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COUPLED FIXED POINTS FOR MIXED g-MONOTONE UNDER RATIONAL CONTRACTIVE EXPRESSIONS IN PARTIALLY ORDERED METRIC SPACES

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Abstract. We propose coupled fixed point theorems for maps satisfying contractive conditions involving a rational expression in the setting of partially ordered metric spaces. We also present a result on the existence and uniqueness of coupled fixed points. In particular, it is shown that the results existing in the literature are extend, generalized, unify and improved by using mixed monotone property. Given to support the useability of our results, and to distinguish them from the known ones.

1. Introduction

The well-known Banach contraction theorem plays a major role in solving problems in many branches in pure and applied mathematics. A great number of generalizations of the Banach contraction principle were obtained in various directions. The study of mixed monotone mapping is an active area of research due to its wide scope of application the theory of mixed monotone mapping in ordered Banach space was extensively investigated in [18]. Many authors generalized this theorem to ordered metric spaces. The first such result was given by Ran and Reurings [14] who presented its applications to linear and nonlinear matrix equations. Subsequently, Nieto and Rodríguez-López [12] extended this result for non-decreasing mappings and applied it to obtain a unique solution for a periodic boundary value problem.

Guo and Lakshmikantham [3] introduced the notion of a coupled fixed point for two mappings. Bhaskar and Lakshmikantham [2] proved some interesting coupled fixed point theorems for mappings satisfying a mixed monotone property and coupled coincidence point in partial ordered metric spaces. Coupled common fixed point and coincidence point problems were first addressed by Lakshmikantham and Ciric [7] in which the authors extended the work of Bhaskar and Lakshmikantham [2] by defining the mixed g-monotone property

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and proved the existence and uniqueness of a coupled coincidence point for such mapping satisfying the mixed g-monotone property in partially ordered metric spaces. Following this result other coupled coincidence point results appeared in [1] and [20]. Subsequently, several authors obtained many results of this kind (see, e.g., [4, 6, 7, 10, 11, 13, 16, 17, 18, 19]). These results have a lot of applications, e.g., in proving existence of solutions of periodic boundary value problems (e.g., [1, 2]) as well as particular integral equations (e.g., [5, 8, 9]).

In this paper we establish coupled coincidence point results for two mappings $F : X \times X \to X$ and $g : X \to X$ and prove some coupled fixed point results in the frame of partially ordered metric space.

2. Main results

Recall the following definitions.

Definition 1. [3] Let X be a nonempty set and let $F : X \times X \to X$. A point $(x, y) \in X \times X$ is said to be a *coupled fixed point* of F if

$$F(x, y) = x \text{ and } F(y, x) = y.$$

Definition 2. [3] Let $(X, \preceq)$ be a partially ordered set. A mapping $F : X\times X \to X$ is said to have *mixed monotone property* if the following two conditions are satisfied:

$$(\forall x_1, x_2, y \in X)\ x_1 \preceq x_2 \ \to F(x_1, y) \preceq F(x_2, y),$$
$$(\forall x, y_1, y_2 \in X)\ y_1 \preceq y_2 \ \to F(x, y_1) \succeq F(x, y_2).$$

Definition 3. [7]Let $(X, \leq, d)$ be a partially ordered set $F : X \times X \to X$ and $g : X \to X$ be a mapping. Then a function F is said to have the mixed g-monotone property if $F(x, y)$ is monotone g-nondecreasing in x in its first argument and is monotone g- non-increasing in y in its second argument that is for any $x, y \in X$,

$$g(x_1) \leq g(x_2) \to F(x_1, y) \leq F(x_2, y) \text{ for all } x_1, x_2 \in X$$

and

$$g(y_1) \leq g(y_2) \to F(x, y_1) \geq F(x, y_2) \text{ for all } y_1, y_2 \in X.$$

Note that if g is the identity mapping, then F is said to have the mixed monotone property

Definition 4. [16]An element $x, y \in X \times X$ is called a coupled coincidence point of the mapping $F : X \times X \to X$ and $g : X \to X$ if

$$gx = F(x, y) \text{ and } gy = F(y, x).$$

We remark that if g is the identity mapping, then (x, y) is called a coupled fixed point of the mapping F.

Definition 5. [16]An element $x, y \in X \times X$ is called a coupled common fixed point of the mapping $F : X \times X \to X$ and $g : X \to X$ if

$$gx = F(x, y) = x \text{ and } gy = F(y, x) = y.$$

Definition 6. [7] Let X be a nonempty set. Then we say that the mappings $F : X \times X \to X$ and $g : X \to X$ are commutative if

$$gF(x, y) = F(gx, gy).$$

Definition 7. [23] Let X be a non-empty set, $F : X \times X \to X$ and $g : X \to X$. Then F and g are said to be coincidentally commuting if they commute at their coupled coincidence points, that is, if $gx = F(x, y)$ and $gy = F(y, x)$ for some $(x, y)X \times X$, then

$$gF(x, y) = F(gx, gy) \text{ and } gF(y, x) = F(gy, gx).$$

We will prove now coupled coincidence point results which generalize the result of Nashine[24].

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be two continuous mappings having the mixed monotone property and satisfying*

(1)
$$\begin{aligned} d(F(x, y), F(u, v)) &\le \frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] + \beta M((gx, gy), (gu, gv)) \\ &+ \frac{\gamma}{2}[d(gx, F(x, y)) + d(gu, F(u, v)) + d(gy, F(y, x)) + d(gv, F(v, u))] \\ &+ \frac{\delta}{2}[d(gx, F(u, v)) + d(gy, F(v, u)) + d(gu, F(x, y)) + d(gv, F(y, x))], \end{aligned}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq gu$ and $g(y) \succeq g(v)$. suppose $F(X \times X \subseteq g(X))$ and g is continuous and commute with F and also suppose either F is continuous or X has the following property:

(i) *If a non-decreasing sequence $\{gx_n\} \to x$, then $g(x_n) \preceq gx$ for all n,*
(ii) *If a non-decreasing sequence $\{gy_n\} \to y$, then $g(y_n) \succeq gy$ for all n,*

(2)
$$\begin{aligned} M((x, y), (u, v)) = \min\Bigg\{ & d(gx, F(x, y))\,\frac{2 + d(gu, F(u, v)) + d(gv, F(v, u))}{2 + d(x, u) + d(y, v)}, \\ & d(gu, F(u, v))\,\frac{2 + d(gx, F(x, y)) + d(gy, F(y, x))}{2 + d(x, u) + d(y, v)} \Bigg\} \end{aligned}$$

and $\alpha, \beta, \gamma, \delta \ge 0$ with $\alpha + \beta + 2\gamma + 2\delta < 1$. If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0) \quad \text{and} \quad g(y_0) \succeq F(y_0, x_0). \tag{3}$$

Then there exists $x_0, y_0 \in X$, $gx = F(x, y)$ and $gy = F(y, x)$, that is, F and g have a coupled coincidence point $(\bar{x}, \bar{y}) \in X \times X$.

