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Comparison of Rigorous Design Procedure with Approximate Design Procedure for Variable Sampling Plans Indexed by Quality Loss

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ABSTRACT

Traditional acceptance sampling plans have focused on the proportion of nonconforming items as an attribute criterion for quality. In today's modern quality management under high quality production environments, the reduction of the deviation from a target value in a quality characteristic has become the most important purpose. In consequence, various inspection plans for the purpose of reducing the deviation from the target value in the quality characteristic have been investigated. In this case, a concept of the quality loss evaluated by the deviation from the target value has been accepted as the variable evaluation criterion of quality. Further, some quality measures based on the quality loss have been devised; e.g. the process loss and the process capability index. Then, as one of inspection plans based on the quality loss, the rigorous design procedure for the variable sampling plan having desired operating characteristics (VS-OC plan) indexed by the quality loss has been proposed by Yen and Chang in 2009. By the way, since the estimator of the quality loss obeys the non-central chi-square distribution, the rigorous design procedure for the VS-OC plan indexed by the quality loss is complicated. In particular, the rigorous design procedure for the VS-OC plan requires a large number of the repetitive and complicated numerical calculation about the non-central chi-square distribution. On the other hand, an approximate design procedure for the VS-OC plan has been proposed before the proposal of the above rigorous design procedure. The approximate design procedure for the VS-OC plan has been constructed by combining Patnaik approximation relating the non-central chi-square distribution to the central chi-square distribution and Wilson-Hilferty approximation relating the central chi-square distribution to the standard normal distribution. Then, the approximate design procedure has been devised as a convenient procedure without complicated and repetitive numerical calculations. In this study, through some comparisons between the rigorous and approximate design procedures, the applicability of the approximate design procedure has been confirmed.

Keywords: Non-Central Chi-Square Distribution, Operating Characteristics, Patnaik Approximation, Wilson-Hilferty Approximation

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1. INTRODUCTION

Quality of lots has been traditionally evaluated based on attribute property such as the proportion or number of nonconforming items in lots. As a recent study based on the traditional evaluation based on attribute property, Subramani and Balamurali (2016) considered the attribute single sampling plan indexed by the number of nonconforming items in a lot.

On the other hand, Taguchi (1985) proposed the quality loss as a measure to evaluate the quality of items based on variable property. The quality loss is known as one of the key elements in the Taguchi method, and evaluated by the function based on the deviation from the ideal quality of item. Further, some quality measures based on the quality loss have been devised; e.g. the process loss and the process capability index. Then, it can be seen that the process loss and the process capability index are essentially reduced to the quality loss.

By the appearance of the quality loss, some sampling plans based on these quality measures have been designed newly. For example, Seifi and Nezhad (2016) developed some sampling plans for resubmitted lots based on the process capability index. And also, Yen and Cheng (2009) considered the rigorous variable sampling plan having desired operating characteristics (VS-OC plan) indexed by the quality loss. Their rigorous design procedure needs to solve two nonlinear simultaneous equations including the evaluation of the non-central chi-square distributions. However, because it is difficult to evaluate the cumulative distribution function (CDF) and percentile of the non-central chi-square distribution, it is also difficult to solve the above simultaneous equations in a general way. Therefore, the sampling plan in Yen and Cheng (2009) was considered under some specified conditions.

On the other hand, Arizono *et al.* (1997) considered the VS-OC plan indexed by the quality loss before the proposal of the rigorous design procedure by Yen and Cheng (2009). Specifically, Arizono *et al.* (1997) constructed the design procedure of the VS-OC plan indexed by the quality loss by combining Patnaik approximation (See Patnaik, 1949) and Wilson-Hilferty approximation (See Wilson and Hilferty, 1931).Then, Patnaik approximation has been known as an approximation method relating the non-central chi-square distribution to the central chi-square distribution, and Wilson-Hilferty approximation has been developed as an approximation method relating the central chi-square distribution to the standard normal distribution. As the consequence, the design procedure by Arizono *et al.* (1997) for the VS-OC plan indexed by the quality loss has been devised as a convenient procedure without a large number of the repetitive and complicated numerical calculation.

