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A CRANK-NICOLSON CHARACTERISTIC FINITE ELEMENT METHOD FOR SOBOLEV EQUATIONS

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Abstract. A Crank-Nicolson characteristic finite element method is introduced to construct approximate solutions of a Sobolev equation with a convection term. The higher order of convergences in the temporal direction and in the spatial direction in L^2 normed space are verified for the Crank-Nicolson characteristic finite element method.

1. Introduction

In this paper, we consider the following Sobolev equation with a convection term: Find $u(\boldsymbol{x}, t)$ defined on $\Omega \times [0, T]$ such that

$$\begin{aligned} c(\boldsymbol{x})u_t + \boldsymbol{d}(\boldsymbol{x}) \cdot \nabla u - \nabla \cdot (a(u)\nabla u) - \nabla \cdot (b(u)\nabla u_t) & \\ = f(\boldsymbol{x}, t, u), &\quad \text{in } \Omega \times (0, T], \\ u(\boldsymbol{x}, t) = 0, &\quad \text{on } \partial\Omega \times (0, T], \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), &\quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^m$, $1 \le m \le 3$, is a bounded convex domain with boundary $\partial\Omega$ and $c, \boldsymbol{d}, a, b$ and f are known functions. The study of Sobolev equations is very important because Sobolev equations describe physical phenomena such as thermodynamics [20], the migration of the moisture in soil [17], the flow of fluids through fissured rock [2] and other applications. For the existence, uniqueness, and regularity of the solutions of the Sobolev equation (1.1), we refer to [3, 4, 20].

For Sobolev equations without a convection term, many mathematicians achieve the numerical results by classical finite element methods [1, 6, 10, 11, 12] or least-squares methods [9, 15, 16, 21, 22] or mixed finite element methods [8] or discontinuous finite element methods [13, 14, 18, 19]. But in many situations, the convection term $\boldsymbol{d}(\boldsymbol{x}) \cdot \nabla u$ exists and $\boldsymbol{d}(\boldsymbol{x})$ is large in order to describe a

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convection dominated diffusion. To treat the time derivative term and the convection term effectively, we use a characteristic method which is natural from the physical point of view and works well for convection dominated diffusion problems as shown in [5, 7]. In [7], the author construct a characteristic finite element method to obtain the higher order of convergence in the spatial variable and the first order of convergence in the temporal variable. But the latter makes the former meaningless.

In this paper, a Crank-Nicolson characteristic finite element method is introduced to construct approximate solutions of the Sobolev equation with a convection term and establish the higher order of convergences in the temporal direction and in the spatial direction in L^2 normed space for the Crank-Nicolson characteristic finite element method. Our paper is organized as follows. In Section 2, we present the smoothness assumptions for $u(\boldsymbol{x},t)$, the conditions for the given functions, and basic notations. In Section 3, we construct finite element spaces and derive basic approximation properties. In Section 4, we construct a Crank-Nicolson characteristic finite element approximation of $u(\boldsymbol{x},t)$ and obtain the higher order of convergence in L^2 and H^1 normed spaces.

2. Assumptions and notations

Throughout this paper, $W^{s,p}(\Omega)$ denotes the Sobolev space equipped with its norm $\|\cdot\|_{s,p}$ for an $s \geq 0$ and $1 \leq p \leq \infty$. For our convenience, we simply denote $H^s(\Omega)$ and $L^2(\Omega)$, instead of $W^{s,2}(\Omega)$ and $H^0(\Omega)$, respectively. And we denote $\|\cdot\|$, $\|\cdot\|_\infty$, and $\|\cdot\|_s$, instead of $\|\cdot\|_{0,2}$, $\|\cdot\|_{0,\infty}$, and $\|\cdot\|_{s,2}$, respectively. Let $\boldsymbol{H}^s(\Omega) = \{\boldsymbol{w} = (w_1, w_2, \ldots, w_m) \mid w_i \in H^s(\Omega), 1 \leq i \leq m\}$ be a Sobolev space equipped with its norm $\|\boldsymbol{w}\|_s^2 = \sum\limits_{i=1}^{m} \|w_i\|_s^2$ and $H_0^1(\Omega) = \{w \in H^1(\Omega) \mid w(\boldsymbol{x}) = 0 \text{ on } \partial\Omega\}$. For a given Banach space X and $t_1, t_2 \in [0,T]$, we introduce Sobolev spaces with the corresponding norms:

$$W^{s,p}(t_1,t_2;X) = \Big\{ w(\boldsymbol{x},t) \mid \|\frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X \in L^p(t_1,t_2), 0 \leq \beta \leq s \Big\},$$

where

$$\|w\|_{W^{s,p}(t_1,t_2;X)} = \begin{cases} \Big(\sum_{\beta=0}^{s} \int_{t_1}^{t_2} \|\frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X^p dt \Big)^{1/p}, & 1 \leq p < \infty, \\ max_{0\leq\beta\leq s} \operatorname{esssup}_{t\in(t_1,t_2)} \|\frac{\partial^\beta w}{\partial t^\beta}(\cdot,t)\|_X, & p = \infty. \end{cases}$$

We simply write $L^p(X)$ and $W^{s,p}(X)$ instead of $W^{0,p}(0,T;X)$ and $W^{s,p}(0,T;X)$, respectively.

Assume that $u(\boldsymbol{x},t)$ and $c(\boldsymbol{x})$, $\boldsymbol{d}(\boldsymbol{x}) = (d_1(\boldsymbol{x}), d_2(\boldsymbol{x}), \ldots, d_m(\boldsymbol{x}))^T$, $a(u), b(u)$ and $f(\boldsymbol{x},t,u)$ satisfy the following assumptions:

(A1) There exists a positive constant $\tilde{K}$ such that $\|u(\boldsymbol{x},t)\|_{L^\infty(L^\infty)} \leq \tilde{K}$.

(A2) There exist constants c_*, c^*, d^*, a_*, a^*, b_*, and b^* such that $0 < c_* \le c(\boldsymbol{x}) \le c^*$, $0 < |\boldsymbol{d}(\boldsymbol{x})| \le d^*$, $0 < a_* \le a(u) \le a^*$, $0 < b_* \le b(u) \le b^*$, for all $\boldsymbol{x} \in \Omega$ and $t \in [0,T]$, where $|\boldsymbol{d}(\boldsymbol{x})| = \sum_{i=1}^{m} d_i^2(\boldsymbol{x})$.

(A3) $a(u), a_u(u), a_{uu}(u), b(u), b_u(u)$ and $b_{uu}(u)$ are bounded.

(A4) $f(\boldsymbol{x}, t, u)$ is locally Lipschitz continuous in the third variable, i.e. if $|u(\boldsymbol{x}, t) - u^*| \le \tilde{K}$ then $|f(\boldsymbol{x}, t, u(\boldsymbol{x}, t)) - f(\boldsymbol{x}, t, u^*)| \le K(u, \tilde{K})|u(\boldsymbol{x}, t) - u^*|$.

