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PARITY BRACKET POLYNOMIAL VIA SOME PARITY OF VIRTUAL LINKS

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Abstract. Manturov inroduced the parity bracket polynomial by using a parity of virtual knots, which is an extension of Jones-Kauffman polynomial. We extend Manturov's result to virtual links, so that we obtain the parity bracket polynomial for virtual links and give some examples.

1. Introduction

Virtual knot theory is introduced by Kauffman as a generalization of classical knot theory so that if two classical link diagrams are equivalent as virtual links, then they are equivalent as classical links [4]. A virtual link diagram is a link diagram in $\mathbb{R}^2$ possibly with some encircled crossings without over/under information, called virtual crossings. A virtual link is the equivalence class of such a link diagram by generalized Reidemeister moves, which consist of (classical) Reidemeister moves of type R_1, R_2 and R_3 and virtual Reidemeister moves of type VR_1, VR_2, VR_3 and the *semivirtual move* VR_4 as shown in Figure 1.

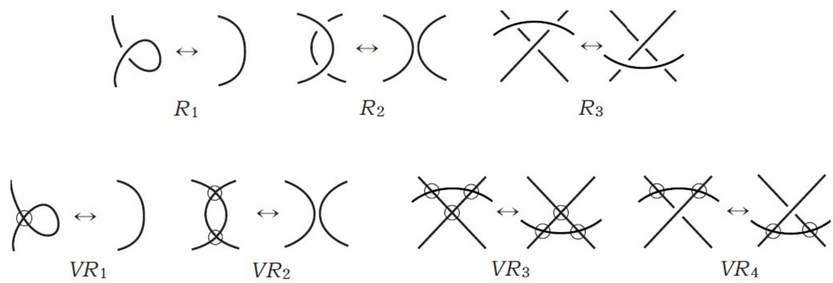

Figure 1. Generalized Reidemeister moves

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Manturov [5] investigated *parity* properties of virtual knots so that every real crossing is declared to be *even* or *odd* according to a certain rule. In particular, he strengthens known invariants such as the Kauffman polynomial for virtual knots by using parities of real crossings so that he introduces the parity bracket polynomial for virtual knots.

Im and Park [2] found a parity of virtual links with certain properties, and defined a multi-variable polynomial invariant for virtual links. Also, Im, Park and Shin [3, 6] obtain some polynomial invariants of virtual links by using a parity.

In this paper we extend the Manturov's parity bracket polynomial of virtual knots to the case of virtual links and give some examples.

This paper is organized as follows. In Section 2, we review a parity of virtual link diagrams introduced in [2] which is preserved under the generalized Reidemeister moves. In section 3, we give main result and some examples.

2. Preliminaries

We give a brief review about a *parity* of virtual link diagrams introduced by Im and Park [2] so that we can define some polynomial invariants for virtual links by using this parity.

A real crossing of a virtual link diagram D is said to be a *self crossing* if the crossing belongs to the one component of D, and we denote the set of self crossings of D by $S(D)$. A real crossing of D is a *mixed crossing* if the crossing belongs to different components of D, and we denote the set of *mixed crossings* of D by $M(D)$. If we denote the set of real crossings of D by $C(D)$, then we have $C(D) = S(D) \sqcup M(D)$.

Recall a *Gauss code* for a virtual knot diagram is a sequence of labels for the crossings with each label repeated twice to indicate a walk along the diagram from a given starting point and returning to that point. In the case of multiple link components, we mean a sequence labels, each repeated twice and inserted by partition symbols $|$ to indicate the component circuits for the code.

For every real crossing of a virtual knot diagram, the *Gaussian parity*, which is even or odd, is determined. The *parity* is also defined for crossings of *free knots*. For free knots, see Manturov [5].

As a generalization of the Gaussian parity for virtual knots, a *parity* of a virtual link diagram is defined so that real crossings of a virtual link diagram can be labeled as *even* or *odd*.

Definition 1. [2] Let $D = D_1 \cup D_2 \cup \cdots \cup D_n$ be a virtual link diagram of a n-component virtual link. We define a *parity* of real crossings of D in order as follows.