Proof. Denote $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Then $g(x_0) \preceq g(x_1)$ and $g(y_0) \succeq g(y_1)$, by (3). Further denote

$$g(x_2) = gF(x_1, y_1) = F(g(x_1), g(y_1)) = F(F(x_0, y_0), F(y_0, x_0)) = F^2(x_0, y_0)$$

and

$$g(y_2) = gF(y_1, x_1) = F(g(y_1), g(x_1)) = F(F(y_0, x_0), F(x_0, y_0)) = F^2(y_0, x_0).$$

Due to the g-mixed monotone property of F and g, we have

$$g(x_2) = F(x_1, y_1) \succeq F((x_0), (y_1)) \succeq F((x_0), (y_0)) = g(x_1)$$

and

$$g(y_2) = F(y_1, x_1) \preceq F((y_0), (x_1)) \preceq F(y_0, x_0) = g(y_1).$$

Further, for $n = 1, 2, \dots,$ we let

$$g(x_{n+1}) = F^{n+1}(g(x_0), g(y_0)) = F(F^n(g(x_0), g(y_0)), F^n(g(y_0), g(x_0)))$$

and

$$g(y_{n+1}) = F^{n+1}(g(y_0), g(x_0)) = F(F^n(g(y_0), g(x_0)), F^n(g(x_0), g(y_0))).$$

We check easily that

$$g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq \cdots \leq g(x_n) \preceq \cdots \tag{4}$$

and

$$g(y_0) \succeq g(y_1) \succeq g(y_2) \succeq \cdots \succeq g(y_n) \succeq \cdots . \tag{5}$$

If $g(x_{n+1}) = g(x_n)$ and $g(y_{n+1}) = g(y_n)$ for some n, then $F(x_n, y_n) = g(x_n)$ and $F(y_n, x_n) = g(y_n)$, hence (gx_n, gy_n) is a coupled coincidence point of F and g. Suppose, further, that

$$g(x_n) \neq g(x_{n+1}) \text{ or } g(y_n) \neq g(y_{n+1}) \text{ for each } n \in \mathbb{N}_0.$$

Now, we claim that, for $n \in \mathbb{N}_0$,

$$\begin{aligned} &d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n)) \\ &\leq \left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right)^n [d(g(x_1), g(x_0)) + d(g(y_1), g(y_0))]. \end{aligned} \tag{6}$$

Indeed, for $n = 1$, using $g(x_1) \succeq g(x_0)$, $g(y_1) \preceq g(y_0)$ and (1), we get:

$$
\begin{aligned}
&(7)\\
&d(g(x_2), g(x_1))\\
&= d(F(x_1, y_1), F(x_0, y_0))\\
&\leq \frac{\alpha}{2}[d(g(x_1), g(x_0)) + d(g(y_1), g(y_0))] + \beta M((g(x_1), g(y_1)), (g(x_0), g(y_0)))\\
&\quad + \frac{\gamma}{2}[d(g(x_1), F(x_1, y_1)) + d(g(x_0), F(x_0, y_0)) + d(g(y_1), F(y_1, x_1)) + d(g(y_0), F(y_0, x_0))]\\
&\quad + \frac{\delta}{2}[d(g(x_1), F(x_0, y_0)) + d(g(y_1), F(y_0, x_0)) + d(g(x_0), F(x_1, y_1)) + d(g(y_0), F(y_1, x_1))]\\
&\leq \frac{\alpha}{2}[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))]\\
&\quad + \beta d(g(x_1), F(x_1, y_1))\,\frac{2 + d(g(x_0), F(x_0, y_0)) + d(g(y_0), F(y_0, x_0))}{2 + d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))}\\
&\quad + \frac{\gamma}{2}[d(g(x_1), g(x_2)) + d(g(x_0), g(x_1)) + d(g(y_1), g(y_2)) + d(g(y_0), g(y_1))]\\
&\quad + \frac{\delta}{2}[d(g(x_1), g(x_1)) + d(g(y_1), g(y_1)) + d(g(x_0), g(x_2)) + d(g(y_0), g(y_2))]\\
&\leq \frac{\alpha}{2}[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))] + \beta d(g(x_1), g(x_2))\\
&\quad + \frac{\gamma + \delta}{2}[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1)) + d(g(x_1), g(x_2)) + d(g(y_1), g(y_2))].
\end{aligned}
$$

Similarly, using that $d(g(y_2), g(y_1)) = d(F(y_1, x_1), F(y_0, x_0)) = d(F(y_0, x_0), F(y_1, x_1))$ and

$$
\begin{aligned}
M((x_1, y_1), (x_0, y_0)) &\leq d(g(y_1), F(y_1, x_1))\,\frac{2 + d(g(y_0), F(y_0, x_0)) + d(g(x_0), F(x_0, y_0))}{2 + d(g(y_0), g(y_1)) + d(g(x_0), g(x_1))}\\
&= d(g(y_1), g(y_2)),
\end{aligned}
$$

we get

$$
\begin{aligned}
&(8)\\
&d(g(y_2), g(y_1))\\
&\leq \frac{\alpha}{2}[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))] + \beta d(g(y_1), g(y_2))\\
&\quad + \frac{\gamma + \delta}{2}[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1)) + d(g(x_1), g(x_2)) + d(g(y_1), g(y_2))].
\end{aligned}
$$

Adding (7) and (8), we have

$$
d(g(x_2), g(x_1)) + d(g(y_2), g(y_1)) \leq \left(\frac{\alpha + \gamma + \delta}{1 - \beta - \gamma - \delta}\right)[d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))].
$$

In a similar way, proceeding by induction, if we assume that (6) holds, we get that

$$\begin{aligned} & d(g(x_{n+2}), g(x_{n+1})) + d(g(y_{n+2}), g(y_{n+1})) \\ & \leq \left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right) [d(g(x_{n+1}), g(x_n)) + d(g(y_{n+1}), g(y_n))] \\ & \leq \left(\frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta}\right)^{n+1} [d(g(x_0), g(x_1)) + d(g(y_0), g(y_1))]. \end{aligned}$$

Hence, by induction, (6) is proved.

Set

$$h_n := d(g(x_n), g(x_{n+1})) + d(g(y_n), g(y_{n+1})), \quad n \in \mathbb{N}$$

and $\Delta := \frac{\alpha+\gamma+\delta}{1-\beta-\gamma-\delta} < 1$. Then, the sequence $\{h_n\}$ is decreasing and

$$h_n \leq \Delta^n h_0.$$

By assumption (4), $h_n > 0$ for $n \in \mathbb{N}_0$. Then, for each $n \geq m$ we have

$$d(g(x_n), g(x_m)) \leq d(g(x_n), g(x_{n-1})) + d(g(x_{n-1}), g(x_{n-2})) + \cdots + d(g(x_{m+1}), g(x_m))$$

and

$$d(g(y_n), g(y_m)) \leq d(g(y_n), g(y_{n-1})) + d(g(y_{n-1}), g(y_{n-2})) + \cdots + d(g(y_{m+1}), g(y_m)).$$

Therefore,

$$\begin{aligned} d(g(x_n), g(x_m)) + d(g(y_n), g(y_m)) & \leq h_{n-1} + h_{n-2} + \cdots + h_m \\ & \leq (\Delta^{n-1} + \Delta^{n-2} + \cdots + \Delta^m) h_0 \\ & \leq \frac{\Delta^m}{1-\Delta} h_0 \end{aligned}$$

which implies that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in X since $0 \leq \Delta < 1$. Since (X, d) is a complete metric space, there exists $(\bar{x}, \bar{y}) \in X \times X$ such that

$$\lim_{n\to\infty} F(\bar{x}_n, \bar{y}_n) = \lim_{n\to\infty} g(\bar{x}_n) = \bar{x} \quad \textit{and} \lim_{n\to\infty} F(\bar{y}_n, \bar{x}_n) = \lim_{n\to\infty} g(\bar{y}_n) = \bar{y}. \quad (9)$$

Finally, we claim that $(\bar{x}, \bar{y})$ is a coupled coincidence point of F and g. Indeed, from $g(\bar{x}_{n+1}) = F(\bar{x}_n, \bar{y}_n)$ and $g(\bar{y}_{n+1}) = F(\bar{y}_n, \bar{x}_n)$, using (9) and the continuity of g, we get

$$\lim_{n\to\infty} g(g(\bar{x}_n)) = g\bar{x} \quad \textit{and} \lim_{n\to\infty} g(g(\bar{y}_n)) = g\bar{y}$$

Taking the limit as $n \to \infty$, using (9) and the continuity of F we get

$$\begin{aligned} g(\bar{x}) &= \lim_{n\to\infty} g(g(\bar{x}_{n+1})) = \lim_{n\to\infty} F(g(\bar{x}_n), g(\bar{y}_n)) \\ &= F(\lim_{n\to\infty} (g\bar{x}_n), \lim_{n\to\infty} (g\bar{y}_n)) \quad = F(\bar{x}, \bar{y}) \end{aligned}$$

$$\begin{aligned} g(\bar{y}) &= \lim_{n\to\infty} g(g(\bar{y}_{n+1})) = \lim_{n\to\infty} F(g(\bar{y}_n), g(\bar{x}_n)) \\ &= F(\lim_{n\to\infty} (g\bar{y}_n), \lim_{n\to\infty} (g\bar{x}_n)) \quad = F(\bar{y}, \bar{x}). \end{aligned}$$

It immediately follows that $g\bar{x} = F(\bar{x}, \bar{y})$ and $g\bar{y} = F(\bar{y}, \bar{x})$. This completes the proof of the theorem. □

In the next theorem, we will substitute the continuity hypothesis on F by an additional property satisfied by the space $(X, d, \preceq)$.