In this study, under the more general condition than the condition considered by Yen and Cheng (2009), the rigorous design procedure for the VS-OC plan indexed by the quality loss is constructed first. Specifically, remark that the general condition considered in this study is identical to the condition considered by Arizono *et al.* (1997). Then, we propose the calculation algorithm incorporating the confluent hypergeometric function for the rigorous design procedure in Yen and Cheng (2009). Further, the approximate design procedure by Arizono *et al.* (1997) for the same sampling plan is summarized. Thereafter, we compare the rigorous design procedure with the approximate design procedure. Then, through some comparisons between the rigorous and approximate design procedures, the applicability of the approximate design procedure by Arizono *et al.* (1997) has been confirmed. Remark that the main aim in this study is to demonstrate the applicability of the approximate design procedure by Arizono *et al*. (1997).

2. BRIEF OF DESIGN PROBLEM OF VS−OC PLAN INDEXED BY QUALITY LOSS

In the Taguchi method, the loss caused by the deviation from the target value in a quality characteristic is expressed as a quadratic form with respect to the difference between the actual value and the target value. When the target value of the quality of item is specified as T and the lot quality obeys a normal distribution $N(\mu, \sigma^2)$, the expected loss can be evaluated as

$$\begin{aligned} E[k(x-T)^2] &= k\{\sigma^2 + (\mu - T)^2\} \\ &= k\tau^2, \end{aligned} \tag{1}$$

where x denotes the measured actual value of characteristic and k denotes the proportional coefficient based on the functional limit of quality and the monetary loss brought by the item which cannot fulfill its function.

From Eq. (1), we see that τ^2 is a loss when $k = 1$ and that the quality loss can be evaluated as τ^2 without loss of generality. Note that there are innumerable combinations of mean μ and variance σ^2 giving the same value of τ^2.

Then, the quality loss τ^2 can be treated as the new evaluation measure of quality instead of the traditional attribute property such as the proportion of nonconforming items. In this case, the VS-OC plan indexed by the quality loss which is the new quality measure should be considered. Accordingly, the VS-OC plan indexed by quality loss has been considered by Arizono *et al.* (1997).

Let τ_0^2 be the specified acceptance loss, τ_1^2 the specified rejection loss, where $\tau_0^2 < \tau_1^2$. And then, the combinations (μ_i, σ_i^2), $i = 0, 1$, mean the combinations of mean and variance causing the same loss τ_i^2, respectively. Then, the combinations of mean μ_i and variance σ_i^2 giving the same value of τ_i^2 exist infinitely on the respective semicircle lines in the (μ, σ) half-plane such as

$$\tau_0^2 = \sigma_0^2 + (\mu_0 - T)^2, \tag{2}$$

$$\tau_1^2 = \sigma_1^2 + (\mu_1 - T)^2. \tag{3}$$

The VS-OC plan can be constructed as

$$\begin{cases} \text{if } \tau^2 \le \tau_0^2, \text{ then the lot should be accepted} \\ \qquad\qquad \text{with the producer's risk } \alpha, \\ \text{if } \tau^2 \ge \tau_1^2, \text{ then the lot should be rejected} \\ \qquad\qquad \text{with the consumer's risk } \beta. \end{cases} \tag{4}$$

Then, it is necessary to estimate the quality loss τ^2.

Let x_j, $j = 1, 2, \cdots, n$, be random samples from a normal distribution $N(\mu, \sigma^2)$. Then, the estimator $\hat{\tau}^2$ of the expected loss can be defined by

$$\hat{\tau}^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - T)^2 = s^2 + (\bar{x} - T)^2, \tag{5}$$

where $\bar{x}$ and s^2 denote the maximum likelihood estmators of μ and σ^2 calculated as follows:

$$\bar{x} = \frac{1}{n}\sum_{j=1}^{n}x_j, \quad s^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})^2.$$

