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SIMPLE LOOPS ON 2-BRIDGE SPHERES IN HECKOID ORBIFOLDS FOR THE TRIVIAL KNOT

Donghi Lee and Makoto Sakuma

Abstract. In this paper, we give a necessary and sufficient condition for an essential simple loop on a 2-bridge sphere in an even Heckoid orbifold for the trivial knot to be null-homotopic, peripheral or torsion in the orbifold. We also give a necessary and sufficient condition for two essential simple loops on a 2-bridge sphere in an even Heckoid orbifold for the trivial knot to be homotopic in the orbifold.

1. Introduction

In [8], following Riley's work [15], we introduced the *Heckoid group* $G(r;n)$ *of index* n *for a* 2*-bridge link,* $K(r)$*, of slope* $r \in \mathbb{Q}$, as the orbifold fundamental group of the *Heckoid orbifold* $\boldsymbol{S}(r;n)$ *of index* n *for* $K(r)$. Here n is an integer or a half-integer greater than 1. The Heckoid group and the Heckoid orbifold are said to be *even* or *odd* according to whether n is an integer or a half-integer. When $K(r)$ is the trivial knot and n is an integer greater than 1, the even Heckoid orbifold $\boldsymbol{S}(r;n) \cong \boldsymbol{S}(0;n)$ is as illustrated in Figure 1, and the even Heckoid group $G(r;n) \cong G(0;n)$ is isomorphic to the index 2 subgroup $\langle P, SPS^{-1}\rangle$ of the classical *Hecke group* $\langle P, S\rangle$ introduced in [4] (cf. [9, Remark 2.5]), where

$$P = \begin{pmatrix} 1 & 2\cos\frac{\pi}{2n} \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

In [9, Theorem 2.3], we gave a complete characterization of those essential simple loops on a 2-bridge sphere in an even Heckoid orbifold $\boldsymbol{S}(r;n)$ which are null-homotopic in $\boldsymbol{S}(r;n)$. Furthermore, in a series of papers [10, 11], we gave a necessary and sufficient condition for two essential simple loops on a 2-bridge sphere in $\boldsymbol{S}(r;n)$ to be homotopic in $\boldsymbol{S}(r;n)$, and a necessary and sufficient condition for an essential simple loop on a 2-bridge sphere in $\boldsymbol{S}(r;n)$ to be peripheral or torsion in $\boldsymbol{S}(r;n)$. However these results were obtained for the generic case when r is non-integral and we deferred the results for the special

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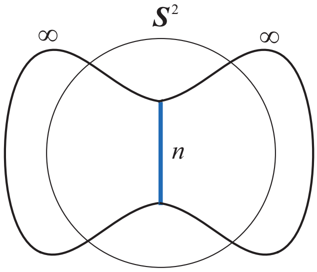

FIGURE 1. The even Heckoid orbifold $\boldsymbol{S}(0;n)$ of index n for the trivial knot

case when r is integral (cf. [9, Remark 2.5]). The purpose of this note is to obtain similar results for the remaining special case when r is integral, that is, $K(r)$ is a trivial knot.

This paper is organized as follows. In Section 2, we describe our main result. In Section 3, we recall the upper presentation of an even Heckoid group, over which we introduce van Kampen diagrams and annular diagrams. Finally, Section 4 is devoted to the proof of Theorem 2.3.

2. Main result

We quickly recall notation and basic facts introduced in [8]. The *Conway sphere* $\boldsymbol{S}$ is the 4-times punctured sphere which is obtained as the quotient of $\mathbb{R}^2 - \mathbb{Z}^2$ by the group generated by the π-rotations around the points in $\mathbb{Z}^2$. For each $s \in \hat{\mathbb{Q}} := \mathbb{Q} \cup \{\infty\}$, let α_s be the simple loop in $\boldsymbol{S}$ obtained as the projection of a line in $\mathbb{R}^2 - \mathbb{Z}^2$ of slope s. We call s the *slope* of the simple loop α_s.

For each $r \in \hat{\mathbb{Q}}$, the *2-bridge link $K(r)$ of slope r* is the sum of the rational tangle $(B^3, t(\infty))$ of slope ∞ and the rational tangle $(B^3, t(r))$ of slope r. Recall that $\partial(B^3 - t(\infty))$ and $\partial(B^3 - t(r))$ are identified with $\boldsymbol{S}$ so that α_∞ and α_r bound disks in $B^3 - t(\infty)$ and $B^3 - t(r)$, respectively. By van-Kampen's theorem, the link group $G(K(r)) = \pi_1(S^3 - K(r))$ is obtained as follows:

$$G(K(r)) = \pi_1(S^3 - K(r)) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r \rangle\rangle.$$

On the other hand, if r is a rational number and $n \geq 2$ is an integer, then the even Heckoid orbifold $\boldsymbol{S}(r;n)$ contains a Conway sphere $\boldsymbol{S}$, and the even Hekoid group $G(r;n)$, which is defined as the orbifold fundamental group of $\boldsymbol{S}(r;n)$, has the following description as the quotient of the fundamental group of the Conway sphere $\boldsymbol{S}$ (see [8, p. 242]):

$$G(r;n) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r^n \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r^n \rangle\rangle.$$

We are interested in the following naturally arising question.