Theorem 2.2. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X. Assume that there exist $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + 2\gamma + 2\delta < 1$ such that*

$$\begin{aligned} d(F(x,y), F(u,v)) \leq & \frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] + \beta M((gx, gy), (gu, gv)) \\ & + \frac{\gamma}{2}[d(gx, F(x,y)) + d(gu, F(u,v)) + d(gy, F(y,x)) + d(gv, F(v,u))] \\ & + \frac{\delta}{2}[d(gx, F(u,v)) + d(gy, F(v,u)) + d(gu, F(x,y)) + d(gv, F(y,x))] \end{aligned}$$

for all $(x,y), (u,v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$, where

$$\begin{aligned} M((x,y),(u,v)) = \min\Bigg\{ & d(gx, F(x,y)) \, \frac{2 + d(gu, F(u,v)) + d(gv, F(v,u))}{2 + d(x,u) + d(y,v)}, \\ & d(gu, F(u,v)) \, \frac{2 + d(gx, F(x,y)) + d(gy, F(y,x))}{2 + d(x,u) + d(y,v)} \Bigg\}. \end{aligned}$$

Suppose that there exist $x_0, y_0 \in X$ such that

$$g(x_0) \preceq F(x_0, y_0) \quad \text{and} \quad g(y_0) \succeq F(y_0, x_0).$$

Further we suppose that $F(X \times X) \subseteq g(X)$, g is continuous non-decreasing, g and F are Finally, assume that X has the following properties:

(i) *if a nondecreasing sequence $\{g(x_n)\}$ in X converges to $x \in X$, then $g(x_n) \preceq g(x)$ for all n,*
(ii) *if a non increasing sequence $\{g(y_n)\}$ in X converges to $y \in X$, then $g(y_n) \succeq g(y)$ for all n.*

Then, F and g have coupled coincidence point $(\bar{x}, \bar{y}) \in X \times X$.

Proof. Following the proof of Theorem 2.1, we only have to show that $(\bar{x}, \bar{y})$ is a coupled coincidence point of F and g. We have

$$d(F(\bar{x},\bar{y}), g\bar{x}) \leq d(F(\bar{x},\bar{y}), gx_{n+1}) + d(gx_{n+1}, g\bar{x}) = d(F(\bar{x},\bar{y}), F(gx_n, gy_n)) + d(gx_{n+1}, g\bar{x}). \tag{10}$$

Since the nondecreasing sequence $\{g(x_n)\}$ converges to $\bar{x}$ and the nonincreasing sequence $\{g(y_n)\}$ converges to $\bar{y}$, by (i)–(ii), we have:

$$g\bar{x} \succeq gx_n \quad \text{and} \quad g\bar{y} \preceq gy_n, \quad \forall n.$$

Now, from the contractive condition (1), we have:

$$\begin{aligned}
d(F(\bar{x},\bar{y}),F(x_n,y_n)) &\le \frac{\alpha}{2}[d(g\bar{x},gx_n)+d(g\bar{y},gy_n)]+\beta M((g\bar{x},g\bar{y}),(gx_n,gy_n))\\
&+\frac{\gamma}{2}[d(g\bar{x},F(\bar{x},\bar{y}))+d(gx_n,F(x_n,y_n))+d(g\bar{y},F(\bar{y},\bar{x}))+d(gy_n,F(y_n,x_n))]\\
&+\frac{\delta}{2}[d(g\bar{x},F(x_n,y_n))+d(g\bar{y},F(y_n,x_n))+d(gx_n,F(\bar{x},\bar{y}))+d(gy_n,F(\bar{y},\bar{x}))]\\
&\le \frac{\alpha}{2}[d(\bar{x},x_n)+d(\bar{y},y_n)]\\
&+\beta d(g\bar{x},F(\bar{x},\bar{y}))\,\frac{2+d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})}{2+d(g\bar{x},gx_n)+d(g\bar{y},gy_n)}\\
&+\frac{\gamma}{2}[d(g\bar{x},F(\bar{x},\bar{y}))+d(gx_n,gx_{n+1})+d(g\bar{y},F(\bar{y},\bar{x}))+d(gy_n,gy_{n+1})]\\
&+\frac{\delta}{2}[d(g\bar{x},gx_{n+1})+d(g\bar{y},gy_{n+1})+d(gx_n,F(\bar{x},\bar{y}))+d(gy_n,F(\bar{y},\bar{x}))].
\end{aligned}$$

Then, from (10), we get:

$$\begin{aligned}
d(F(\bar{x},\bar{y}),g\bar{x})) &\le d(gx_{n+1},g\bar{x})\\
&+\frac{\alpha}{2}[d(\bar{x},x_n)+d(\bar{y},y_n)]+\beta d(g\bar{x},F(\bar{x},\bar{y}))\,\frac{2+d(gx_n,gx_{n+1})+d(gy_n,gy_{n+1})}{2+d(g\bar{x},gx_n)+d(g\bar{y},gy_n)}\\
&+\frac{\gamma}{2}[d(g\bar{x},F(\bar{x},\bar{y}))+d(gx_n,gx_{n+1})+d(g\bar{y},F(\bar{y},\bar{x}))+d(gy_n,gy_{n+1})]\\
&+\frac{\delta}{2}[d(g\bar{x},gx_{n+1})+d(g\bar{y},gy_{n+1})+d(gx_n,F(\bar{x},\bar{y}))+d(gy_n,F(\bar{y},\bar{x}))].
\end{aligned}$$

Taking limit as $n\to\infty$, we have

$$d((F(\bar{x},\bar{y}),g\bar{x}))\le\beta d(g\bar{x},F(\bar{x},\bar{y}))+\frac{\gamma+\delta}{2}[d(g\bar{x},F(\bar{x},\bar{y}))+d(g\bar{y},F(\bar{y},\bar{x}))]. \quad (11)$$

Similarly,

$$d(g\bar{y},F(\bar{y},\bar{x}))\le\beta d(g\bar{y}F(\bar{y},\bar{x}))+\frac{\gamma+\delta}{2}[d(g\bar{x},F(\bar{x},\bar{y}))+d(g\bar{y},F(\bar{y},\bar{x}))]. \quad (12)$$

Adding (11) and (12), we have

$$\begin{aligned}
d(g\bar{x},F(\bar{x},\bar{y}))+d(g\bar{y},F(\bar{y},\bar{x})) &\le(\beta+\gamma+\delta)[d(g\bar{x},F(\bar{x},\bar{y}))+d(g\bar{y},F(\bar{y},\bar{x}))]\\
&\le(\alpha+\beta+2\gamma+2\delta)[d(g\bar{x},F(\bar{x},\bar{y}))+d(g\bar{y},F(\bar{y},\bar{x}))].
\end{aligned}$$

Since $0\le\alpha+\beta+2\gamma+2\delta<1$, we obtain $d(F(\bar{x},\bar{y}),g\bar{x})=0$ and $d(g\bar{y},F(\bar{y},\bar{x}))$, i.e., $F(\bar{x},\bar{y})=g\bar{x}$ and $F(\bar{y},\bar{x})=g\bar{y}$. This completes the proof of the theorem. □

Now we shall prove an uniqueness theorem for the coupled coincidence point. Note that, if $(X,\preceq)$ is a partially ordered set, then we endow the product space $X\times X$ with the following partial order:

$$\text{for } (x,y),(u,v)\in X\times X,\quad (u,v)\preceq(x,y)\to gx\preceq gu,\ gy\succeq gv.$$

Theorem 2.3. *Assume that*

$$\forall (x,y),(x^*,y^*)\in X\times X,\ \exists\,(u,v)\in X\times X \text{ such that} \tag{13}$$

$(F(u,v),F(v,u))$ is comparable to $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ and $(F(x^*,y^*),F(y^*,x^*))$. Then F and g have unique coupled coincidence point that is there exists a unique $(x,y)\in X\times X$ such that $g\bar{x}=F(\bar{x},\bar{y})$ and $g\bar{y}=F(\bar{y},\bar{x})$, $gx^*=F(x^*,y^*)$ and $gy^*=F(y^*,x^*)$.

