finbow and G. MacGillivray, *Hamiltonicity of Bell and Stirling Colour Graphs*, manuscript 2014.

[10] R. Haas, *The canonical coloring graph of trees and cycles*, Ars Math. Contemp., **5**(2012), 149–157.

[11] T. Ito, K. Kawamura, H. Ono, and X. Zhou, *Reconfiguration of list $L(2,1)$-labelings in a graph*, Theoretical Computer Science, **9702**(2014), `DOI: 10.1016/j.tcs.2014.04.011`.

[12] M. Jerrum, *A very simple algorithm for estimating the number of k-colorings of a low-degree graph*, Random Structures Algorithms, **7**(1995), 157–165.

[13] B. Lucier and M. Molloy, *The Glauber dynamics for colorings of bounded degree trees*, SIAM J. Disc. Math., **25**(2011), 827–853.

[14] C. Savage and P. Winkler, *Montone Gray codes and the middle levels problem*, J. Combin. Theory Ser. A, **70**(1995), 230–248.