Question 2.1. For r a rational number and n an integer greater than 1, consider the even Heckoid orbifold $\boldsymbol{S}(r;n)$ for the 2-bridge link $K(r)$.

(1) Which essential simple loop α_s on $\boldsymbol{S}$ is null-homotopic in $\boldsymbol{S}(r;n)$?
(2) For two distinct essential simple loops α_s and $\alpha_{s'}$ on $\boldsymbol{S}$, when are they homotopic in $\boldsymbol{S}(r;n)$?
(3) Which essential simple loop α_s on $\boldsymbol{S}$ is peripheral or torsion in $\boldsymbol{S}(r;n)$?

This question originated from Minsky's question [3, Question 5.4], and was completely solved in the series of papers [8, 9, 10, 11] for the generic case when r is non-integral, that is, $K(r)$ is not a trivial knot. See [7] for an overview of these works.

We note that (1) a loop in the orbifold $\boldsymbol{S}(r;n)$ is *null-homotopic* in $\boldsymbol{S}(r;n)$ if and only if it determines the trivial conjugacy class of the Heckoid group $G(r;n)$, and (2) two loops in $\boldsymbol{S}(r;n)$ are *homotopic* in $\boldsymbol{S}(r;n)$ if and only if they determine the same conjugacy class in $G(r;n)$ (see [1, 2] for the concept of homotopy in orbifolds). We say that a loop in $\boldsymbol{S}(r;n)$ is *peripheral* if and only if it is homotopic to a loop in the paring annulus naturally associated with $\boldsymbol{S}(r;n)$ (see [8, Section 6]), i.e., it represents the conjugacy class of a power of a meridian of $G(r;n)$. We also say that a loop in $\boldsymbol{S}(r;n)$ is *torsion* if it represents the conjugacy class of a non-trivial torsion element of $G(r;n)$. If we identify $G(r;n)$ with a Kleinian group generated by two parabolic transformations (see [8, Theorem 2.2]), then a loop $\boldsymbol{S}(r;n)$ is peripheral or torsion if and only if it corresponds to a parabolic transformation or a non-trivial elliptic transformation accordingly. Thus Question 2.1 can be interpreted as a question on the even Heckoid group $G(r;n)$.

Let $\mathcal{D}$ be the *Farey tessellation* of the upper half plane $\mathbb{H}^2$. Then $\hat{\mathbb{Q}}$ is identified with the set of the ideal vertices of $\mathcal{D}$. Let Γ_∞ be the group of automorphisms of $\mathcal{D}$ generated by reflections in the edges of $\mathcal{D}$ with an endpoint ∞. For r a rational number and n an integer or a half-integer greater than 1, let $C_r(2n)$ be the group of automorphisms of $\mathcal{D}$ generated by the parabolic transformation, centered on the vertex r, by $2n$ units in the clockwise direction, and let $\Gamma(r;n)$ be the group generated by Γ_∞ and $C_r(2n)$. The answer to Question 2.1 obtained in [8, 9, 10, 11], for the general case when r is non-integral, is given in terms of the action of $\Gamma(r;n)$ on $\partial\mathbb{H}^2 = \hat{\mathbb{R}}$. The answer to the remaining case when r is an integer is also given in a similar way.

Observe that, when $n \geq 2$ is an integer, the group $\Gamma(0;n)$ is the free product of three cyclic groups of order 2 generated by the reflections in the Farey edges $\langle\infty, 0\rangle$, $\langle\infty, 1\rangle$ and $\langle 0, 1/n\rangle$ (see Figure 2). In fact, the region, R, of $\mathbb{H}^2$ bounded by these three Farey edges is a fundamental domain for the action of $\Gamma(0;n)$ on $\mathbb{H}^2$. Note that the intersection of the closure of R with $\partial\mathbb{H}^2$ is the disjoint union of the discrete set $\{\infty, 0\}$ and the closed interval $I(0;n) := [1/n, 1]$. The following two theorems give a complete answer to Question 2.1 for the remaining special case.

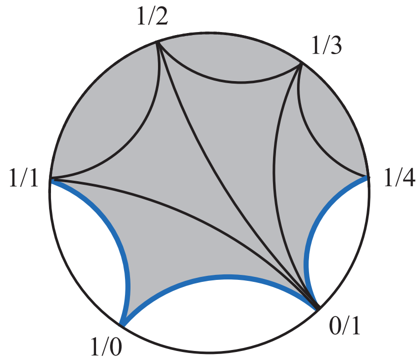

FIGURE 2. The fundamental domain of $\Gamma(0;n)$ in the Farey tessellation (the shaded domain) for $n = 4$

Theorem 2.2. *Suppose that n is an integer greater than 1. Then for any $s \in \hat{\mathbb{Q}}$, there is a unique rational number $s_0 \in I(0;n) \cup \{\infty, 0\}$ such that s is contained in the $\Gamma(r;n)$-orbit of s_0. Moreover the conjugacy classes α_s and α_{s_0} in $G(0;n)$ are equal. In particular, (i) if $s_0 = \infty$, then α_s is the trivial conjugacy class in $G(0;n)$, and (ii) if $s_0 = 0$, then α_s is torsion in $G(0;n)$.*