Adding (13) to the hypotheses of Theorem 2.1, we obtain the uniqueness of the coupled coincidence point of F and g.

Proof. From Theorem 2.1 we know that there exists the set of coupled coincidence point of F and g is non empty, suppose that $(\bar{x},\bar{y})$ and (x^*,y^*) are coupled coincidence point of F and g, that is $g\bar{x}=F(\bar{x},\bar{y})$ and $g\bar{y}=F(\bar{y},\bar{x})$, $gx^*=F(x^*,y^*)$ and $gy^*=F(y^*,x^*)$. which is obtained as $g\bar{x}=\lim_{n\to\infty}F^n(x_0,y_0)$ and $g\bar{y}=\lim_{n\to\infty}F^n(y_0,x_0)$. Then we have to show that

$$d(g\bar{x},gx^*)+d(g\bar{y},gy^*)=0. \tag{14}$$

We distinguish two cases.

Case I: $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ is comparable with $(F(x^*,y^*),F(y^*,x^*))$ with respect to the ordering in $X\times X$. Let, e.g., $g\bar{x}\preceq gx^*$ and $g\bar{y}\succeq gy^*$. Then, we can apply the contractive condition (1) to obtain

$$\begin{aligned} d(g\bar{x},gx^*)&=d(F(\bar{x},\bar{y}),F(x^*,y^*))\\ &\le \frac{\alpha}{2}[d(g\bar{x},gx^*)+d(g\bar{y},gy^*)]+\delta[d(g\bar{x},gx^*)+d(g\bar{y},gy^*)], \end{aligned}$$

and

$$\begin{aligned} d(g\bar{y},gy^*)&=d(F(\bar{y},\bar{x}),F(y^*,x^*))=d(F(y^*,x^*),F(\bar{y},\bar{x}))\\ &\le \frac{\alpha}{2}[d(g\bar{x},gx^*)+d(g\bar{y},gy^*)]+\delta[d(g\bar{x},gx^*)+d(g\bar{y},gy^*)]. \end{aligned}$$

Adding up, we get that

$$d(g\bar{x},gx^*)+d(g\bar{y},gy^*)\le(\alpha+2\delta)[d(g\bar{x},gx^*)+d(g\bar{y},gy^*)].$$

Since $0\le\alpha+2\delta<1$, (14) holds.

Case II: $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ is not comparable with $(F(x^*,y^*),F(y^*,x^*))$. In this case, By assumption there exists $(u,v)\in X\times X$ that is comparable both to $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ and $(F(x^*,y^*),F(y^*,x^*))$. Then, for all $n\in\mathbb{N}$, $(F^n(u,v),F^n(v,u))$ is comparable both to $(F^n(\bar{x},\bar{y}),F^n(\bar{y},\bar{x}))=(g\bar{x},g\bar{y})$ and $(F^n(x^*,y^*),F^n(y^*,x^*))=(gx^*,gy^*)$. We have

$$\begin{aligned} d(g\bar{x},gx^*)+d(g\bar{y},gy^*)&=d(F^n(\bar{x},\bar{y}),F^n(x^*,y^*))+d(F^n(\bar{y},\bar{x}),F^n(y^*,x^*))\\ &\le d(F^n(\bar{x},\bar{y}),F^n(u,v))+d(F^n(u,v),F^n(x^*,y^*))\\ &\quad+d(F^n(\bar{y},\bar{x}),F^n(v,u))+d(F^n(v,u),F^n(y^*,x^*))\\ &\le(\alpha^n+2\delta^n)[d(g\bar{x},u)+d(g\bar{y},v)+d(gx^*,u)+d(gy^*,v)]. \end{aligned}$$

Since $0<\alpha,\delta<1$, (14) holds.

We deduce that in all cases (14) holds. This implies that $(g\bar{x}, g\bar{y}) = (gx^*, gy^*)$ and the uniqueness of the coupled fixed point of F is proved.

□

Theorem 2.4. *In addition to the hypotheses of Theorem 2.1 (resp. Theorem 2.2), suppose that x_0, y_0 in X are comparable. Then $g\bar{x} = g\bar{y}$.*

Proof. Suppose that $gx_0 \preceq gy_0$. We claim that

$$gx_n \preceq gy_n, \ \forall\, n \in \mathbb{N}. \tag{15}$$

From the mixed monotone property of F and g, we have

$$gx_1 = F(x_0, y_0) \preceq F(y_0, y_0) \preceq F(y_0, x_0) = gy_1.$$

Assume that $gx_n \preceq gy_n$ for some n. Now,

$$\begin{aligned} gx_{n+1} &= F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)) \\ &= F(x_n, y_n) \preceq F(y_n, y_n) \preceq F(y_n, x_n) = gy_{n+1}. \end{aligned}$$

Hence, (15) holds.

Now, using (15) and the contractive condition, we get

$$\begin{aligned} &d(g\bar{x}, g\bar{y}) \\ &\le d(g\bar{x}, gx_{n+1}) + d(gx_{n+1}, gy_{n+1}) + d(gy_{n+1}, g\bar{y}) \\ &= d(g\bar{x}, gx_{n+1}) + d(F(y_n, x_n), F(x_n, y_n)) + d(gy_{n+1}, g\bar{y}) \\ &\le d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) + \beta M((gy_n, gx_n), (gx_n, gy_n)) \\ &\quad + \frac{\gamma}{2}[d(gx_n, F(x_n, y_n)) + d(gy_n, F(y_n, x_n)) + d(gy_n, F(y_n, x_n)) + d(gx_n, F(x_n, y_n)))] \\ &\quad + \frac{\delta}{2}[d(gx_n, F(y_n, x_n)) + d(gy_n, F(x_n, y_n)) + d(gy_n, F(x_n, y_n)) + d(gx_n, F(y_n, x_n))] \\ &\le d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) \\ &\quad + \beta d(gy_n, gy_{n+1}) \frac{2 + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})}{2 + 2d(gy_n, gx_n)} \\ &\quad + \gamma[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] + \delta[d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1})] \\ &\le d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) \\ &\quad + \beta d(gy_n, gy_{n+1})[2 + d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \\ &\quad + \gamma[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] + \delta[d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1})]. \end{aligned}$$

Passing to the limit as $n \to \infty$, we get that

$$d(g\bar{x}, g\bar{y}) \le (\alpha + 2\delta) d(g\bar{x}, g\bar{y}).$$

Since $0 \le \alpha + 2\delta < 1$, this implies that $d(g\bar{x}, g\bar{y}) = 0$, i.e., $g\bar{x} = g\bar{y}$. This completes the proof of the theorem. □

Remark 1. If we put

$$\mathcal{T}(x) = F(x, x), \ \forall\, x \in X,$$

then for $x = y$ and $u = v$, the contractive condition (1) reduces to the condition for a single map (in the case without order) of Rhoades from [15, Corollary 15].

We illustrate our results by the following example which also distinguishes these result from the known ones.