Since the statistic $n\hat{\tau}^2/\sigma^2$ is written as

$$\frac{n\hat{\tau}^2}{\sigma^2} = \frac{ns^2}{\sigma^2} + \frac{n(\bar{x} - T)^2}{\sigma^2}, \tag{6}$$

it is seen that the statistic $n\hat{\tau}^2/\sigma^2$ obeys the non-central chi-square distribution with n degrees of freedom and non-centrality parameter $n\xi$, where ξ is defined as

$$\xi = \frac{(\mu - T)^2}{\sigma^2}. \tag{7}$$

The design problem of the VS-OC plan indexed by the quality loss is to determine the acceptance criterion c and sample size n which satisfy the producer's risk α and the consumer's risk β in any combination of mean and variance giving by Eq. (2) and Eq. (3). Then, we have the following inequalities:

$$\Pr\left\{\hat{\tau}^2 < c \,|\, \forall(\mu_0, \sigma_0^2) \in \Omega(\tau_0^2)\right\} \ge 1 - \alpha, \tag{8}$$

$$\Pr\left\{\hat{\tau}^2 < c \,|\, \forall(\mu_1, \sigma_1^2) \in \Omega(\tau_1^2)\right\} \le \beta, \tag{9}$$

where $\Omega(\tau^2)$ represents the set consisting of the combinations in (μ, σ^2) satisfying the quality loss Then, Eq. (8) can be reduced as

$$\min_{\forall(\mu_0, \sigma_0^2) \in \Omega(\tau_0^2)} \Pr\left\{\frac{n\hat{\tau}^2}{\sigma_0^2} < \frac{nc}{\sigma_0^2} \,\middle|\, \forall(\mu_0, \sigma_0^2)\right\} = 1 - \alpha. \tag{10}$$

Since the statistic $n\hat{\tau}^2/\sigma_0^2$ obeys the non-central chi-square distribution with n degrees of freedom and non-centrality parameter $n\xi_0 = n(\mu_0 - T)^2/\sigma_0^2$ under τ_0^2, the acceptance criterion c should be given by

$$c = \max_{\forall(\mu_0, \sigma_0^2) \in \Omega(\tau_0^2)} \left\{\frac{\sigma_0^2 \chi^2(\alpha; n, n\xi_0)}{n}\right\}, \tag{11}$$

where $\chi^2(\alpha; n, n\xi_0)$ denotes the upper 100α percentile of the non-central chi-square distribution with n degrees of freedom and non-centrality parameter $n\xi_0$.

In the same way, we also obtain the following relation under τ_1^2 from Eq. (9):

$$\max_{\forall(\mu_1, \sigma_1^2) \in \Omega(\tau_1^2)} \Pr\left\{\frac{n\hat{\tau}^2}{\sigma_1^2} < \frac{nc}{\sigma_1^2} \,\middle|\, \forall(\mu_1, \sigma_1^2)\right\} = \beta. \tag{12}$$

In this case, the statistic $n\hat{\tau}^2/\sigma_1^2$ obeys the non-central chi-square distribution with n degrees of freedom and non-centrality parameter $n\xi_1 = n(\mu_1 - T)^2/\sigma_1^2$ under τ_1^2. Therefore, the acceptance criterion c should be obtained as

$$c = \min_{\forall(\mu_1, \sigma_1^2) \in \Omega(\tau_1^2)} \left\{\frac{\sigma_1^2 \chi^2(1-\beta; n, n\xi_1)}{n}\right\}, \tag{13}$$

where $\chi^2(1-\beta; n, n\xi_1)$ denotes the upper $100(1-\beta)$ percentile of the non-central chi-square distribution with n degrees of freedom and non-centrality parameter $n\xi_1$.

Therefore, under the constraint that the relations in Eq. (8) and Eq. (9) should be satisfied simultaneously, the following inequality is derived:

$$\max_{\forall(\mu_0, \sigma_0^2) \in \Omega(\tau_0^2)} \left\{\frac{\sigma_0^2 \chi^2(\alpha; n, n\xi_0)}{n}\right\} \le \min_{\forall(\mu_1, \sigma_1^2) \in \Omega(\tau_1^2)} \left\{\frac{\sigma_1^2 \chi^2(1-\beta; n, n\xi_1)}{n}\right\}. \tag{14}$$

If the required sample size n is obtained under Eq. (14), the acceptance criterion c can be also given by using Eq. (11) (or Eq. (13)).