For each $(\boldsymbol{x}, t)$, we let $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x}, t)$ be the unit vector such that $\frac{\partial u}{\partial \boldsymbol{\nu}} = \frac{c(\boldsymbol{x})}{\psi(\boldsymbol{x})}\frac{\partial u}{\partial t} + \frac{\boldsymbol{d}(\boldsymbol{x})}{\psi(\boldsymbol{x})} \cdot \nabla u$, where $\psi(\boldsymbol{x}) = [c(\boldsymbol{x})^2 + |\boldsymbol{d}(\boldsymbol{x})|^2]^{\frac{1}{2}}$. Then the Sobolev equation (1.1) can be rewritten as follows

$$
\begin{aligned}
\psi(\boldsymbol{x})\frac{\partial u}{\partial \boldsymbol{\nu}} - \nabla \cdot (a(u)\nabla u) - \nabla \cdot (b(u)\nabla u_t) &= f(\boldsymbol{x}, t, u), \text{ in } \Omega \times (0, T], \\
u(\boldsymbol{x}, t) &= 0, \text{ on } \partial\Omega \times (0, T], \\
u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}), \text{ in } \Omega.
\end{aligned} \tag{2.1}
$$

Now the variational formulation of the equation (2.1) is given as follows: Find $u(\boldsymbol{x}, t) \in H_0^1(\Omega)$ such that

$$
\begin{aligned}
(\psi(\boldsymbol{x})\frac{\partial u}{\partial \boldsymbol{\nu}}, \tau) + (a(u)\nabla u, \nabla \tau) &+ (b(u)\nabla u_t, \nabla \tau) \\
&= (f(x, t, u), \tau), \ \forall \tau \in H_0^1(\Omega), \\
u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}).
\end{aligned} \tag{2.2}
$$

3. Finite element spaces and an elliptic projection

For $h > 0$, let $\{S_h^r\}$ be a family of finite dimensional subspaces of $H_0^1(\Omega)$ satisfying the following approximation and inverse properties: for $\phi \in H_0^1(\Omega) \cap W^{s,p}(\Omega)$, there exist a positive constant K_1, independent of h, ϕ, and r, and a sequence $P_h\phi \in S_h^r$ such that for any $0 \le q \le s$ and $1 \le p \le \infty$,

$$\|\phi - P_h\phi\|_{q,p} \le K_1 h^{\mu - q}\|\phi\|_{s,p},$$

where $\mu = \min(r+1, s)$ and

$$\|\varphi\|_1 \le K_1 h^{-1}\|\varphi\| \text{ and } \|\varphi\|_\infty \le K_1 h^{-\frac{m}{2}}\|\varphi\|, \ \forall \varphi \in S_h^r.$$

Now we introduce bilinear forms A and B defined on $H_0^1(\Omega) \times H_0^1(\Omega)$ as follows

$$A(u : v, w) = (a(u)\nabla v, \nabla w), \quad B(u : v, w) = (b(u)\nabla v, \nabla w). \tag{3.1}$$

By the assumption (A2), it is clear that there exists a unique $\tilde{u}(t) \in S_h^r$ satisfying

$$
\begin{aligned}
A(u : u - \tilde{u}, \chi) + B(u : u_t - \tilde{u}_t, \chi) &= 0, \qquad \forall \chi \in S_h^r, \\
(\tilde{u}(0), \chi) &= (u_0, \chi), \quad \forall \chi \in S_h^r.
\end{aligned} \tag{3.2}
$$

Now letting $\eta = u - \tilde{u}$, we obtain some estimates for η, η_t, η_{tt}, and η_{ttt} whose proofs can be found in [13, 14].

Theorem 3.1. *Let $u_0 \in H^s(\Omega)$, $u_t, u_{tt}, u_{ttt} \in H^s(\Omega)$, and $u_t \in L^2(H^s(\Omega))$. Then there exists a constant K, independent of h, such that*

(**i**) $\|\eta\| + h\|\eta\|_1 \leq Kh^{\mu}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s)$,

(**ii**) $\|\eta_t\| + h\|\eta_t\|_1 \leq Kh^{\mu}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s)$,

(**iii**) $\|\eta_{tt}\|_1 \leq Kh^{\mu-1}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s + \|u_{tt}\|_s)$,

(**iv**) $\|\eta_{ttt}\|_1 \leq Kh^{\mu-1}(\|u_t\|_{L^2(H^s(\Omega))} + \|u_0\|_s + \|u_t\|_s + \|u_{tt}\|_s + \|u_{ttt}\|_s)$,

where $\mu = \min(r+1, s)$ *and* $s \geq 2$.

Throughout this paper, K denotes a generic positive constant depending on the domain $\Omega, \tilde{K}, u(\boldsymbol{x}, t)$ but independent of the discretization magnitudes of spatial and temporal directions. So any two Ks in the different places don't need to be the same.

4. The optimal $L^\infty(L^2(\Omega))$ and $L^\infty(H^1(\Omega))$ error estimates

Let N be a positive integer, $\Delta t = T/N$ and $t^n = n\Delta t$, for $0 \leq n \leq N$. By the definitions of bilinear forms A and B, (3) can be rewritten as follows:

$$\begin{aligned} &\Big(\psi(\boldsymbol{x})\frac{\partial u(t^{n-\frac{1}{2}})}{\partial \boldsymbol{\nu}}, \chi\Big) + A(u(t^{n-\frac{1}{2}}) :\ u(t^{n-\frac{1}{2}}), \chi) \\ &\quad + B(u(t^{n-\frac{1}{2}}) :\ u_t(t^{n-\frac{1}{2}}), \chi) = (f(u(t^{n-\frac{1}{2}})), \chi), \quad \forall \chi \in S_h^r, \end{aligned} \tag{4.1}$$

where $f(u(t^{n-\frac{1}{2}})) = f(\boldsymbol{x}, t^{n-\frac{1}{2}}, u(t^{n-\frac{1}{2}}))$ And so (4.1) can be rewritten as follows:

$$\begin{aligned} &\Big(c(\boldsymbol{x})\frac{\check{u}^n - \hat{u}^{n-1}}{\Delta t}, \chi\Big) + A(u(t^{n-\frac{1}{2}}) : u^{n-\frac{1}{2}}, \chi) \\ &\qquad + B(u(t^{n-\frac{1}{2}}) :\ \frac{u^n - u^{n-1}}{\Delta t}, \chi) \\ &= (f(u(t^{n-\frac{1}{2}})), \chi) + (c(\boldsymbol{x})\frac{\check{u}^n - \hat{u}^{n-1}}{\Delta t} - \psi(\boldsymbol{x})\frac{\partial u(t^{n-\frac{1}{2}})}{\partial \boldsymbol{\nu}}, \chi) \\ &\quad + A(u(t^{n-\frac{1}{2}}) :\ u^{n-\frac{1}{2}} - u(t^{n-\frac{1}{2}}), \chi) \\ &\quad + B(u(t^{n-\frac{1}{2}}) :\ \frac{u^n - u^{n-1}}{\Delta t} - u_t(t^{n-\frac{1}{2}}), \chi),\ \forall \chi \in S_h^r, \end{aligned} \tag{4.2}$$

where $\check{u}^n = u^n(\check{\boldsymbol{x}})$, $\hat{u}^{n-1} = u^{n-1}(\hat{\boldsymbol{x}})$, $\check{\boldsymbol{x}} = \boldsymbol{x} + \frac{1}{2}\tilde{\boldsymbol{d}}(\boldsymbol{x})\Delta t$, $\hat{\boldsymbol{x}} = \boldsymbol{x} - \frac{1}{2}\tilde{\boldsymbol{d}}(\boldsymbol{x})\Delta t$, $\tilde{\boldsymbol{d}}(\boldsymbol{x}) = \frac{\boldsymbol{d}(\boldsymbol{x})}{c(\boldsymbol{x})}$, and $u^{n-\frac{1}{2}} = \frac{1}{2}(u^n + u^{n-1})$.