Example 2.5. *Let $X = R$, $d(x, y) = |x - y|$, $y \preceq x$ if and only if $y \leq x$ and we have two mapping $F : X \times X \to X$ and $g : X \to X$ defined by $F(x, y) = (x+y)/8$ and $g(x) = x/2$ with the standard metric and ordered by the relation $\preceq$. Suppose that $gx \preceq gu$ and $gy \succeq gv$.*

Let $\alpha, \beta, \gamma, \delta$ be nonnegative numbers satisfying $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + 2\gamma + 2\delta < 1$, and denote by L and R, respectively, the left-hand and right-hand side of contraction condition (3.1). It is easy to check that all the condition of Theorem 3.1 and 3.4 are satisfied for $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha+\beta+2\gamma+2\delta < 1$ and that $(0, 0)$ is an unique coupled fixed point of F. we note that function F has mixed monotone property, that is $F(x, y)$ is monotone nondecreasing in x and monotone non-increasing in y. For example, if $(x, y) = (4, 3)$,$(u, v) = (2, 3)$ for all $(x, y), (u, v) \in X \times X$ and $x \succeq u$, $y \preceq v$ then

$$L = d(F(x, y), F(u, v)) = d(\frac{x+y}{8}, \frac{u+v}{8}) = |\frac{4+3}{8} - \frac{2+3}{8}| = \frac{2}{8} = 0.25$$

$$\begin{aligned} R = &\frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] + \beta M((gx, gy), (gu, gv)) \\ &+ \frac{\gamma}{2}[d(gx, F(x, y)) + d(gu, F(u, v)) + d(gy, F(y, x)) + d(gv, F(v, u))] \\ &+ \frac{\delta}{2}[d(gx, F(u, v)) + d(gy, F(v, u)) + d(gu, F(x, y)) + d(gv, F(y, x))] \end{aligned}$$

$$\begin{aligned} R = &\frac{\alpha}{2}[d(\frac{x}{2}, \frac{u}{2}) + d(\frac{y}{2}, \frac{v}{2})] + \beta M((\frac{x}{2}, \frac{y}{2}), (\frac{u}{2}, \frac{v}{2})) \\ &+ \frac{\gamma}{2}[d(\frac{x}{2}, \frac{x+y}{8}) + d(\frac{u}{2}, \frac{u+v}{8}) + d(\frac{y}{2}, \frac{y+x}{8}) + d(\frac{v}{2}, \frac{v+u}{8})] \\ &+ \frac{\delta}{2}[d(\frac{x}{2}, \frac{u+v}{8}) + d(\frac{y}{2}, \frac{v+u}{8}) + d(\frac{u}{2}, \frac{x+y}{8}) + d(\frac{v}{2}, \frac{y+x}{8})]. \end{aligned}$$

on putting the value of $\alpha, \beta, \gamma, \delta$.

$$\begin{aligned} = &\frac{1}{8}\frac{1}{2}[|\frac{x-u}{2}| + |\frac{y-v}{2}|] + 0M \\ &+ \frac{1}{4}\frac{1}{2}[|\frac{x}{2} - \frac{x+y}{8}| + |\frac{u}{2} - \frac{u+v}{8}| + |\frac{y}{2} - \frac{y+x}{8}| + |\frac{v}{2} - \frac{v+u}{8}|] \\ &+ \frac{1}{12}\frac{1}{2}[|\frac{x}{2} - \frac{u+v}{8}| + |\frac{y}{2} - \frac{v+u}{8}| + |\frac{u}{2} - \frac{x+y}{8}| + |\frac{v}{2} - \frac{y+x}{8}|] \end{aligned}$$

$$
\begin{aligned}
R &= \frac{1}{16} + \frac{1}{8}[\frac{24}{8}] + \frac{1}{24}[\frac{24}{8}] \\
&= \frac{9}{16} = 0.56
\end{aligned}
$$

where $\alpha = 1/8$,$\beta = 0$,$\gamma = 1/4$ and $\delta = 1/12$, satisfying the condition $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + 2\gamma + 2\delta < 1$, on putting all those values and applying the given conditions in above right hand side equation, we get R=0.56, which is greater than the value of L, this implies that $L \leq R$ and the given contraction condition is satisfied.

Example 2.6. *Let $X = [0, +\infty)$, $d(x, y) = |x - y|$, $x \preceq y$ be equipped with the standard metric and ordered by the relation $\preceq$ given by*

$$x \preceq y \iff x = y \vee (x, y \in [0, 1] \wedge x \leq y).$$

Consider the (continuous) mapping $F : X \times X \to X$ given by

$$F(x, y) = \begin{cases} \dfrac{x^2 - y^2}{8}, & x \geq y \\ 0, & x < y. \end{cases}$$

and a mapping $g : X \to x$ defined by $gx = x^2$. Then we have $gx \preceq gu$ and $gy \succeq gv$. Let $\alpha, \beta, \gamma, \delta$ be nonnegative numbers satisfying $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + 2\gamma + 2\delta < 1$, and denote by L and R, respectively, the left-hand and right-hand side of contraction condition (1). It is easy to check that all the condition of Theorem 2.1 is satisfied for $\alpha, \beta, \gamma, \delta \geq 0$ with $\alpha + \beta + 2\gamma + 2\delta < 1$ and that $(0, 0)$ is an unique coupled fixed point of F. we note that function F has mixed monotone property, that is $F(x, y)$ is monotone nondecreasing in x and monotone non-increasing in y. For example, if $(x, y) = (2, 3)$,$(u, v) = (1, 2)$ for all $(x, y), (u, v) \in X \times X$ and suppose that $x \succeq u$, $y \preceq v$ then consider the following possible cases.

1) $x, u, y, v \in [0, 1]$ and hence $x \geq u$, $y \leq v$. we get that in each case

$$L \leq \frac{x - y}{4} \leq \frac{\alpha}{2}[d(x, u) + d(y, v)] \leq R.$$

For example, if $0 \leq y \leq u \leq x \leq v \leq 1$ then

$$L = d(F(x, y), F(u, v)) = d\left(\frac{x^2 - y^2}{8}, 0\right) = |\frac{x^2 - y^2}{8} - 0| \leq |\frac{x - y}{4}| = 0.25;$$

the other cases are treated similarly.

2a) $x, u \in [0, 1]$ and $y = v > 1$; then $L = 0$ and the condition is satisfied.

2b) $y, v \in [0, 1]$ and $x = u > 1$; then

$$L = d\left(\frac{x^2 - y^2}{8}, \frac{u^2 - v^2}{8}\right) = \frac{x^2 - y^2}{8} - \frac{x^2 - v^2}{8} = \frac{v^2 - y^2}{8} \leq \frac{v - y}{4}$$

$$\begin{aligned}R = &\frac{\alpha}{2}[d(gx,gu)+d(gy,gv)]+\beta M((gx,gy),(gu,gv))\\ &+\frac{\gamma}{2}[d(gx,F(x,y))+d(gu,F(u,v))+d(gy,F(y,x))+d(gv,F(v,u))]\\ &+\frac{\delta}{2}[d(gx,F(u,v))+d(gy,F(v,u))+d(gu,F(x,y))+d(gv,F(y,x))]\end{aligned}$$

$$\begin{aligned}R = &\frac{\alpha}{2}[d(x^2,u^2)+d(y^2,v^2)]+\beta M(x^2,y^2),(u^2,v^2)\\ &+\frac{\gamma}{2}[d(x^2,\frac{x^2-y^2}{8})+d(u^2,\frac{u^2-v^2}{8})+d(y^2,\frac{y^2-x^2}{8})+d(v^2,\frac{v^2-u^2}{8})]\\ &+\frac{\delta}{2}[d(x^2,\frac{u^2-v^2}{8})+d(y^2,\frac{v^2-u^2}{8})+d(u^2,\frac{x^2-y^2}{8})+d(v^2,\frac{y^2-x^2}{8})]\end{aligned}$$

On putting the value of $\alpha,\beta,\gamma,\delta$.

$$\begin{aligned}R = &\frac{1}{8\times 2}[|x^2-u^2|+y^2-v^2]+0M(x^2,y^2),(u^2,v^2)\\ &+\frac{1}{4\times 2}[d(x^2,\frac{x-y}{4})+d(u^2,\frac{u-v}{4})+d(y^2,\frac{y-x}{4})+d(v^2,\frac{v-u}{4})]\\ &+\frac{1}{12\times 2}[d(x^2,\frac{u-v}{4})+d(y^2,\frac{v-u}{4})+d(u^2,\frac{x-y}{4})+d(v^2,\frac{y-x}{4})]\end{aligned}$$

$$\begin{aligned}R = &\frac{1}{16}[|4-1|+|9-4|]+\frac{1}{8}[\frac{|4x^2-x+y}{4}|+|\frac{4u^2-u+v}{4}|+|\frac{4y^2-y+x}{4}|+|\frac{4v^2-v+u}{4}|]\\ &+\frac{1}{2}[|\frac{4x^2-u+v}{4}|+|\frac{4y^2-v+u}{4}|+|\frac{4u^2-x+y}{4}|+|\frac{4v^2-y+x}{4}|]\end{aligned}$$

$$\begin{aligned}R &= \frac{1}{16}[8]+\frac{1}{8}[\frac{72}{4}]+\frac{1}{24}[\frac{72}{4}]\\ &= \frac{14}{4} = 3.5\end{aligned}$$

where $\alpha=1/8$,$\beta=0$,$\gamma=1/4$ and $\delta=1/12$, satisfying the condition $\alpha,\beta,\gamma,\delta\geq 0$ with $\alpha+\beta+2\gamma+2\delta<1$, on putting all those values in above right hand side equation, we get R=3.5, which is greater than the value of L, this implies that $L\leq R$ and the given contraction condition is satisfied.