However, to that end, we have to calculate the percentiles $\chi^2(\alpha; n, n\xi_0)$ and $\chi^2(1-\beta; n, n\xi_1)$ of the respective non-central chi-square distributions. In addition, both the maximum value of $\sigma_0^2\chi^2(\alpha; n, n\xi_0)$ for any combinations $(\mu_0, \sigma_0^2) \in \Omega(\tau_0^2)$ and the minimum value of $\sigma_1^2\chi^2(1-\beta; n, n\xi_1)$ for any combinations $(\mu_1, \sigma_1^2) \in \Omega(\tau_1^2)$ have to be searched for. Remark that a large number of the repetitive and complicated numerical calculation about the non-central chi-square distribution is needed because the innumerable combinations $(\mu_0, \sigma_0^2) \in \Omega(\tau_0^2)$ and $(\mu_1, \sigma_1^2) \in \Omega(\tau_1^2)$ exist respectively.

On the other hand, Yen and Chen (2009) specified the critical situation as $\xi_0 = \xi_1 = 0$ based on some numerical examples and discussed the design procedure under this specified critical condition. However, under the conditions of $\xi_0 = 0$ and $\xi_1 = 0$, it is not guaranteed theoretically that $\sigma_0^2\chi^2(\alpha; n, n\xi_0)$ is always maximized and $\sigma_1^2\chi^2(1-\beta; n, n\xi_1)$ is always minimized. In addition, the calculation methods of the CDF and the percentile of the non-central chi-square distribution were described in Yen and Cheng (2009).Therefore, we propose the calculation methods, based on the confluent hypergeometric function, of the CDF and the percentile of the non-central chi-square distribution to design the rigorous VS-OC plan indexed by the quality loss in Yen and Cheng (2009).

3. PROPOSAL OF CALCULATION ALGORITHM FOR PERCENTILE OF NON-CENTRAL CHI-SQUARE DISTRIBUTION BASED ON CONFLUENT HYPERGEOMETRIC FUNCTION

The probability density function (PDF) $f(x; k, \lambda)$ of the non-central chi-square distribution with k degrees of freedom and non-centrality parameter λ is given by

$$f(x; k, \lambda) = \sum_{j=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^j}{j!} g(x; k+2j), \tag{15}$$

where $g(x; k)$ means the PDF of the central chi-square distribution with k degrees of freedom. Then, the CDF $P(x; k, \lambda)$ of the non-central chi-square distribution with k degrees of freedom and non-centrality parameter λ can be written as

$$P(x; k, \lambda) = \sum_{j=0}^{\infty} \frac{e^{-\lambda/2}(\lambda/2)^j}{j!} Q(x; k+2j), \tag{16}$$

where $Q(x; k)$ is the CDF of the central chi-square distribution with k degrees of freedom which is given by

$$Q(x; k) = \frac{\gamma(k/2, x/2)}{\Gamma(k/2)}, \tag{17}$$

where $\gamma(s, z)$ is the lower incomplete gamma function and $\Gamma(z)$ is the complete gamma function. The lower incomplete gamma function $\gamma(s, z)$ and the complete gamma function $\Gamma(z)$ are respectively given by

$$\gamma(s, z) = \int_0^z t^{s-1} e^{-t} dt, \tag{18}$$

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt. \tag{19}$$

Consequently, it is necessary to calculate the infinite series including the incomplete gamma function $\gamma(s, z)$ and the complete gamma function $\Gamma(z)$ iteratively. In such a case, an advanced software package for numerical evaluation is needed in general.

However, we propose the alternative approach for numerical evaluation of the CDF of the non-central chi-square distribution under a common environment composed of standard personal computers and standard programming languages such as C.

Eq. (18) can be expanded as

$$\begin{aligned}\gamma(s, z) &= \int_0^z t^{s-1} e^{-t} dt \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m}{s+m} \frac{z^{s+m}}{m!} \\ &= \frac{z^s}{s} \frac{\Gamma(s+1)}{\Gamma(s)} \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{\Gamma(s+1+m)} \frac{(-z)^m}{m!} \\ &= \frac{z^s}{s} {}_1F_1(s, s+1, -z),\end{aligned} \tag{20}$$

where ${}_1F_1(\alpha, \gamma, z)$ is the confluent hypergeometric function which is defined as

$${}_1F_1(\alpha, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+m)}{\Gamma(\gamma+m)} \frac{z^m}{m!}. \tag{21}$$