We now introduce a Crank-Nicolson characteristic finite element scheme: Find $\{u_h^n\} \in S_h^r$ such that for $n = 1, 2, \ldots, N$

$$\begin{aligned} &\Big(c(\boldsymbol{x})\frac{\check{u}_h^n - \hat{u}_h^{n-1}}{\Delta t}, \chi\Big) + (a(u_h^{n-\frac{1}{2}})\nabla u_h^{n-\frac{1}{2}}, \nabla\chi) \\ &\quad + \Big(b(u_h^{n-\frac{1}{2}})\frac{\nabla u_h^n - \nabla u_h^{n-1}}{\Delta t}, \nabla\chi\Big) = (f(u_h^{n-\frac{1}{2}}), \chi), \quad \forall \chi \in S_h^r, \\ &\qquad u_h^0(\boldsymbol{x}) = \tilde{u}(\boldsymbol{x}, 0), \end{aligned} \tag{4.3}$$

where $\check{u}_h^n = u_h^n(\check{\boldsymbol{x}})$, $\hat{u}_h^{n-1} = u_h^{n-1}(\hat{\boldsymbol{x}})$, $\check{\boldsymbol{x}} = \boldsymbol{x} + \frac{1}{2}\tilde{\boldsymbol{d}}(\boldsymbol{x})\Delta t$, $\hat{\boldsymbol{x}} = \boldsymbol{x} - \frac{1}{2}\tilde{\boldsymbol{d}}(\boldsymbol{x})\Delta t$, $u_h^{n-\frac{1}{2}} = \frac{1}{2}(u_h^n + u_h^{n-1})$, $\tilde{\boldsymbol{d}}(\boldsymbol{x}) = \frac{\boldsymbol{d}(\boldsymbol{x})}{c(\boldsymbol{x})}$, and $f(u_h^{n-\frac{1}{2}}) = (f(\boldsymbol{x}, t^{n-\frac{1}{2}}, u_h^{n-\frac{1}{2}})$. By the definitions of bilinear forms A and B, (4.3) can be rewritten as follows:

$$\begin{aligned}\Big(c(\boldsymbol{x})\frac{\check{u}_h^n - \hat{u}_h^{n-1}}{\Delta t}, \chi\Big) + A(u_h^{n-\frac{1}{2}} : u_h^{n-\frac{1}{2}}, \chi) + B(u_h^{n-\frac{1}{2}} : \frac{u_h^n - u_h^{n-1}}{\Delta t}, \chi) \\ = (f(u_h^{n-\frac{1}{2}}), \chi), \qquad \forall \chi \in S_h^r.\end{aligned} \tag{4.4}$$

Lemma 4.1. *Let $u_0 \in H^s(\Omega)$, $u, u_t, u_{tt}, u_{ttt} \in L^\infty(H^s(\Omega)) \cap L^\infty(W^{1,\infty}(\Omega))$, $u_t \in L^2(H^s(\Omega))$ and $s \geq 2$. If $\mu \geq 1 + \frac{m}{2}$, then the following statements hold:*

$$\max\{\|\eta\|_\infty,\ \|\nabla\eta\|_\infty,\ \|\nabla\partial_t\eta\|_\infty,\ \|\nabla\eta_t\|_\infty,\ \|\nabla\eta_{tt}\|_\infty,\ \|\nabla\eta_{ttt}\|_\infty\} \leq \tilde{K}.$$

Proof. By the approximation property, the inverse property, Theorem 3.1, and the fact that $\mu \geq 1 + \frac{m}{2}$, we obtain

$$\begin{aligned}\|\nabla\eta\|_\infty &\leq \|\nabla(u - P_h u)\|_\infty + \|\nabla(P_h u - \tilde{u})\|_\infty \\ &\leq \|\nabla(u - P_h u)\|_\infty + Ch^{-\frac{m}{2}}\|\nabla(P_h u - \tilde{u})\| \\ &\leq \|\nabla(u - P_h u)\|_\infty + Ch^{-\frac{m}{2}}(\|\nabla(u - P_h u)\| + \|\nabla(u - \tilde{u})\|) \\ &\leq K_1[\|u\|_{1,\infty} + h^{-\frac{m}{2}+\mu-1}(\|u\|_s + \|u_t\|_{L^2(H^s)} + \|u_0\|_s)],\end{aligned}$$

so that $\|\nabla\eta\|_\infty$ is bounded by $\tilde{K}$. By similar arguments, we can prove the boundedness of $\|\eta\|_\infty, \|\nabla\partial_t\eta\|_\infty, \|\nabla\eta_t\|_\infty, \|\nabla\eta_{tt}\|_\infty$ and $\|\nabla\eta_{ttt}\|_\infty$. □

For our convenience of error analysis, we denote $\xi = u_h - \tilde{u}$, $\partial_t\xi^n = \frac{\xi^n - \xi^{n-1}}{\Delta t}$.

Theorem 4.1. *In addition to the assumptions of Lemma 4.1, if $u \in L^\infty(H^3(\Omega))$, $\mu \geq 1 + \frac{m}{2}$, and $\Delta t = O(h)$, then*

$$\|\nabla\xi^1\|^2 + \Delta t(\|\partial_t\xi^1\|^2 + \|\nabla\partial_t\xi^1\|^2) \leq K\Delta t(h^{2\mu} + (\Delta t)^4),$$

where $\mu = \min(r+1, s)$.

Proof. To obtain our result, we subtract (4.2) from (4.4) with $n = 1$ and take $\chi = \partial_t \xi^1$ to get

$$\begin{aligned}
&(c(\boldsymbol{x})\partial_t\xi^1, \partial_t\xi^1) + A(u_h^{\frac{1}{2}} \,:\, \xi^{\frac{1}{2}}, \partial_t\xi^1) + B(u_h^{\frac{1}{2}} \,:\, \partial_t\xi^1, \partial_t\xi^1) \\
=&\Big(c(\boldsymbol{x})\frac{\xi^1 - \check{\xi}^1}{\Delta t}, \partial_t\xi^1\Big) + \Big(c(\boldsymbol{x})\frac{\hat{\xi}^0 - \xi^0}{\Delta t}, \partial_t\xi^1\Big) - \Big(c(\boldsymbol{x})\frac{\eta^1 - \check{\eta}^1}{\Delta t}, \partial_t\xi^1\Big) \\
&+ \Big(c(\boldsymbol{x})\frac{\eta^1 - \eta^0}{\Delta t}, \partial_t\xi^1\Big) - \Big(c(\boldsymbol{x})\frac{\hat{\eta}^0 - \eta^0}{\Delta t}, \partial_t\xi^1\Big) + A(u_h^{\frac{1}{2}} \,:\, \eta^{\frac{1}{2}}, \partial_t\xi^1) \\
&+ \Big[A(u(t^{\frac{1}{2}}) \,:\, u^{\frac{1}{2}}, \partial_t\xi^1) - A(u_h^{\frac{1}{2}} \,:\, u^{\frac{1}{2}}, \partial_t\xi^1)\Big] \\
&+ B(u_h^{\frac{1}{2}} \,:\, \frac{\eta^1 - \eta^0}{\Delta t}, \partial_t\xi^1) \\
&+ \Big[B(u(t^{\frac{1}{2}}) \,:\, \frac{u^1 - u^0}{\Delta t}, \partial_t\xi^1) - B(u_h^{\frac{1}{2}} \,:\, \frac{u^1 - u^0}{\Delta t}, \partial_t\xi^1)\Big] \\
&+ (f(u_h^{\frac{1}{2}}) - f(u(t^{\frac{1}{2}})), \partial_t\xi^1) + A(u(t^{\frac{1}{2}}) :\, u(t^{\frac{1}{2}}) - u^{\frac{1}{2}}, \partial_t\xi^1) \\
&+ B(u(t^{\frac{1}{2}}) :\, u_t(t^{\frac{1}{2}}) - \frac{u^1 - u^0}{\Delta t}, \partial_t\xi^1) \\
&+ \Big(\psi(\boldsymbol{x})\frac{\partial u(t^{\frac{1}{2}})}{\partial \boldsymbol{\nu}} - c(\boldsymbol{x})\frac{\breve{u}^1 - \bar{u}^0}{\Delta t}, \partial_t\xi^1\Big) \\
=&\sum_{i=1}^{13} I_i.
\end{aligned} \tag{4.5}$$