First, for every mixed crossing $c \in M(D)$ which belongs to $D_i \cap D_j (i \neq j)$, c is *odd* if the number of real crossings of $D_i \cap D_j$ is odd. If the number of real crossings of $D_i \cap D_j$ is odd for $i \neq j$, we replace all virtual crossings of $D_i \cap D_j$ by real crossings whose signs are chosen arbitrarily and obtain a new virtual link diagram $D' = D'_1 \cup D'_2 \cup \cdots \cup D'_n$ (if the number of real crossings of $D_i \cap D_j$ is even for all $i \neq j$, then $D = D'$). Then the number of real crossings of $D'_i \cap D'_j$ is even for any $i \neq j$.

Now consider $c \in M(D)$ which belongs to $D_i \cap D_j (i \neq j)$ and the number of real crossings of $D_i \cap D_j$ is even. If there is a *Gauss code* of the virtual link diagram D' so that the number of labels between two appearances of d for all real crossings $d \in M(D')$ is even, then c is *even*. Otherwise, c is *odd*.

Next, for every self crossing $c \in S(D)$, c is *even*(*odd*) if there is a Gauss code of the virtual link diagram D' so that the number of labels between two appearances of c is *even*(*odd*), respectively.

For classical links, all crossings are *even*, but in virtual links real crossings can be *even* or *odd*. Also, the *parity* of virtual link diagrams can be affected under the generalized Reidemeister moves as follows.

Lemma 2.1. [2] *Let D_1 and D_2 be equivalent virtual link diagrams. Suppose D_1 and D_2 are obtained from each other by a single generalized Reidemeister move, and the number of real crossings in D_2 is less or equal to that in D_1. Then the followings hold.*

(1) If D_2 is obtained from D_1 by a R_1-move, then the real crossing of D_1 involved in the R_1-move is even.

(2) If D_2 is obtained from D_1 by a R_2-move, then both real crossings involved in the R_2-move have the same parity.

(3) If D_2 is obtained from D_1 by a R_3-move, then the three real crossings in the diagram D_1 and the corresponding three real crossings in D_2 which are involved in the R_3-move have the same parity.

(4) If D_2 is obtained from D_1 by a VR_4-move, then the corresponding real crossings in D_1 and D_2 which are involved in the VR_4-move have the same parity.

(5) For each generalized Reidemeister move from D_1 to D_2 there is a one-to-one correspondence between the real crossings in D_1 and the crossings in D_2 which are not involved in the Reidemeister move. The corresponding real crossings have the same parity.

Remark 1. For virtual knots, the parity given in Definition 2.1 is the same as the Gaussian parity of virtual knots defined by Manturov [5].

From now on, we assume that all virtual links are oriented if there is no special mention.

3. Parity polynomial invariant for virtual links

In this section we extend the parity bracket polynomial for virtual knots introduced by Manturov [5] to the case of virtual links.

We begin with the following Lemma.

Lemma 3.1. *Let D and D' be virtual link diagrams with n components. Assume D' is obtained from D by applying a R_3-move and at least one real crossing* *among three real crossings in a local disk for a R_3-move belongs to a single component. Then the number of even crossings of $D(D')$ in a local disk for a R_3-move is not equal to 2, respectively.*

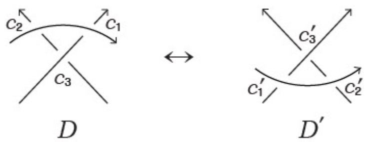

FIGURE 2. A R_3-move

Proof. Let c_1, c_2 and c_3 be real crossings between the top and the middle arcs, the top and the bottom arcs and the middle and the bottom arcs of D in the process of R_3-move, respectively as in Figure 2. We denote the corresponding three real crossings of D' by c_1', c_2' and c_3'. If c_1, c_2 and c_3 belong to $S(D)$, then the number of even crossings among them is either 1 or 3.

Now we assume that $c_1, c_2 \in M(D)$ and $c_3 \in S(D)$. Then $c_3(c_3')$ is in $D_i(D_i')$ and $c_1, c_2(c_1, c_2)$ are in $D_i \cap D_j (D_i' \cap D_j')$ for some components of $D(D')$, respectively. If the number of real crossings of $D_i \cap D_j (D_i' \cap D_j')$ is odd, then $c_1(c_1')$ and $c_2(c_2')$ are odd, respectively. As a result the conclusion follows. Thus we assume the number of real crossings of $D_i \cap D_j (D_i' \cap D_j')$ is even, respectively.

By Lemma 2.2, c_3 and c_3' have the same parity.