3) $x=u>1$ and $y=v>1$; then obviously $L=0$.

Thus, F satisfies all the assumptions of the given theorems and it has an unique coupled fixed point (which is $(0,0)$).

Theorem 2.7. *Let $(X,d,\preceq)$ be a partially ordered metric space. Let $F: X\times X\to X$ and $g: X\to X$ be two continuous mappings such that F has the mixed*

g-monotone property on X and satisfying

(16)
$$\begin{aligned} &d(F(x,y),F(u,v)) \\ &\leq \frac{\alpha}{2}[d(gx,gu)+d(gy,gv)]+\beta N((gx,gy),(gu,gv)) \\ &\quad+\frac{\gamma}{2}[d(gx,F(x,y))+d(gu,F(u,v))+d(gy,F(y,x))+d(gv,F(v,u))], \end{aligned}$$

for all $(x,y),(u,v)\in X\times X$ with $gx\preceq gu$ and $gy\succeq gv$, when $D_1=d(gx,F(u,v))+d(gu,F(x,y))\neq 0$ and $D_2=d(gy,F(v,u))+d(gv,F(y,x))\neq 0$, where

(17)
$$\begin{aligned} &N((x,y),(u,v)) \\ &=\min\left\{\frac{d^2(gx,F(u,v))+d^2(gu,F(x,y))}{d(gx,F(u,v))+d(gu,F(x,y))},\frac{d^2(gy,F(v,u))+d^2(gv,F(y,x))}{d(gy,F(v,u))+d(gv,F(y,x))}\right\} \end{aligned}$$

and $\alpha,\beta,\gamma\geq 0$ with $\alpha+2\beta+2\gamma<1$. Further,

$$d(F(x,y),F(u,v))=0 \quad \text{if} \quad D_1=0 \text{ or } D_2=0. \tag{18}$$

We assume that there exist $x_0,y_0\in X$ such that

$$gx_0\preceq F(x_0,y_0) \quad \text{and} \quad gy_0\succeq F(y_0,x_0). \tag{19}$$

Then, F and g have a coupled coincidence fixed point $(\bar{x},\bar{y})\in X\times X$.

Proof. Following the proof of Theorem 2.1, we can construct sequences $\{gx_n\}$ and $\{gy_n\}$ satisfying conditions (4) and (5).

Now, we claim that, for $n\in\mathbb{N}$,

$$d(gx_{n+1},gx_n)+d(gy_{n+1},gy_n)\leq\left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)^n[d(gx_1,gx_0)+d(gy_1,gy_0)].$$

Indeed, for $n=1$, consider the following possibilities.

Case I: $gx_0\neq gx_2$ and $gy_0\neq gy_2$. Then $d(gx_1,F(x_0,y_0))+d(gx_0,F(x_1,y_1))\neq 0$ and $d(gy_1,F(y_0,x_0))+d(gy_0,F(y_1,x_1))\neq 0$. Hence, using $gx_1\preceq gx_0$,

$gy_1 \succeq gy_0$ and (16), we get:

(20)

$$\begin{aligned}
&d(gx_2, gx_1)\\
&= d(F(x_1, y_1), F(x_0, y_0))\\
&\leq \frac{\alpha}{2}[d(gx_1, gx_0) + d(gy_1, gy_0)] + \beta N(gx_1, gy_1), (gx_0, gy_0))\\
&\quad + \frac{\gamma}{2}[d(gx_1, F(x_1, y_1)) + d(gx_0, F(x_0, y_0)) + d(gy_1, F(y_1, x_1)) + d(gy_0, F(y_0, x_0))]\\
&\leq \frac{\alpha}{2}[d(gx_0, gx_1) + d(gy_0, gy_1)] + \beta \frac{d^2(gx_1, F(x_0, y_0)) + d^2(gx_0, F(x_1, y_1))}{d(gx_1, F(x_0, y_0)) + d(gx_0, F(x_1, y_1))}\\
&\quad + \frac{\gamma}{2}[d(gx_1, gx_2) + d(gx_0, gx_1) + d(gy_1, gy_2) + d(gy_0, gy_1)]\\
&\leq \frac{\alpha}{2}[d(gx_0, gx_1) + d(gy_0, gy_1)] + \beta[d(gx_0, gx_1) + d(gx_1, gx_2)]\\
&\quad + \frac{\gamma}{2}[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gx_1, gx_2) + d(gy_1, gy_2)].
\end{aligned}$$

Similarly, using that $d(gy_2, gy_1) = d(F(y_1, x_1), F(y_0, x_0)) = d(F(y_0, x_0), F(y_1, x_1))$ and

$$\begin{aligned}
N((y_1, x_1), (y_0, x_0)) &\leq \frac{d^2(gy_1, F(y_0, x_0) + d^2(gy_0, F(y_1, x_1))}{d(gy_1, F(y_0, x_0)) + d(gy_0, F(y_1, x_1))}\\
&= d(gy_0, gy_2) \leq d(gy_0, gy_1) + d(gy_1, gy_2),
\end{aligned}$$

we get

(21)

$$\begin{aligned}
d(gy_2, gy_1) \leq \frac{\alpha}{2}&[d(gx_0, gx_1) + d(gy_0, gy_1)] + \beta[d(gy_0, gy_1) + d(gy_1, gy_2)]\\
&+ \frac{\gamma}{2}[d(gx_0, gx_1) + d(gy_0, gy_1) + d(gx_1, gx_2) + d(gy_1, gy_2)].
\end{aligned}$$

Adding (20) and (21), we have

$$d(gx_2, gx_1) + d(gy_2, gy_1) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}\right)[d(gx_0, gx_1) + d(gy_0, gy_1)]. \quad (22)$$

Case II: $gx_0 = gx_2$ and $gy_0 \neq gy_2$. The first equality implies that $d(gx_1, F(x_0, y_0)) + d(gx_0, F(x_1, y_1)) = 0$, and hence $d(gx_1, gx_2) = d(F(x_0, y_0), F(x_1, y_1)) = 0$, by (18). This means that $gx_0 = gx_1 = gx_2$. From $gy_0 \neq gy_2$, as in the first case, we get that (21) holds true. As a consequence

$$d(gy_1, gy_2) \leq \frac{\frac{\alpha}{2} + \beta + \frac{\gamma}{2}}{1 - \beta - \frac{\gamma}{2}} d(gy_0, gy_1) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(gy_0, gy_1),$$

since $\frac{\frac{\alpha}{2} + \beta + \frac{\gamma}{2}}{1 - \beta - \frac{\gamma}{2}} \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma}$. But then $d(gx_0, gx_1) = d(gx_1, gx_2) = 0$ implies that (22) holds.

The case $gx_0 \neq gx_2$ and $gy_0 = gy_2$ is treated analogously.

Case III: If $gx_0 = gx_2$ and $gy_0 = gy_2$, then $d(gx_1, F(x_0, y_0)) + d(gx_0, F(x_1, y_1)) = 0$ and $d(gy_1, F(y_0, x_0)) + d(gy_0, F(y_1, x_1)) = 0$. Hence, (18) implies that $gx_1 = gx_2 = gx_3$ and $gy_1 = gy_2 = gy_3$, and so (22) holds trivially.