By the assumption (A2) and the fact that $\xi^0 = 0$, we estimate the left hand-side of (4.5) as follows:

$$c_*\|\partial_t\xi^1\|^2 + b_*\|\nabla\partial_t\xi^1\|^2 + \frac{a_*}{2\Delta t}\|\nabla\xi^1\|^2 \le \sum_{i=1}^{13} I_i. \tag{4.6}$$

By the assumption (A2), the Cauchy-Schwartz inequality, and the fact that $\xi^0 = 0$, we estimate $I_1 \sim I_5$ as follows: for an $\epsilon > 0$

$$\begin{aligned}
I_1 &\le \epsilon\|\partial_t\xi^1\|^2 + K\|\nabla\xi^1\|^2, \\
I_2 &\le \epsilon\|\partial_t\xi^1\|^2 + K\|\nabla\xi^0\|^2, \\
I_3 &\le \epsilon(\|\partial_t\xi^1\|^2 + \|\nabla\partial_t\xi^1\|^2) + K\|\eta^1\|^2, \\
I_4 &\le \epsilon\|\partial_t\xi^1\|^2 + K\|\eta_t^1\|^2, \\
I_5 &\le \epsilon(\|\partial_t\xi^1\|^2 + \|\nabla\partial_t\xi^1\|^2) + K\|\eta^0\|^2.
\end{aligned}$$

To get the bound for the sum of I_6 and I_8, by using (3.2), we split it into four terms as follows:

$$\begin{aligned} I_6+I_8 =&([a(u(t^{\frac{1}{2}}))-a(u_h^{\frac{1}{2}})]\nabla\eta^{\frac{1}{2}},\nabla\partial_t\xi^1) \\ &+\Big([b(u(t^{\frac{1}{2}}))-b(u_h^{\frac{1}{2}})]\frac{\nabla\eta^1-\nabla\eta^0}{\Delta t},\nabla\partial_t\xi^1\Big) \\ &+(a(u(t^{\frac{1}{2}}))[\nabla\eta(t^{\frac{1}{2}})-\nabla\eta^{\frac{1}{2}}],\nabla\partial_t\xi^1) \\ &+\Big(b(u(t^{\frac{1}{2}}))\Big[\nabla\eta_t(t^{\frac{1}{2}})-\frac{\nabla\eta^1-\nabla\eta^0}{\Delta t}\Big],\nabla\partial_t\xi^1\Big) \\ \equiv& J_1+J_2+J_3+J_4. \end{aligned}$$

Note that

$$\begin{aligned} \|u_h^{\frac{1}{2}}-u(t^{\frac{1}{2}})\| &= \|u_h^{\frac{1}{2}}-\tilde{u}^{\frac{1}{2}}+\tilde{u}^{\frac{1}{2}}-\tilde{u}(t^{\frac{1}{2}})+\tilde{u}(t^{\frac{1}{2}})-u(t^{\frac{1}{2}})\| \\ &= \|\xi^{\frac{1}{2}}+\tilde{u}^{\frac{1}{2}}-\tilde{u}(t^{\frac{1}{2}})-\eta(t^{\frac{1}{2}})\| \\ &\le K(\|\xi^1\|+h^{\mu}+(\Delta t)^2). \end{aligned} \tag{4.7}$$

By Lemma 4.1 and (4.7), the bounds for J_1 and J_2 can be obtained as follows:

$$J_1 \le \epsilon\|\nabla\partial_t\xi^1\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4)$$

and

$$J_2 \le \epsilon\|\nabla\partial_t\xi^1\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4).$$

Since $\|\nabla\eta(t^{\frac{1}{2}})-\nabla\eta^{\frac{1}{2}}\|+\|\nabla\eta_t(t^{\frac{1}{2}})-\frac{\nabla\eta^1-\nabla\eta^0}{\Delta t}\| \le K(\Delta t)^2$ by the Taylor expansion and Theorem 3.1, we have

$$J_3+J_4 \le \epsilon\|\nabla\partial_t\xi^1\|^2+K(\Delta t)^4.$$

Hence we get

$$I_6+I_9 \le 3\epsilon\|\nabla\partial_t\xi^1\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4).$$

By (4.7), the estimates for I_7, I_{10}, and I_{11} can be obtained as follows:

$$\begin{aligned} I_7 &\le \epsilon\|\nabla\partial_t\xi^1\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4), \\ I_{10} &\le \epsilon\|\nabla\partial_t\xi^1\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4), \\ I_{11} &\le \epsilon\|\partial_t\xi^1\|^2+K(\|\xi^1\|^2+h^{2\mu}+(\Delta t)^4). \end{aligned}$$

Now we estimates for I_8 and I_{13} as follows:

$$\begin{aligned} I_8 &\le \epsilon\|\nabla\partial_t\xi^1\|^2+K(\Delta t)^4, \\ I_{13} &\le \epsilon\|\nabla\partial_t\xi^1\|^2+K(\Delta t)^4. \end{aligned}$$

By the Taylor expansion, there exist $t_\theta^1 \in (t^{\frac{1}{2}}, t^1), t_\theta^0 \in (t^0, t^{\frac{1}{2}}), \check{\boldsymbol{x}}_{\theta i} \in (\check{\boldsymbol{x}}, \boldsymbol{x})$, and $\hat{\boldsymbol{x}}_{\theta i} \in (\hat{\boldsymbol{x}}, \boldsymbol{x}),\ 1 \le i \le 3$, satisfying