From Eq. (17)-Eq. (21), Eq. (16) can be rewritten as

$$P(x; k, \lambda) = \sum_{j=0}^{\infty} \frac{e^{-\frac{\lambda}{2}} \left(\frac{\lambda}{2}\right)^j}{j!} \frac{\left(\frac{x}{2}\right)^{k/2+j}}{\Gamma\left(\frac{k}{2}+j+1\right)} \times {}_1F_1\left(\frac{k}{2}+j, \frac{k}{2}+j+1, -\frac{x}{2}\right). \tag{22}$$

Further, we consider the calculation algorithm for the confluent hypergeometric function ${}_1F_1(\alpha, \gamma, z)$. Then, we define the numerical sequence $A_m(z)$ such as

$$A_m(z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+m)}{\Gamma(\gamma+m)} \frac{z^n}{m!}. \tag{23}$$

If $m \geq 1$, the general term $A_m(z)$ is calculated with

$$A_m(z) = A_{m-1}(z) \times \frac{\alpha+m-1}{\gamma+m-1} \frac{z}{m}. \tag{24}$$

Therefore, the confluent hypergeometric function can be calculated recursively with

$${}_1F_1(\alpha, \gamma, z) = \sum_{m=0}^{\infty} A_m(z). \tag{25}$$

By using Eq. (22)-Eq. (25), the CDF of the non-central chi-square distribution can be evaluated with required precision.

Because the value of the CDF can be calculated, we can obtain the percentile of the non-central chi-square distribution under a common environment composed of standard personal computers and standard programming languages such as C by using a search technique such as the bisection method. Then, the required sample size n satisfying the relation of Eq. (14) can be found, so the rigorous design procedure for the VS-OC plan indexed by the quality loss can be constructed in this study. However, it is found that the rigorous design procedure for the VS-OC plan indexed by the quality loss needs a large number of the repetitive and complicated numerical calculation about the non-central chi-square distribution.

4. APPROXIMATE DESIGN PROCEDURE CONSTRUCTED BY ARIZONO *ET AL.*

Arizono *et al*. (1997) have proposed the approximate design procedure for the VS-OC plan indexed by the quality loss. This approximate design procedure has been constructed by combining Patnaik approximation relating the non-central chi-square distribution to the central chi-square distribution and Wilson-Hilferty approximation relating the central chi-square distribution and the standard normal distribution.

Arizono *et al.* (1997) have discussed about the following statistic ρ:

$$\rho = \frac{2E\left[\frac{n\hat{\tau}^2}{\sigma^2}\right]}{V\left[\frac{n\hat{\tau}^2}{\sigma^2}\right]}\frac{n\hat{\tau}^2}{\sigma^2} = \frac{1+\xi}{1+2\xi}\frac{n\hat{\tau}^2}{\sigma^2}. \quad (26)$$

Based on the non-central chi-square distribution with n degrees of freedom and non-centrality parameter $n\xi$, the mean and variance of statistic ρ are given by

$$E[\rho] = \frac{1+\xi}{1+2\xi}E\left[\frac{n\hat{\tau}^2}{\sigma^2}\right] = \frac{n(1+\xi)^2}{1+2\xi}, \quad (27)$$

$$V[\rho] = \left(\frac{1+\xi}{1+2\xi}\right)^2 V\left[\frac{n\hat{\tau}^2}{\sigma^2}\right] = \frac{2n(1+\xi)^2}{1+2\xi}. \quad (28)$$

It is seen that the mean and variance of statistic ρ coincide with those of the central chi-square distribution with ϕ degrees of freedom, where

$$\phi = \frac{n(1+\xi)^2}{1+2\xi}. \quad (29)$$

Accordingly, the central chi-square distribution with ϕ degrees of freedom in Eq. (29) can be employed as the approximate distribution of ρ. This approximation technique has been proposed by Patnaik (1949). Further, it is easy to derive that the function ϕ is the monotonous increasing function in ξ. Then, it can be presented that $\phi \geq n$. It can be easily known that the minimum value $\phi_{\min} = n$ is given by the condition of $\xi = 0$.

From Eq. (5)-Eq. (7), Eq. (26) and Eq. (29), the statistic ρ can be rewritten as $\rho = \phi\hat{\tau}^2/\tau^2$. Hereby, the distribution of estimator $\hat{\tau}^2$ is specified approximately as follows:

$$\hat{\tau}^2 \sim \frac{\tau^2}{\phi}\chi_\phi^2, \quad (30)$$

where χ_ϕ^2 means the central chi-square distribution with ϕ degrees of freedom. Note that ϕ is the function composing of μ and σ^2. Hence, the distribution of $\hat{\tau}^2$ is not unique even if the value of τ^2 is same.