If c_3 is an even crossing and there is a Gauss code $(\cdots|c_3c_1\cdots c_3c_2\cdots|c_1c_2\cdots|\cdots)$ so that the number of labels between two appearances of d for all real crossings $d \in M(D)$ is even, both c_1 and c_2 are even. Otherwise, both c_1 and c_2 are odd. Because of the corresponding Gauss code $(\cdots|c_1'c_3'\cdots c_2'c_3'\cdots|c_2'c_1'\cdots|\cdots)$ for D', we obtain that c_1' and c_2' are both either even or odd.

If $c_3(c_3')$ is an odd crossing, there is no Gauss code for $D(D')$ so that the number of labels between two appearances of all crossings of $M(D)(M(D'))$ is even, respectively. Thus, $c_1(c_1')$ and $c_2(c_2')$ are both odd, respectively.

As a consequence, the conclusion follows. $\square$

Now, we define the parity bracket polynomial for a virtual link diagram with n components as follows.

Let $D = \cup_{i=1}^{n} D_i$ be an unoriented virtual link diagram with n components. For each pair (i, j) $(i < j)$, we define a parity state of a virtual link diagram $D_i \cup D_j$ to be a labeled virtual graph obtained from $D_i \cup D_j$ by using a parity of virtual link diagrams in Section 2 as follows: For each odd crossing in $D_i \cup D_j$ replace the crossing by a graphical node. For each even crossing in $D_i \cup D_j$ replace the crossing by one of its two possible smoothings, and label the smoothing site by A or A^{-1} in the usual way. See Figure 3.

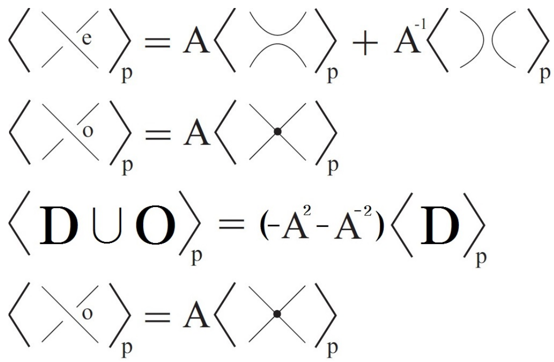

FIGURE 3. Parity bracket polynomial

The resulting virtual graphs are taken up to the virtual equivalence and up to the reduction rule, shown in Figure 3. The reduction rule is simply a R_2-move reducing the number of nodes in a graphical state. Then the graphical states containing nodes are irreducible states and give as graphical coefficients to the polynomial.

Definition 2. Let $D = \cup_{i=1}^{n} D_i$ be an unoriented virtual link diagram with n components. Then a polynomial $< D >_P$ for D is defined by

$$< D >_P = \sum_{i<j} \sum_{s_{(i,j)}} A^{\alpha-\beta}(-A^2 - A^{-2})^{\gamma-1} G(s_{(i,j)}).$$

In this formula, $\alpha(\beta)$ is the number of all A-*splices* (A^{-1}-*splices*) of $s_{(i,j)}$, respectively, γ is the number of components in the state $s_{(i,j)}$ and $G(s_{(i,j)})$ is the union of reduced graphical states containing graphical nodes. We call this polynomial the *even bracket polynomial.*

Theorem 3.2. *If D and D' are equivalent unoriented virtual link diagrams by all generalized Reidemeister moves but R_1-moves, then $< D >_P = < D' >_P$.*

Proof. We may assume that the number of classical crossings of D is less than or equal to that of D'. If $D(D')$ has one component, it has been proven by Manturov [5]. Thus we assume that $D(D')$ has at least two components. For $i < j$ and a state $s(s')$ of $D_i \cup D_j (D'_i \cup D'_j)$, let $\alpha_s(\alpha'_{s'})$, $\beta_s(\beta'_{s'})$ and $\gamma_s(\gamma'_{s'})$ be the number of A-splices, A^{-1}-splices and the number of components of $s(s')$, respectively.

For a R_2-move, we consider two cases according to the parities of two new classical crossings a and b of $D'_i \cup D'_j$.

Case 1: a and b are both even crossings of $D'_i \cup D'_j$

By splicing at a and b of D' and mimicking the proof of the original Kauffman polynomial for virtual links, it is immediate that $< D >_P = < D' >_P$.

Case 2: a and b are both odd crossings of $D'_i \cup D'_j$

By the reduction rule in Figure, we have $< D >_P = < D' >_P$.