$$
\begin{aligned}
&\psi(\boldsymbol{x})\frac{\partial u(t^{\frac{1}{2}})}{\partial \boldsymbol{\nu}} - c(\boldsymbol{x})\frac{\check{u}^1 - \hat{u}^0}{\Delta t} \\
=&c(\boldsymbol{x})u_t(t^{\frac{1}{2}}) + \boldsymbol{d}(\boldsymbol{x}) \cdot \nabla u(t^{\frac{1}{2}}) \\
&- c(\boldsymbol{x})\frac{1}{\Delta t}\Big[(\check{u}(t^{\frac{1}{2}}) + \frac{\Delta t}{2}\check{u}_t(t^{\frac{1}{2}}) + \frac{(\Delta t)^2}{8}\check{u}_{tt}(t^{\frac{1}{2}}) + \frac{(\Delta t)^3}{48}\check{u}_{ttt}(t_\theta^1)) \\
&\qquad - (\hat{u}(t^{\frac{1}{2}}) - \frac{\Delta t}{2}\hat{u}_t(t^{\frac{1}{2}}) + \frac{(\Delta t)^2}{8}\hat{u}_{tt}(t^{\frac{1}{2}}) - \frac{(\Delta t)^3}{48}\hat{u}_{ttt}(t_\theta^0))\Big] \\
=&c(\boldsymbol{x})u_t(t^{\frac{1}{2}}) + \boldsymbol{d}(\boldsymbol{x}) \cdot \nabla u(t^{\frac{1}{2}}) \\
&- c(\boldsymbol{x})\frac{1}{\Delta t}\Big[(u(t^{\frac{1}{2}}) + \frac{1}{2}\Delta t\tilde{\boldsymbol{d}} \cdot \nabla u(t^{\frac{1}{2}}) + \frac{1}{8}(\tilde{\boldsymbol{d}}\Delta t)^2 \cdot \nabla^2 u(t^{\frac{1}{2}}) \\
&\quad + \frac{1}{48}(\tilde{\boldsymbol{d}}\Delta t)^3 \cdot \nabla^3 u(\check{\boldsymbol{x}}_{\theta 1}, t^{\frac{1}{2}})) \\
&\quad + \frac{\Delta t}{2}(u_t(t^{\frac{1}{2}}) + \frac{1}{2}\tilde{\boldsymbol{d}}\Delta t \cdot \nabla u_t(t^{\frac{1}{2}}) + \frac{1}{8}(\tilde{\boldsymbol{d}}\Delta t)^2 \cdot \nabla^2 u_t(\check{\boldsymbol{x}}_{\theta 2}, t^{\frac{1}{2}})) \\
&\quad + \frac{(\Delta t)^2}{8}(u_{tt}(t^{\frac{1}{2}}) + \frac{1}{2}\tilde{\boldsymbol{d}}\Delta t \cdot \nabla u_{tt}(\check{\boldsymbol{x}}_{\theta 3}, t^{\frac{1}{2}})) + \frac{1}{48}(\Delta t)^3\check{u}_{ttt}(t_\theta^1) \\
&\quad - (u(t^{\frac{1}{2}}) - \frac{1}{2}\tilde{\boldsymbol{d}}\Delta t \cdot \nabla u(t^{\frac{1}{2}}) + \frac{1}{8}(\tilde{\boldsymbol{d}}\Delta t)^2 \cdot \nabla^2 u(t^{\frac{1}{2}}) \\
&\quad - \frac{1}{48}(\tilde{\boldsymbol{d}}\Delta t)^3 \cdot \nabla^3 u(\hat{\boldsymbol{x}}_{\theta 1}, t^{\frac{1}{2}})) \\
&\quad + \frac{\Delta t}{2}(u_t(t^{\frac{1}{2}}) - \frac{1}{2}\tilde{\boldsymbol{d}}\Delta t \cdot \nabla u_t(t^{\frac{1}{2}}) + \frac{1}{8}(\tilde{\boldsymbol{d}}\Delta t)^2 \cdot \nabla^2 u_t(\hat{\boldsymbol{x}}_{\theta 2}, t^{\frac{1}{2}})) \\
&\quad - \frac{(\Delta t)^2}{8}(u_{tt}(t^{\frac{1}{2}}) - \frac{\Delta t}{2}\tilde{\boldsymbol{d}} \cdot \nabla u_{tt}(\hat{\boldsymbol{x}}_{\theta 3}, t^{\frac{1}{2}})) + \frac{1}{48}(\Delta t)^3\hat{u}_{ttt}(t_\theta^0)\Big] \\
=&-c(\boldsymbol{x})(\Delta t)^2\Big[\frac{1}{48}\tilde{\boldsymbol{d}}^3 \cdot \nabla^3 u(\check{\boldsymbol{x}}_{\theta 1}, t^{\frac{1}{2}}) + \frac{1}{16}\tilde{\boldsymbol{d}}^2 \cdot \nabla^2 u_t(\check{\boldsymbol{x}}_{\theta 2}, t^{\frac{1}{2}}) \\
&\qquad + \frac{1}{16}\tilde{\boldsymbol{d}} \cdot \nabla u_{tt}(\check{\boldsymbol{x}}_{\theta 3}, t^{\frac{1}{2}}) + \frac{1}{48}\check{u}_{ttt}(t_\theta^1) \\
&\qquad + \frac{1}{48}\tilde{\boldsymbol{d}}^3 \cdot \nabla^3 u(\hat{\boldsymbol{x}}_{\theta 1}, t^{\frac{1}{2}}) + \frac{1}{16}\tilde{\boldsymbol{d}}^2 \cdot \nabla^2 u_t(\hat{\boldsymbol{x}}_{\theta 2}, t^{\frac{1}{2}}) \\
&\qquad + \frac{1}{16}\tilde{\boldsymbol{d}} \cdot \nabla u_{tt}(\hat{\boldsymbol{x}}_{\theta 3}, t^{\frac{1}{2}}) + \frac{1}{48}\hat{u}_{ttt}(t_\theta^0)\Big]
\end{aligned}
$$

where $\boldsymbol{d}^j \cdot (\nabla^j u^{n+1}) = \sum_{l=0}^{j} \binom{j}{l} d_1^{j-l} d_2^l \frac{\partial^j u^{n+1}}{\partial x_1^{j-l} \partial x_2^l}$ for $j = 2$ and $j = 3$ when $m = 2$ and we use similar notations when $m = 3$. From the assumptions of u, u_t, u_{tt}, and u_{ttt}, we get

$$|I_{12}| \le K(\Delta t)^4 + \epsilon\|\partial_t \xi^1\|^2.$$

Now using the estimates for $I_1 \sim I_{13}$, we obtain from (4.6)

$$\begin{aligned} &c_*\|\partial_t\xi^1\|^2 + b_*\|\nabla\partial_t\xi^1\|^2 + \frac{b_*}{2\Delta t}\|\nabla\xi^1\|^2 \\ &\leq 7\epsilon\|\partial_t\xi^1\|^2 + 9\epsilon\|\nabla\partial_t\xi^1\|^2 + K(\|\xi^1\|^2 + \|\nabla\xi^1\|^2 + h^{2\mu} + (\Delta t)^4), \end{aligned}$$

which yields that for sufficiently small ϵ

$$\begin{aligned} &c_*\Delta t\|\partial_t\xi^1\|^2 + b_*\Delta t\|\nabla\partial_t\xi^1\|^2 + b_*\|\nabla\xi^1\|^2 \\ &\leq K\Delta t(\|\xi^1\|^2 + \|\nabla\xi^1\|^2 + h^{2\mu} + (\Delta t)^4). \end{aligned}$$

By Poincare's inequality, we have $\|\xi^1\|^2 + \|\nabla\xi^1\|^2 \leq K\|\nabla\xi^1\|^2$. If we choose Δt sufficiently small so that $b_* - K\Delta t > 0$, then we get

$$c_*\Delta t\|\partial_t\xi^1\|^2 + b_*\Delta t\|\nabla\partial_t\xi^1\|^2 + \alpha\|\nabla\xi^1\|^2 \leq K\Delta t(h^{2\mu} + (\Delta t)^4)$$

and so we have

$$\|\nabla\xi^1\|^2 + \Delta t(\|\partial_t\xi^1\|^2 + \|\nabla\partial_t\xi^1\|^2) + \leq K\Delta t[h^{2\mu} + (\Delta t)^4], \tag{4.8}$$

which completes the proof. □

Theorem 4.2. *In addition to the assumptions of Lemma 4.1, if* $u \in L^\infty(H^3(\Omega))$, $\mu \geq 1 + \frac{m}{2}$, *and* $\Delta t = O(h)$, *then*

$$\max_{0\leq n\leq N}\Big[\|u^n - u_h^n\| + h\|\nabla(u^n - u_h^n)\|\Big] \leq K(h^\mu + (\Delta t)^2),$$

where $\mu = \min(r+1, s)$.

Proof. To establish this theorem, we prove the following statement by mathematical induction: There exist $0 < \tilde{h} < 1$ and $0 < \tilde{\Delta t} < 1$ such that

$$\|\nabla\xi^n\|^2 + \Delta t(\|\partial_t\xi^n\|^2 + \|\nabla\partial_t\xi^n\|^2) \leq K(h^{2\mu} + (\Delta t)^4) \tag{4.9}$$

for any $0 < h < \tilde{h}$, $0 < \Delta t < \tilde{\Delta t}$ and $n = 0, 1, \ldots, N$. For our convenience, we abuse the notations such as $Eu_h^0 = 0$ and $\xi^{-1} = 0$. Since $\xi^0 = 0$, (4.9) trivially holds for $n = 0$. And by Theorem 4.1, (4.9) holds for $n = 1$. Now we assume that (4.9) holds with $n \leq l-1$. Notice that $\|\xi^n\|_\infty \leq \tilde{K}$, $0 \leq n \leq l-1$.