Thus, (20) holds for $n = 1$. In a similar way, proceeding by induction, if we assume that (20) holds, we get that

$$\begin{aligned} d(gx_{n+2}, gx_{n+1}) + d(gy_{n+2}, gy_{n+1}) &\leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)[d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)] \\ &\leq \left(\frac{\alpha+\beta+\gamma}{1-\beta-\gamma}\right)^{n+1}[d(gx_0, gx_1) + d(gy_0, gy_1)]. \end{aligned}$$

Hence, by induction, (20) is proved.

Using similar arguments as in the proof of Theorem 2.1, we have the desired result.

This completes the proof of the theorem. □

In the next theorem, we will substitute the continuity hypothesis on F and g by an additional property satisfied by the space $(X, d, \preceq)$.

Theorem 2.8. *Let $(X, d, \preceq)$ be a partially ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings having the mixed g-monotone property. Assume that there exist $\alpha, \beta, \gamma \geq 0$ with $\alpha + 2\beta + 2\gamma < 1$ such that*

$$\begin{aligned} &d(F(x, y), F(u, v)) \\ &\leq \frac{\alpha}{2}[d(gx, gu) + d(gy, gv)] + \beta N((gx, gy), (gu, gv)) \\ &\quad + \frac{\gamma}{2}[d(gx, F(x, y)) + d(gu, F(u, v)) + d(gy, F(y, x)) + d(gv, F(v, u))], \end{aligned}$$

for all $(x, y), (u, v) \in X \times X$ with $gx \preceq gu$ and $gy \succeq gv$, when $D_1 = d(gx, F(u, v)) + d(gu, F(x, y)) \neq 0$ and $D_2 = d(gy, F(v, u)) + d(gv, F(y, x)) \neq 0$, where

$$\begin{aligned} &N((x, y), (u, v)) \\ &= \min\left\{\frac{d^2(gx, F(u, v)) + d^2(gu, F(x, y))}{d(gx, F(u, v)) + d(gu, F(x, y))}, \frac{d^2(gy, F(v, u)) + d^2(gv, F(y, x))}{d(gy, F(v, u)) + d(gv, F(y, x))}\right\}. \end{aligned}$$

Further, $d(F(x, y), F(u, v)) = 0$ if $D_1 = 0$ or $D_2 = 0$.

Suppose that there exist $x_0, y_0 \in X$ such that

$$gx_0 \preceq F(x_0, y_0) \quad \text{and} \quad gy_0 \succeq F(y_0, x_0).$$

Finally, assume that X has the following properties:

(i) *if a nondecreasing sequence $\{gx_n\}$ in X converges to $x \in X$, then $gx_n \preceq gx$ for all n,*
(ii) *if a non-increasing sequence $\{gy_n\}$ in X converges to $y \in X$, then $gy_n \succeq gy$ for all n.*

Then, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Following the proof of Theorem 2.7, we only have to show that $(g\bar{x}, g\bar{y})$ is a coupled coincidence point of F and g. Suppose this is not the case, i.e., $F(\bar{x}, \bar{y}) \neq g\bar{x}$ or $F(\bar{y}, \bar{x}) \neq g\bar{y}$ (e.g., let the first one of these holds). We have

$$d(F(\bar{x}, \bar{y}), g\bar{x}) \leq d(F(\bar{x}, \bar{y}), gx_{n+1}) + d(gx_{n+1}, g\bar{x}) = d(F(\bar{x}, \bar{y}), F(x_n, y_n)) + d(gx_{n+1}, g\bar{x}). \tag{23}$$

Since the nondecreasing sequence $\{gx_n\}$ converges and $g(x_n) \to \bar{x}$ and the nonincreasing sequence $\{gy_n\}$ converges and $g(y_n) \to \bar{y}$, by (i)–(ii), we have:

$$g\bar{x} \preceq gx_n \quad \text{and} \quad g\bar{y} \succeq gy_n, \quad \forall\, n.$$

Now, from the contractive condition, we have:

$$\begin{aligned} d(F(\bar{x}, \bar{y}), F(x_n, y_n)) &\leq \frac{\alpha}{2}[d(g\bar{x}, gx_n) + d(g\bar{y}, gy_n)] + \beta N((g\bar{x}, g\bar{y}), (gx_n, gy_n)) \\ &\quad + \frac{\gamma}{2}[d(g\bar{x}, F(\bar{x}, \bar{y})) + d(gx_n, F(x_n, y_n)) + d(g\bar{y}, F(\bar{y}, \bar{x})) + d(gy_n, F(y_n, x_n))] \\ &\leq \frac{\alpha}{2}[d(g\bar{x}, gx_n) + d(g\bar{y}, gy_n)] + \beta \frac{d^2(g\bar{x}, gx_{n+1}) + d^2(gx_n, F(\bar{x}, \bar{y}))}{d(g\bar{x}, gx_{n+1}) + d(gx_n, F(\bar{x}, \bar{y}))} \\ &\quad + \frac{\gamma}{2}[d(g\bar{x}, F(\bar{x}, \bar{y})) + d(gx_n, gx_{n+1}) + d(g\bar{y}, F(\bar{y}, \bar{x})) + d(gy_n, gy_{n+1})]. \end{aligned}$$

We note that the case $d(g\bar{x}, gx_{n+1}) + d(gx_n, F(\bar{x}, \bar{y})) = 0$ is impossible, since otherwise the condition (18) would imply $g\bar{x} = F(\bar{x}, \bar{y})$, which is excluded. Then, from (23), we get:

$$\begin{aligned} d(F(\bar{x}, \bar{y}), g\bar{x}) &\leq d(gx_{n+1}, g\bar{x}) + \frac{\alpha}{2}[d(g\bar{x}, gx_n) + d(g\bar{y}, gy_n)] \\ &\quad + \beta \frac{d^2(g\bar{x}, gx_{n+1}) + d^2(gx_n, F(\bar{x}, \bar{y}))}{d(g\bar{x}, gx_{n+1}) + d(gx_n, F(\bar{x}, \bar{y}))} \\ &\quad + \frac{\gamma}{2}[d(g\bar{x}, F(\bar{x}, \bar{y})) + d(gx_n, gx_{n+1}) + d(g\bar{y}, F(\bar{y}, \bar{x})) + d(gy_n, gy_{n+1})]. \end{aligned}$$

Taking limit as $n \to \infty$ (and again using that $F(\bar{x}, \bar{y}) \neq g\bar{x}$), we have

$$d(F(\bar{x}, \bar{y}), g\bar{x}) \leq \beta d(g\bar{x}, F(\bar{x}, \bar{y})) + \frac{\gamma}{2}[d(g\bar{x}, F(\bar{x}, \bar{y})) + d(g\bar{y}, F(\bar{y}, \bar{x}))]. \tag{24}$$

Now, if $g\bar{y} = F(\bar{y}, \bar{x})$, using that $\beta + \frac{\gamma}{2} < 1$, it follows immediately that $g\bar{x} = F(\bar{x}, \bar{y})$, a contradiction. If this is not the case, we similarly get

$$d(g\bar{y}, F(\bar{y}, \bar{x})) \leq \beta d(g\bar{y}, F(\bar{y}, \bar{x})) + \frac{\gamma}{2}[d(g\bar{x}, F(\bar{x}, \bar{y})) + d(g\bar{y}, F(\bar{y}, \bar{x}))]. \tag{25}$$

Adding (24) and (25), we have

$$\begin{aligned} d(g\bar{x}, F(\bar{x}, \bar{y})) + d(g\bar{y}, F(\bar{y}, \bar{x})) &\leq (\beta + \gamma)[d(g\bar{x}, F(\bar{x}, \bar{y})) + d(g\bar{y}, F(\bar{y}, \bar{x}))] \\ &\leq (\alpha + 2\beta + 2\gamma)[d(g\bar{x}, F(\bar{x}, \bar{y})) + d(g\bar{y}, F(\bar{y}, \bar{x}))]. \end{aligned}$$

Since $0 \leq \alpha + 2\beta + 2\gamma < 1$, we obtain $d(F(\bar{x}, \bar{y}), g\bar{x}) = 0$ and $d(g\bar{y}, F(\bar{y}, \bar{x})) = 0$, i.e., $F(\bar{x}, \bar{y}) = g\bar{x}$ and $F(\bar{y}, \bar{x}) = g\bar{y}$, again a contradiction. This completes the proof of the theorem. □

Theorem 2.9. *Assume that*

$$\forall (x,y),(x^*,y^*) \in X \times X,\ \exists\,(u,v) \in X \times X \text{ such that} \tag{26}$$

$(F(u,v),F(v,u))$ *is comparable to* $(F(x,y),F(y,x))$ *and* $(F(x^*,y^*),F(y^*,x^*))$. *Then* F *and* g *have unique coupled coincidence point that is there exists an unique* $(x,y) \in X \times X$ *such that* $gx = F(x,y)$ *and* $gy = F(y,x)$, $gx^* = F(x^*,y^*)$ *and* $gy^* = F(y^*,x^*)$. *Adding (26) to the hypotheses of Theorem 2.7, we obtain the uniqueness of the coupled coincidence point of* F *and* g.