Therefore, Eq. (10) is written such as:

$$\min_{\forall(\mu_0,\sigma_0^2)\in\Omega(\tau_0^2)} \Pr\left\{\hat{\tau}^2 < \tau_0^2\frac{\chi_{\phi_0}^2(\alpha)}{\phi_0} \,\middle|\, \forall(\mu_0,\ \sigma_0^2)\right\} = 1-\alpha, \quad (31)$$

where $\chi_\phi^2(\alpha)$ denotes the upper 100α percentile of the central chi-square distribution with ϕ degrees of freedom, and then, $\phi_0 = n(1+\xi_0)^2/(1+2\xi_0)$ and $\xi_0 = (\mu_0 - T)^2/\sigma_0^2$. Therefore, the acceptance criterion c in consideration of the producer's risk α can be evaluated by the following relation:

$$c = \max_{\forall(\mu_0,\sigma_0^2)\in\Omega(\tau_0^2)} \tau_0^2\frac{\chi_{\phi_0}^2(\alpha)}{\phi_0}. \quad (32)$$

Further, by using Wilson-Hilferty approximation, $\chi_{\phi_0}^2(\alpha)/\phi_0$ is approximated as

$$\frac{\chi_{\phi_0}^2(\alpha)}{\phi_0} = \left\{1 - \frac{2}{9\phi_0} + u_\alpha\sqrt{\frac{2}{9\phi_0}}\right\}^3, \quad (33)$$

where u_α $(\alpha < 0.5)$ denotes the upper 100α percentile of the standard normal distribution. The right side of Eq. (33) is the monotonous decreasing function of ϕ_0 for $\alpha \leq 0.2525$ and ϕ_0 is the monotonous increasing function of ξ_0. Therefore, it is found that $\chi_{\phi_0}^2(\alpha)/\phi_0$ decreases with increasing ϕ_0. Consequently, the critical value should be decided under the condition of $\mu_0 = T$ and $\sigma_0^2 = \tau_0^2$, that is, $\xi_0 = 0$. Therefore, we have

$$\max_{\forall(\mu_0,\sigma_0^2)\in\Omega(\tau_0^2)} \tau_0^2\frac{\chi_{\phi_0}^2(\alpha)}{\phi_0} = \tau_0^2\frac{\chi_n^2(\alpha)}{n}. \quad (34)$$

Similarly, from Eq. (12), the acceptance criterion c in consideration of the consumer's risk β can be evaluated by

$$c = \min_{\forall(\mu_1,\sigma_1^2)\in\Omega(\tau_1^2)} \tau_1^2\frac{\chi_{\phi_1}^2(1-\beta)}{\phi_1}, \quad (35)$$

where $\phi_1 = n(1+\xi_1)^2/(1+2\xi_1)$ and $\xi_1 = (\mu_1 - T)^2/\sigma_1^2$. Then, $\chi_{\phi_1}^2(1-\beta)/\phi_1$ is approximated as

$$\frac{\chi_{\phi_1}^2(1-\beta)}{\phi_1} = \left\{1 - \frac{2}{9\phi_1} + u_{1-\beta}\sqrt{\frac{2}{9\phi_1}}\right\}^3, \quad (36)$$

where $u_{1-\beta}$ $(\beta < 0.5)$ denotes the upper $100(1-\beta)$ percentile of the standard normal distribution. Further, since $\chi^2_{\phi_1}(1-\beta)/\phi_1$ is the monotonous increasing function for ϕ_1, the critical value should be decided under the condition of $\mu_1 = T$ and $\sigma_1^2 = \tau_1^2$, that is, $\xi_1 = 0$. Therefore, we have

$$\max_{\forall(\mu_1,\sigma_1^2)\in\Omega(\tau_1^2)} \tau_1^2 \frac{\chi^2_{\phi_1}(1-\beta)}{\phi_1} = \tau_1^2 \frac{\chi^2_n(1-\beta)}{n}. \quad (37)$$

As mentioned in Eq. (34) and Eq. (37), the design conditions of $\xi_0 = 0$ and $\xi_1 = 0$ have been derived theoretically in Arizono *et al*. (1997).