For a R_3-move, let c_1, c_2 and c_3 be classical crossings between the top and the middle arcs, the top and the bottom arcs, and the middle and the bottom arcs of $D_i \cup D_j$ in the process of R_3-move, respectively. We denote the corresponding three classical crossings of D' by c'_1, c'_2 and c'_3, see Figure 4. Then by Lemma 2.2, c_i and c'_i have the same parities for $i = 1, 2, 3$. Now by Lemma 3.1, the case that two of c_i's $(i = 1, 2, 3)$ are even cannot happen. So we consider the following three cases according to the parities of three crossings in a local disc.

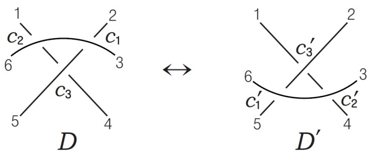

FIGURE 4. A R_3-move

Case 1: c_i and c'_i are even crossings of D and D', respectively for $i = 1, 2, 3$. Then it is immediate that $< D >_P = < D' >_P$ by following the same proof of the Kauffman polynomial.

Case 2: c_i and c'_i are odd crossings of D and D', respectively for $i = 1, 2, 3$.

Since D has at least two components, this case can be possible [2]. By replacing odd crossings by nodes, we have the same virtual graphs and $< D >_P = < D' >_P$.

Case 3: one of c_i's is even and the others are odd.

By splicing at even crossings of D and D', it is straightforward that $\alpha_s = \alpha'_{s'}$, $\beta_s = \beta'_{s'}$ and $\gamma_s = \gamma'_{s'}$ for an even state s of D and the corresponding even state s' of D'. Hence, we have $< D >_P = < D' >_P$.

Therefore the conclusion follows. □

We normaize the polynomial by the writhe to obtain a virtual link invariant.

Definition 3. The normalized parity bracket polynomial of a virtual link diagram D is defined as

$$P_D(A) = (-A^3)^{-w(D)} < D >_P .$$

Theorem 3.3. *The normalized parity bracket polynomial is an invariant of virtual links.*

Proof. The writhe normalization makes the parity polynomial invariant under R_1-move. The conclusion follows from Theorem 3.3. □

Example 1. *Let D be the Kishino knot diagram as shown in Figure 5.*

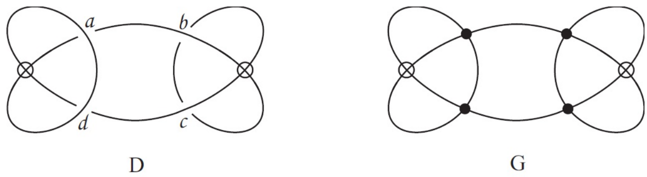

FIGURE 5. Kishino knot diagram and reduced graph

Then it is easy to check that every real crossing of D is odd. By replacing odd crossings with nodes and the reduction rule, we have the irreducible virtual graph G and $P_D(A) = G$. Thus the Kishino knot diagram is non-trivial.

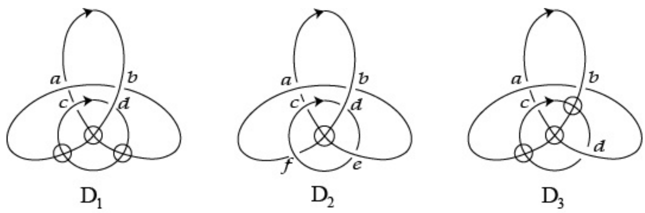

FIGURE 6. Different virtual link diagrams

Example 2. *Let D_1, D_2 and D_3 be virtual link diagrams in Figure 6.*

Since there is a Gauss code $(abcabd|dc)$ of D_1, all real crossings are even. Thus the normalized parity bracket polynomial and the Jones Kauffman polynomial are the same.

Next, since there is no Gauss code of D_2 so that the number of labels between two appearances of x for all mixed crossings x is even, then c, d, e and f are odd. Also, there is a Gauss code $(abecabdf|cdef)$ of D_2 so that a and b are odd. Thus, all real crossings of D_2 are odd. Then by computation, we have $P_D(A) = (-A^3)^{(-6)}(-A^2 - A^{-2})$.

On the other hand, for D_3 it is easy to check that a and b are odd, and c and d are even. Thus by computation, we have $P_D(A) = (-A^3)^{(-4)}(-A^4 - A^{-4})$.

As a result, D_1, D_2 and D_3 are different each other.

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