Proof. From Theorem 2.7 we know that there exists the set of coupled coincidence point of F and g is non empty, suppose that $(\bar{x},\bar{y})$ and (x^*,y^*) are coupled coincidence point of F and g, that is $g\bar{x} = F(\bar{x},\bar{y})$ and $g\bar{y} = F(\bar{y},\bar{x})$, $gx^* = F(x^*,y^*)$ and $gy^* = F(y^*,x^*)$. which is obtained as $g\bar{x} = \lim_{n\to\infty} F^n(x_0,y_0)$ and $g\bar{y} = \lim_{n\to\infty} F^n(y_0,x_0)$. Let us prove that

$$d(g\bar{x},gx^*) + d(g\bar{y},gy^*) = 0. \tag{27}$$

We distinguish two cases.

Case I: $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ is comparable with $(F(x^*,y^*),F(y^*,x^*))$ with respect to the ordering in $X \times X$. Let, e.g., $g\bar{x} \preceq gx^*$ and $g\bar{y} \succeq gy^*$. Then, we can apply the contractive condition (16) to obtain

$$\begin{aligned} d(g\bar{x},gx^*) &= d(F(\bar{x},\bar{y}),F(x^*,y^*)) \\ &\le \frac{\alpha}{2}[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)] + \beta d(g\bar{x},gx^*), \end{aligned}$$

and

$$\begin{aligned} d(g\bar{y},gy^*) &= d(F(\bar{y},\bar{x}),F(y^*,x^*)) = d(F(y^*,x^*),F(\bar{y},\bar{x})) \\ &\le \frac{\alpha}{2}[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)] + \beta d(g\bar{y},gy^*). \end{aligned}$$

Adding up, we get that

$$d(g\bar{x},gx^*) + d(g\bar{y},gy^*) \le (\alpha+\beta)[d(g\bar{x},gx^*) + d(g\bar{y},gy^*)].$$

Since $0 \le \alpha + \beta < 1$, (27) holds.

Case II: $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ is not comparable with $(F(x^*,y^*),F(y^*,x^*))$. In this case, By assumption there exists $(u,v) \in X \times X$ that is comparable both to $(F(\bar{x},\bar{y}),F(\bar{y},\bar{x}))$ and $(F(x^*,y^*),F(y^*,x^*))$. Then, for all $n \in \mathbb{N}$, $(F^n(u,v),F^n(v,u))$ is comparable both to $(F^n(\bar{x},\bar{y}),F^n(\bar{y},\bar{x})) = (g\bar{x},g\bar{y})$ and $(F^n(x^*,y^*),F^n(y^*,x^*)) = (gx^*,gy^*)$. We have

$$\begin{aligned} d(g\bar{x},gx^*) + d(g\bar{y},gy^*) &= d(F^n(\bar{x},\bar{y}),F^n(x^*,y^*)) + d(F^n(\bar{y},\bar{x}),F^n(y^*,x^*)) \\ &\le d(F^n(\bar{x},\bar{y}),F^n(u,v)) + d(F^n(u,v),F^n(x^*,y^*)) \\ &\quad + d(F^n(\bar{y},\bar{x}),F^n(v,u)) + d(F^n(v,u),F^n(y^*,x^*)) \\ &\le (\alpha^n + \beta^n)[d(g\bar{x},u) + d(g\bar{y},v) + d(gx^*,u) + d(gy^*,v)]. \end{aligned}$$

Since $0 < \alpha, \beta < 1$, (27) holds.

We deduce that in all cases (27) holds. This implies that $(g\bar{x}, g\bar{y}) = (gx^*, gy^*)$ and the uniqueness of the coupled coincidence point of F and g is proved. □

Our next result is as follows:

Theorem 2.10. *In addition to the hypotheses of Theorem 2.7 (resp. Theorem 2.8), suppose that $g(x_0), g(y_0)$ in X are comparable. Then $g\bar{x} = g\bar{y}$.*

Proof. Suppose that $x_0 \preceq y_0$. We claim that

$$gx_n \preceq gy_n, \ \forall n \in \mathbb{N}. \tag{28}$$

From the mixed monotone property of F, we have

$$gx_1 = F(x_0, y_0) \preceq F(y_0, y_0) \preceq F(y_0, x_0) = gy_1.$$

Assume that $x_n \preceq y_n$ for some n. Now,

$$\begin{aligned} gx_{n+1} &= F^{n+1}(x_0, y_0) = F(F^n(x_0, y_0), F^n(y_0, x_0)) \\ &= F(x_n, y_n) \preceq F(y_n, y_n) \preceq F(y_n, x_n) = gy_{n+1}. \end{aligned}$$

Hence, (28) holds.

Now, using (28) and the contractive condition, we get

$$\begin{aligned} &d(g\bar{x}, g\bar{y}) \le d(g\bar{x}, gx_{n+1}) + d(gx_{n+1}, gy_{n+1}) + d(gy_{n+1}, g\bar{y}) \\ &= d(g\bar{x}, gx_{n+1}) + d(F(y_n, x_n), F(x_n, y_n)) + d(gy_{n+1}, g\bar{y}) \\ &\le d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) \\ &\quad + \beta N((y_n, x_n), (x_n, y_n)) \\ &\quad + \frac{\gamma}{2}[d(gx_n, F(x_n, y_n)) + d(gy_n, F(y_n, x_n)) + d(gy_n, F(y_n, x_n)) + d(gx_n, F(x_n, y_n)))] \\ &\le d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) \\ &\quad + \beta \frac{d^2(gx_n, F(y_n, x_n)) + d^2(gy_n, F(x_n, y_n))}{d(gx_n, F(y_n, x_n)) + d(gy_n, F(x_n, y_n))} + \gamma[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})] \\ &\le d(g\bar{x}, gx_{n+1}) + d(gy_{n+1}, g\bar{y}) + \alpha d(gx_n, gy_n) \\ &\quad + \beta \frac{d^2(gx_n, gy_{n+1}) + d^2(gy_n, gx_{n+1})}{d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1})} \\ &\quad + \gamma[d(gx_n, gx_{n+1}) + d(gy_n, gy_{n+1})], \end{aligned}$$

provided $d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1}) \neq 0$.

Passing to the limit as $n \to \infty$, we get that

$$d(g\bar{x}, g\bar{y}) \le (\alpha + \beta) d(g\bar{x}, g\bar{y}).$$

Since $0 \le \alpha + \beta < 1$, this implies that $d(g\bar{x}, g\bar{y}) = 0$, i.e., $g\bar{x} = g\bar{y}$.

In the case when $d(gx_n, gy_{n+1}) + d(gy_n, gx_{n+1}) = 0$, the conditions of the theorem readily imply that $d(g\bar{x}, g\bar{y}) = 0$. This completes the proof of the theorem. □

Remark 2. The results of this paper can be easily modified in a way to obtain the existence of a coupled coincidence point of the mapping $F : X \times X \to X$ and an additional mapping $G : X \times X \to X$, in the case when F has the g-mixed monotone property (see respective definitions in [7]).

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