Under the consideration as mentioned above, we can define the following conservative decision rule:

$$\begin{cases} \text{if } \hat{\tau}^2 \le \tau_0^2 \dfrac{\chi^2_n(\alpha)}{n}, \text{ then accept the lot,} \\ \text{otherwise reject the lot.} \end{cases} \quad (38)$$

Then, it is evident that the value of the actual producer's risk α^* for any pair of μ_0 and σ_0^2 is less than or equal to the specified producer's risk α. Simultaneously, in order that the value of the actual consumer's risk β^* for any pair of μ_1 and σ_1^2 be less than or equal to the specified consumer's risk β, the following relation has to be satisfied:

$$\frac{\chi^2_n(\alpha)}{n}\tau_0^2 \le \frac{\chi^2_n(1-\beta)}{n}\tau_1^2. \quad (39)$$

Then, we can obtain the sample size n as the smallest integer satisfying the relation as

$$\frac{\tau_0^2}{\tau_1^2} \le \frac{\chi^2_n(1-\beta)}{\chi^2_n(\alpha)}. \quad (40)$$

Then, by applying Wilson-Hilferty approximation to the upper percentile of the central chi-square distribution in Eq. (40), we can also have the closed form of the sample size n as follows:

$$n = \left\lfloor \frac{4}{9(K^2 - K\sqrt{K^2+4}+2)} \right\rfloor, \quad (41)$$

where

$$K = \frac{u_\alpha \tau_0^{2/3} + u_\beta \tau_1^{2/3}}{\tau_1^{2/3} - \tau_0^{2/3}}, \quad (42)$$

and $\lfloor w \rfloor$ means the smallest integer more than w. Finally, the conservative sampling plan can be described as the decision rule of Eq. (38) and the required sample size n deriving from the relation of Eq. (40) or from the closed form of Eq. (41).

5. NUMERICAL EXAMPLES

In order to compare the rigorous and approximate procedures for the VS-OC plans indexed by the quality loss in sections 3 and 4, we show some numerical examples. In what follows, let the producer's and consumer's risks be $\alpha = 0.05$ and $\beta = 0.1$, and let the acceptance loss be $\tau_0^2 = 1.0$.

In Table 1, a part of calculated results for the required sample size n and acceptance criterion c are illustrated when some values of the rejection loss τ_1^2. n_{rigorous} and c_{rigorous} are calculated by the rigorous design procedure using the algorithm based on the confluent hypergeometric function in section 3. Remark that the numerical results based on the rigorous design produce applying the confluent hypergeometric function have included the results derived by the design procedure of Yen and Chang (2009). Further, $n_{\text{approximate}}$ and $c_{\text{approximate}}$ are calculated by the approximate design procedure in section 4, where $c_{\text{approximate}}$ has been based on Eq. (32). As shown in Table 1, about the required sample size, it is found that the sample sizes $n_{\text{approximate}}$ under the approximate design procedure are the same as the sample sizes n_{rigorous} under the rigorous design procedure in all cases. About the acceptance criterion, we find that $c_{\text{approximate}}$ are almost equal to c_{rigorous}.

Table 2 shows the calculated values of the actual producer's risk α^* and the consumer's risk β^* which are evaluated by sampling plans $(n_{\text{approximate}}, c_{\text{approximate}})$ in case of $\tau_0^2 = 1.0$, $\tau_1^2 = 2.5$, $\alpha = 0.05$ and $\beta = 0.1$. As shown in Table 2, we are able to confirm that all combinations of mean and variance (μ_i, σ_i^2) almost satisfy the specified producer's and consumer's risks α and β. In addition, we have confirmed that the strictest case is given in $\mu_0 = T$, $\sigma_0^2 = \tau_0^2$ or $\mu_1 = T$, $\sigma_1^2 = \tau_1^2$.

In the case of $\mu_0 = T$ and $\sigma_0^2 = \tau_0^2$ under $\tau_0^2 = 1.0$, the actual producer's risk α^* is slightly more than the specified value. It is thought that this difference is influenced by the precision of Wilson-Hilferty approximation. Note that there is no influence of Patnaik approximation in this case because the statistic $n\hat{\tau}^2/\sigma_0^2$ obeys the central chi-square distribution because of non-central parameter $\xi_0 = (\mu_0 - T)^2/\sigma_0^2 = 0$.