Subtracting (4.2) from (4.4) with $\chi = \partial_t \xi^n$ for $2 \le n \le l$, we get

$$
\begin{aligned}
&\Big(c(\boldsymbol{x})\partial_t\xi^n, \partial_t\xi^n\Big) + A(u_h^{n-\frac{1}{2}} \ : \ \xi^{n-\frac{1}{2}}, \partial_t\xi^n) + B(u_h^{n-\frac{1}{2}} \ : \ \partial_t\xi^n, \partial_t\xi^n\Big) \\
=&\Big(c(\boldsymbol{x})\frac{\xi^n - \check{\xi}^n}{\Delta t}, \partial_t\xi^n\Big) + \Big(c(\boldsymbol{x})\frac{\hat{\xi}^{n-1} - \xi^{n-1}}{\Delta t}, \partial_t\xi^n\Big) \\
&- \Big(c(\boldsymbol{x})\frac{\eta^n - \check{\eta}^n}{\Delta t}, \partial_t\xi^n\Big) + \Big(c(\boldsymbol{x})\frac{\eta^n - \eta^{n-1}}{\Delta t}, \partial_t\xi^n\Big) \\
&- \Big(c(\boldsymbol{x})\frac{\hat{\eta}^{n-1} - \eta^{n-1}}{\Delta t}, \partial_t\xi^n\Big) + A(u_h^{n-\frac{1}{2}} \ : \ \eta^{n-\frac{1}{2}}, \partial_t\xi^n) \\
&+ B(u_h^{n-\frac{1}{2}} \ : \ \frac{\eta^n - \eta^{n-1}}{\Delta t}, \partial_t\xi^n) \\
&+ [A(u(t^{n-\frac{1}{2}}) \ : \ u^{n-\frac{1}{2}}, \partial_t\xi^n) - A(u_h^{n-\frac{1}{2}} \ : \ u^{n-\frac{1}{2}}, \partial_t\xi^n)] \\
&+ [B(u(t^{n-\frac{1}{2}})) \ : \ \frac{u^n - u^{n-1}}{\Delta t}, \partial_t\xi^n) - B(u_h^{n-\frac{1}{2}} \ : \ \frac{u^n - u^{n-1}}{\Delta t}, \partial_t\xi^n)] \\
&+ (f(u_h^{n-\frac{1}{2}}) - f(u(t^{n-\frac{1}{2}})), \partial_t\xi^n) \\
&+ A(u(t^{n-\frac{1}{2}}) : \ u(t^{n-\frac{1}{2}}) - u^{n-\frac{1}{2}}, \partial_t\xi^n) \\
&+ B(u(t^{n-\frac{1}{2}}) : \ u_t(t^{n-\frac{1}{2}}) - \frac{u^n - u^{n-1}}{\Delta t}, \partial_t\xi^n) \\
&+ \Big(\psi(\boldsymbol{x})\frac{\partial u(t^{n-\frac{1}{2}})}{\partial \boldsymbol{\nu}} - c(\boldsymbol{x})\frac{\breve{u}^n - \bar{u}^{n-1}}{\Delta t}, \partial_t\xi^n\Big) \\
=&\Sigma_{i=1}^{13} R_i.
\end{aligned} \tag{4.10}
$$

Now letting three terms of the left-hand side of (4.10) by L_1, L_2 and L_3, respectively, we estimate the lower bounds of L_1, L_2 and L_3 as follows:

$$
\begin{aligned}
L_1 &= (c(\boldsymbol{x})\partial_t\xi^n, \partial_t\xi^n) \ge c_*\|\partial_t\xi^n\|^2, \\
L_2 &= A(u_h^{n-\frac{1}{2}} : \xi^{n-\frac{1}{2}}, \partial_t\xi^n) \\
&\ge \frac{1}{2\Delta t}(\|\sqrt{a(u_h^{n-\frac{1}{2}})}\nabla\xi^n\|^2 - \|\sqrt{a(u_h^{n-\frac{1}{2}})}\nabla\xi^{n-1}\|^2) \\
&= \frac{1}{2\Delta t}(\|\sqrt{a(u_h^{n-\frac{1}{2}})}\nabla\xi^n\|^2 - \|\sqrt{a(u_h^{n-\frac{3}{2}})}\nabla\xi^{n-1}\|^2) \\
&\quad + \frac{1}{2\Delta t}(\|\sqrt{a(u_h^{n-\frac{3}{2}})}\nabla\xi^{n-1}\|^2 - \|\sqrt{a(u_h^{n-\frac{1}{2}})}\nabla\xi^{n-1}\|^2), \\
L_3 &= B(u_h^{n-\frac{1}{2}} : \partial_t\xi^n, \partial_t\xi^n) \ge b_*\|\nabla\partial_t\xi^n\|^2.
\end{aligned}
$$

By applying the lower bounds of $L_1 \sim L_3$ to (4.10), we get

$$\begin{aligned} & c_*\|\partial_t \xi^n\|^2 + b_*\|\nabla\partial_t\xi^n\|^2 \\ & \quad + \frac{1}{2\Delta t}(\|\sqrt{a(u_h^{n-\frac{1}{2}})}\nabla\xi^n\|^2 - \|\sqrt{a(u_h^{n-\frac{3}{2}})}\nabla\xi^{n-1}\|^2) \\ & \leq \frac{1}{2\Delta t}((a(u_h^{n-\frac{3}{2}}) - a(u_h^{n-\frac{1}{2}}))\nabla\xi^{n-1}, \nabla\xi^{n-1}) + \sum_{i=1}^{13} R_i. \end{aligned} \tag{4.11}$$

Notice that

$$\begin{aligned} u_h^n - u_h^{n-2} &= u_h^n - u^n + u^n - u^{n-2} + u^{n-2} - u_h^{n-2} \\ &= \xi^n - \eta^n + 2\Delta t u_t^n + \eta^{n-2} - \xi^{n-2} \\ &= \Delta t[\partial_t\xi^n + \partial_t\xi^{n-1} - \partial_t\eta^n - \partial_t\eta^{n-1} + 2u_t^n], \end{aligned}$$

$$\|\nabla\xi^{n-1}\|_\infty \leq K h^{-m/2}\|\nabla\xi^{n-1}\|,$$

and

$$\|\partial_t\xi^{n-1}\|_\infty \leq K h^{-m/2}\|\partial_t\xi^{n-1}\|.$$

So, by the assumption (A3), the approximation property, the inverse property, Theorem 3.1, $\Delta t = O(h)$, and (4.9), we have

$$\begin{aligned} & \frac{1}{2\Delta t}((a(u_h^{n-\frac{3}{2}}) - a(u_h^{n-\frac{1}{2}}))\nabla\xi^{n-1}, \nabla\xi^{n-1}) \\ \leq & \frac{K}{4\Delta t}(|u_h^{n-2} - u_h^n|\nabla\xi^{n-1}, \nabla\xi^{n-1}) \\ \leq & K(\|\partial_t\xi^n\| + \|\partial_t\xi^{n-1}\| + \|\partial_t\eta^n\| + \|\partial_t\eta^{n-1}\| + \|u_t^n\|)\nabla\xi^{n-1}, \nabla\xi^{n-1}) \\ \leq & K\|\nabla\xi^{n-1}\|_\infty\|\partial_t\xi^n\|\|\nabla\xi^{n-1}\| \\ & + K(\|\partial_t\xi^{n-1}\|_\infty + \|\partial_t\eta^n\|_\infty + \|\partial_t\eta^{n-1}\|_\infty + \|u_t^n\|_\infty)\|\nabla\xi^{n-1}\|^2 \\ \leq & K\|\nabla\xi^{n-1}\|^2 + \epsilon\|\partial_t\xi^n\|^2. \end{aligned} \tag{4.12}$$