From the above numerical examples, it is shown that the sampling plans derived by the approximate design procedure by Arizono *et al*. (1997) correspond to the rigorous VS-OC plans incorporating the proposed calculation procedure based on the confluent hypergeometric function for the design procedure in Yen and Cheng (2009). In addition, as described in Section 4, the design procedure by Arizono *et al*. (1997) for the VS-OC plan indexed by the quality loss does not require a large number of the repetitive and complicated numerical calculation about the non-central chi-square distribution. Therefore, we have reconfirmed that the approximate design procedure by Arizono *et al*. (1997) has the convenient and superior applicability.

Table 1. Respective sampling plans calculated by rigorous and approximate design procedures

τ_1^2	n_{rigorous}	c_{rigorous}	$n_{\text{approximate}}$	$c_{\text{approximate}}$
5.0	7	2.0096	7	2.0067
4.0	10	1.8307	10	1.8292
3.0	15	1.6664	15	1.6657
2.0	36	1.4166	36	1.4165
1.9	42	1.3839	42	1.3838
1.8	50	1.3501	50	1.3500
1.7	61	1.3153	61	1.3152
1.6	77	1.2790	77	1.2790
1.5	104	1.2385	104	1.2385

Table 2. Actual producer's risk α^* and consumer's risk β^* under approximate sampling plans $(n_{\text{approximate}}, c_{\text{approximate}})$

$\tau_0^2 = 1.0$			$\tau_1^2 = 2.5$		
$(\mu_0 - T)^2$	σ_0^2	α^*	$(\mu_1 - T)^2$	σ_1^2	β^*
0.0000	1.0000	0.0501	0.0000	2.5000	0.0937
0.1667	0.8333	0.0478	0.4167	2.0833	0.0907
0.3333	0.6667	0.0410	0.8333	1.6667	0.0808
0.5000	0.5000	0.0296	1.2500	1.2500	0.0629
0.6667	0.3333	0.0146	1.6667	0.8333	0.0365
0.8333	0.1667	0.0017	2.0833	0.4167	0.0073

6. CONCLUSION

In this study, we have reviewed the design procedure for the VS-OC plan indexed by the quality loss in Yen and Cheng (2009). In this design, we have to solve two nonlinear simultaneous equations including the evaluation of the non-central chi-square distribution. Although it is difficult to solve this problem, we have proposed the calculation algorithm based on the confluent hypergeometric function. This algorithm can be used on standard personal computers and programming languages such as C. On the other hand, the approximate design procedure developed by Arizono *et al.* (1997) has been constructed by combining Patnaik approximation and Wilson-Hilferty approximation.

Then, through some numerical comparisons between results by the rigorous design procedure incorporating the proposed calculation procedure based on the confluent hypergeometric function for the design procedure in Yen and Cheng (2009) and results by the approximate design procedure by Arizono *et al.* (1997), we have found that the sampling plans by the approximate design procedure are almost equal to the sampling plans by the rigorous design procedure. Further, it has been confirmed that the sampling plans by the approximate design procedure provide the enough performance on the producer's and consumer's risks. Therefore, we have verified the high applicability of the approximate design procedure by Arizono *et al.* (1997).

In addition, some sampling plans indexed by the quality loss have been considered separately from the sampling plan considered in this study. For example, Tomohiro *et al.* (2013) have considered the variable repetitive group sampling plan on operating characteristics indexed by the quality loss based on concepts of Sharman (1965) and Arizono *et al.* (1997). And also, Tomohiro *et al.* (2016) proposed the variable sequential sampling plan on operating characteristics indexed by the quality loss based on Wald's sequential probability ratio test (Wald, 1947; Wald and Wolfowitz, 1948). On the other hand, Morita *et al.* (2009), Suzuki *et al.* (2009) and Arizono *et al.* (2014) have considered variable sampling inspection plans with screening. Moreover, Arizono *et al.* (2016) have considered the variable repetitive group sampling plan with screening. These studies have been considered based on approximated evaluations. We can consider the comparison between the rigorous and approximate design procedures for these studies like this study.

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