Hence, by (4.12), (4.11) can be estimated as follows:

$$\begin{aligned} & c_*\|\partial_t \xi^n\|^2 + b_*\|\nabla\partial_t\xi^n\|^2 \\ & \quad + \frac{1}{2\Delta t}(\|\sqrt{a(u_h^{n-\frac{1}{2}})}\nabla\xi^n\|^2 - \|\sqrt{a(u_h^{n-\frac{3}{2}})}\nabla\xi^{n-1}\|^2) \\ & \leq K\|\nabla\xi^{n-1}\|^2 + \epsilon\|\partial_t\xi^n\|^2 + \sum_{i=1}^{13} R_i. \end{aligned} \tag{4.13}$$

By the assumption (A2) and the Taylor expansion, we have the following estimates for R_1 and R_2:

$$R_1 = \Big(c(\boldsymbol{x})\frac{\xi^n - \check{\xi}^n}{\Delta t}, \partial_t \xi^n\Big) \le K\|\nabla \xi^n\|^2 + \epsilon\|\partial_t \xi^n\|^2,$$
$$R_2 = \Big(c(\boldsymbol{x})\frac{\hat{\xi}^{n-1} - \xi^{n-1}}{\Delta t}, \partial_t \xi^n\Big) \le K\|\nabla \xi^{n-1}\|^2 + \epsilon\|\partial_t \xi^n\|^2.$$

Since

$$\begin{aligned}
\frac{\eta^n - \check{\eta}^n}{\Delta t} &= -\frac{1}{2}\nabla \eta(\tilde{\boldsymbol{x}}_1, t^n) \cdot \tilde{\boldsymbol{d}}(\boldsymbol{x}), \\
\frac{\eta^n - \eta^{n-1}}{\Delta t} &= \eta_t(t^n_\theta), \\
\frac{\eta^{n-1} - \hat{\eta}^{n-1}}{\Delta t} &= \frac{1}{2}\nabla \eta(\tilde{\boldsymbol{x}}_2, t^{n-1}) \cdot \tilde{\boldsymbol{d}}(\boldsymbol{x})
\end{aligned}$$

for some $t^n_\theta \in (t^{n-1}, t^n)$, $\tilde{\boldsymbol{x}}_1 \in (\boldsymbol{x}, \check{\boldsymbol{x}})$ and $\tilde{\boldsymbol{x}}_2 \in (\hat{\boldsymbol{x}}, \boldsymbol{x})$, we have the estimates for $R_3 \sim R_5$

$$\begin{aligned}
R_3 &\le K\|\eta^n\|^2 + \epsilon\|\nabla \partial_t \xi^n\|^2 + \epsilon\|\partial_t \xi^n\|^2 \\
&\le Kh^{2\mu} + \epsilon\|\nabla \partial_t \xi^n\|^2 + \epsilon\|\partial_t \xi^n\|^2, \\
R_4 &\le K\|\eta^n_t\|^2 + \epsilon\|\partial_t \xi^n\|^2 \\
&\le Kh^{2\mu} + \epsilon\|\partial_t \xi^n\|^2, \\
R_5 &\le K\|\eta^{n-1}\|^2 + \epsilon\|\nabla \partial_t \xi^n\|^2 + \epsilon\|\partial_t \xi^n\|^2 \\
&\le Kh^{2\mu} + \epsilon\|\nabla \partial_t \xi^n\|^2 + \epsilon\|\partial_t \xi^n\|^2.
\end{aligned}$$

To get the bound for the sum of R_6 and R_7, by using (3.2), we can split it into four terms as follows:

$$\begin{aligned}
R_6 + R_7 &= (a(u_h^{n-\frac{1}{2}})\nabla \eta^{n-\frac{1}{2}}, \nabla \partial_t \xi^n) + (b(u_h^{n-\frac{1}{2}})\frac{\nabla \eta^n - \nabla \eta^{n-1}}{\Delta t}, \nabla \partial_t \xi^n) \\
&= ([a(u_h^{n-\frac{1}{2}}) - a(u(t^{n-\frac{1}{2}})]\nabla \eta^{n-\frac{1}{2}}, \nabla \partial_t \xi^n) \\
&\quad + ([b(u_h^{n-\frac{1}{2}}) - b(u(t^{n-\frac{1}{2}})]\frac{\nabla \eta^n - \nabla \eta^{n-1}}{\Delta t}, \nabla \partial_t \xi^n) \\
&\quad + (a(u(t^{n-\frac{1}{2}}))[\nabla \eta^{n-\frac{1}{2}} - \nabla \eta(t^{n-\frac{1}{2}})], \nabla \partial_t \xi^n) \\
&\quad + (b(u(t^{n-\frac{1}{2}}))[\frac{\nabla \eta^n - \nabla \eta^{n-1}}{\Delta t} - \nabla \eta_t(t^{n-\frac{1}{2}})], \nabla \partial_t \xi^n) \\
&= \sum_{j=1}^{4} T_j.
\end{aligned}$$

Notice that

$$\begin{aligned}
u_h^{n-\frac{1}{2}} - u(t^{n-\frac{1}{2}}) &= \xi^{n-\frac{1}{2}} + \tilde{u}^{n-\frac{1}{2}} - \tilde{u}(t^{n-\frac{1}{2}}) - \eta(t^{n-\frac{1}{2}}),\\
\tilde{u}^{n-\frac{1}{2}} - \tilde{u}(t^{n-\frac{1}{2}}) =& \frac{1}{2}\Big[\tilde{u}(t^{n-\frac{1}{2}}) + \frac{\Delta t}{2}\tilde{u}_t(t^{n-\frac{1}{2}}) + \frac{1}{2}\Big(\frac{\Delta t}{2}\Big)^2 \tilde{u}_{tt}(\tilde{t}_{1,\theta})\Big]\\
&+ \frac{1}{2}\Big[\tilde{u}(t^{n-\frac{1}{2}}) - \frac{\Delta t}{2}\tilde{u}_t(t^{n-\frac{1}{2}}) + \frac{1}{2}\Big(-\frac{\Delta t}{2}\Big)^2 \tilde{u}_{tt}(\tilde{t}_{2,\theta})\Big]\\
&- \tilde{u}(t^{n-\frac{1}{2}})\\
=& O((\Delta t)^2)(\tilde{u}_{tt}(\tilde{t}_{1,\theta}) + \tilde{u}_{tt}(\tilde{t}_{2,\theta})),
\end{aligned}$$

and

$$\begin{aligned}
\nabla\eta^{n-\frac{1}{2}} - \nabla\eta(t^{n-\frac{1}{2}}) &= \frac{1}{2}(\nabla\eta^n + \nabla\eta^{n-1}) - \nabla\eta(t^{n-\frac{1}{2}})\\
&= O((\Delta t)^2)(\nabla\eta_{tt}(t_{1,\theta}) + \nabla\eta_{tt}(t_{2,\theta})),
\end{aligned}$$

for some $t_{1,\theta} \in (t^{n-\frac{1}{2}}, t^n), \tilde{t}_{1,\theta} \in (t^{n-\frac{1}{2}}, t^n)$, $t_{2,\theta} \in (t^{n-1}, t^{n-\frac{1}{2}})$, and $\tilde{t}_{2,\theta} \in (t^{n-1}, t^{n-\frac{1}{2}})$. Hence the estimates for $T_1 \sim T_4$ are obtained as follows:

$$\begin{aligned}
T_1 &\le K\|\nabla\eta^{n-\frac{1}{2}}\|_\infty[(\Delta t)^2 + \|\xi^{n-\frac{1}{2}}\| + \|\eta(t^{n-\frac{1}{2}})\|]\|\nabla\partial_t\xi^n\|\\
&\le K[\|\xi^n\|^2 + \|\xi^{n-1}\|^2 + h^{2\mu} + (\Delta t)^4] + \epsilon\|\nabla\partial_t\xi^n\|^2,\\
T_2 &\le K\|\nabla\partial_t\eta^n\|_\infty[(\Delta t)^2 + \|\xi^{n-\frac{1}{2}}\| + \|\eta(t^{n-\frac{1}{2}})\|]\|\nabla\partial_t\xi^n\|\\
&\le K[\|\xi^n\|^2 + \|\xi^{n-1}\|^2 + h^{2\mu} + (\Delta t)^4] + \epsilon\|\nabla\partial_t\xi^n\|^2,\\
T_3 &\le K(\Delta t)^4 + \epsilon\|\nabla\partial_t\xi^n\|^2,\\
T_4 &\le K(\Delta t)^4 + \epsilon\|\nabla\partial_t\xi^n\|^2,
\end{aligned}$$

which implies that

$$R_6 + R_7 \le K[\|\xi^n\|^2 + \|\xi^{n-1}\|^2 + h^{2\mu} + (\Delta t)^4] + \epsilon\|\nabla\partial_t\xi^n\|^2.$$

The estimate for R_8 is given as follows:

$$\begin{aligned}
R_8 &= ([a(u(t^{n-\frac{1}{2}})) - a(u_h^{n-\frac{1}{2}})]\nabla u^{n-\frac{1}{2}}, \nabla\partial_t\xi^n)\\
&\le K[\|\xi^n\|^2 + \|\xi^{n-1}\|^2 + h^{2\mu} + (\Delta t)^4] + \epsilon\|\nabla\partial_t\xi^n\|^2.
\end{aligned}$$

The estimates for R_9 and R_{10} can be obtained as follows:

$$\begin{aligned}
R_9 &= \Big((b(u(t^{n-\frac{1}{2}})) - b(u_h^{n-\frac{1}{2}}))\frac{\nabla u^n - \nabla u^{n-1}}{\Delta t}, \nabla\partial_t\xi^n\Big)\\
&\le K[\|\xi^n\|^2 + \|\xi^{n-1}\|^2 + h^{2\mu} + (\Delta t)^4] + \epsilon\|\nabla\partial_t\xi^n\|^2
\end{aligned}$$

and

$$\begin{aligned}
R_{10} &= (f(u_h^{n-\frac{1}{2}}) - f(u(t^{n-\frac{1}{2}})), \partial_t\xi^n)\\
&\le K[\|\xi^n\|^2 + \|\xi^{n-1}\|^2 + h^{2\mu} + (\Delta t)^4] + \epsilon\|\nabla\partial_t\xi^n\|^2.
\end{aligned}$$

The estimates for $R_{11} \sim R_{13}$ are obtained as follows:

$$R_{11} =(a(u(t^{n-\frac{1}{2}}))[\nabla u(t^{n-\frac{1}{2}}) - \nabla u^{n-\frac{1}{2}}], \nabla \partial_t \xi^n) \leq K(\Delta t)^4 + \epsilon \|\nabla \partial_t \xi^n\|^2,$$

$$R_{12} =\Big(b(u(t^{n-\frac{1}{2}}))\Big[\nabla u_t(t^{n-\frac{1}{2}}) - \frac{\nabla u^n - \nabla u^{n-1}}{\Delta t}\Big], \nabla \partial_t \xi^n\Big) \leq K(\Delta t)^4 + \epsilon \|\nabla \partial_t \xi^n\|^2,$$

$$R_{13} =\Big(\psi(\boldsymbol{x})\frac{\partial u(t^{n-\frac{1}{2}})}{\partial \boldsymbol{\nu}} - c(\boldsymbol{x})\frac{\check{u}^n - \hat{u}^{n-1}}{\Delta t}, \partial_t \xi^n\Big) \leq K(\Delta t)^4 + \epsilon \|\partial_t \xi^n\|^2.$$

Therefore, using the estimates for $R_1 \sim R_{13}$ in (4.13), we get

$$\begin{aligned}
&c_* \|\partial_t \xi^n\|^2 + b_* \|\nabla \partial_t \xi^n\|^2 \\
&+ \frac{1}{2\Delta t}(\|\sqrt{a(u_h^{n-\frac{1}{2}})}\nabla \xi^n\|^2 - \|\sqrt{a(u_h^{n-\frac{3}{2}})}\nabla \xi^{n-1}\|^2) \\
\leq& K\Big[\|\nabla \xi^n\|^2 + \|\nabla \xi^{n-1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4 + h^{2\mu}\Big] \\
&+ \epsilon \|\partial_t \xi^n\|^2 + \epsilon \|\nabla \partial_t \xi^n\|^2.
\end{aligned}$$

Hence, for sufficiently small ϵ, we obtain

$$\begin{aligned}
&\Delta t \|\partial_t \xi^n\|^2 + \Delta t \|\nabla \partial_t \xi^n\|^2 \\
&\quad + (\|\sqrt{a(u_h^{n-\frac{1}{2}})}\nabla \xi^n\|^2 - \|\sqrt{a(u_h^{n-\frac{3}{2}})}\nabla \xi^{n-1}\|^2) \\
&\leq K\Delta t\Big[\|\nabla \xi^n\|^2 + \|\nabla \xi^{n-1}\|^2 + \|\xi^n\|^2 + \|\xi^{n-1}\|^2 + (\Delta t)^4 + h^{2\mu}\Big].
\end{aligned} \tag{4.14}$$

Now we add both sides of (4.14) from $n = 2$ to l to get

$$\begin{aligned}
&\Delta t \sum_{n=2}^{l} [\|\partial_t \xi^n\|^2 + \|\nabla \partial_t \xi^n\|^2] + \|\sqrt{a(u_h^{l-\frac{1}{2}})}\nabla \xi^n\|^2 \\
&\leq K\Delta t \sum_{n=1}^{l} \Big\{\|\xi^n\|^2 + \|\nabla \xi^n\|^2\Big\} + K\Delta t \sum_{n=0}^{l} \Big\{(\Delta t)^4 + h^{2\mu}\Big\} + K\|\nabla \xi^1\|^2.
\end{aligned}$$

So, by Theorem 4.1 and the Poincare's inequality, we have

$$\begin{aligned}
&\|\nabla \xi^l\|^2 + \Delta t\{\|\partial_t \xi^l\|^2 + \|\nabla \partial_t \xi^l\|^2\} \\
\leq& K\Delta t \sum_{n=1}^{l-1} \{\Delta t(\|\partial_t \xi^n\|^2 + \|\nabla \partial_t \xi^n\|^2) + \|\nabla \xi^n\|^2\} + K\{h^{2\mu} + (\Delta t)^4\}
\end{aligned}$$

for sufficiently small Δt. Therefore, by Gronwall's inequality, we have

$$\|\nabla \xi^l\|^2 + \Delta t\{\|\partial_t \xi^l\|^2 + \|\nabla \partial_t \xi^l\|^2\} \leq K[h^{2\mu} + (\Delta t)^4],$$

which completes the proof of the statement (4.9). By the triangle inequality and the Poincare's inequality, we finally have

$$\|u^l - u_h^l\| + h\|\nabla(u^l - u_h^l)\| \leq K(h^\mu + (\Delta t)^2).$$

Thus the result of this theorem hold. □

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