Theorem 2.3. *Suppose that n is an integer greater than 1. Then the following hold.*

(1) *Any simple loop in $\{\alpha_s \,|\, s \in I(0;n) \cup \{0\}\}$ does not represent the trivial element of $G(0;n)$.*
(2) *The simple loops $\{\alpha_s \,|\, s \in I(0;n)\}$ represent mutually distinct conjugacy classes in $G(0;n)$.*
(3) *There is no rational number $s \in I(0;n)$ for which α_s is peripheral in $G(0;n)$.*
(4) *There is no rational number $s \in I(0;n)$ for which α_s is torsion in $G(0;n)$.*

The proof of Theorem 2.2 is essentially the same as that of [8, Theorem 2.2]. In fact, the first assertion is proved as in [6, Lemma 7.1] by using the fact that R is a fundamental domain for the action of $\Gamma(r;n)$ on $\mathbb{H}^2$. The second assertion is nothing other than [8, Theorem 2.4]. The last assertion follows immediately from the second assertion.

We shall prove Theorem 2.3 with a classical geometric method in combinatorial group theory such as using van Kampen diagrams and annular diagrams over two-generator and one-relator presentations, so-called the upper presentations, of even Heckoid groups.

3. Preliminaries

3.1. Upper presentations of even Heckoid groups

We introduce the upper presentation of an even Heckoid group $G(r;n)$, where r is a rational number and $n \geq 2$ is an integer. Recall that

$$G(r;n) \cong \pi_1(\boldsymbol{S})/\langle\langle \alpha_\infty, \alpha_r^n \rangle\rangle \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r^n \rangle\rangle.$$

Let $\{a,b\}$ be the standard meridian generator pair of $\pi_1(B^3 - t(\infty), x_0)$ as described in [6, Section 3] (see also [5, Section 5]). Then $\pi_1(B^3 - t(\infty))$ is identified with the free group $F(a,b)$. Obtain a word $u_r \in F(a,b) \cong \pi_1(B^3 - t(\infty))$ which is represented by the simple loop α_r. It then follows that

$$G(r;n) \cong \pi_1(B^3 - t(\infty))/\langle\langle \alpha_r^n \rangle\rangle \cong \langle a, b \,|\, u_r^n \rangle.$$

This two-generator and one-relator presentation is called the *upper presentation* of the even Heckoid group $G(r;n)$. It is known by [14, Proposition 1] that there is a nice formula to find u_r as follows. (For a geometric description, see [5, Section 5].)

Lemma 3.1. *Let p and q be relatively prime integers such that $p \geq 1$. For $1 \leq i \leq p-1$, let*

$$\epsilon_i = (-1)^{\lfloor iq/p \rfloor},$$

where $\lfloor x \rfloor$ is the greatest integer not exceeding x.

(1) *If p is odd, then*
$$u_{q/p} = a\hat{u}_{q/p} b^{(-1)^q} \hat{u}_{q/p}^{-1},$$
where $\hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \cdots b^{\epsilon_{p-2}} a^{\epsilon_{p-1}}$.

(2) *If p is even, then*
$$u_{q/p} = a\hat{u}_{q/p} a^{-1} \hat{u}_{q/p}^{-1},$$
where $\hat{u}_{q/p} = b^{\epsilon_1} a^{\epsilon_2} \cdots a^{\epsilon_{p-2}} b^{\epsilon_{p-1}}$.

Remark 3.2. We have $u_0 = ab$. Thus if r is an integer, then

$$G(r;n) \cong G(0;n) \cong \langle a, b \,|\, (ab)^n \rangle.$$

Now we define the cyclic sequence $CS(r)$, which is read from u_r defined in Lemma 3.1, and review an important property of this sequence from [6]. To this end we fix some definitions and notation. Let X be a set. By a *word* in X, we mean a finite sequence $x_1^{\epsilon_1} x_2^{\epsilon_2} \cdots x_n^{\epsilon_n}$ where $x_i \in X$ and $\epsilon_i = \pm 1$. Here we call $x_i^{\epsilon_i}$ the *i-th letter* of the word. For two words u, v in X, by $u \equiv v$ we denote the *visual equality* of u and v, meaning that if $u = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$ and $v = y_1^{\delta_1} \cdots y_m^{\delta_m}$ ($x_i, y_j \in X$; $\epsilon_i, \delta_j = \pm 1$), then $n = m$ and $x_i = y_i$ and $\epsilon_i = \delta_i$ for each $i = 1, \ldots, n$. For example, two words $x_1 x_2 x_2^{-1} x_3$ and $x_1 x_3$ ($x_i \in X$) are *not* visually equal, though $x_1 x_2 x_2^{-1} x_3$ and $x_1 x_3$ are equal as elements of the free group with basis X. The length of a word v is denoted by $|v|$. A word v in X is said to be *reduced* if v does not contain xx^{-1} or $x^{-1}x$ for any $x \in X$. A word is said to be *cyclically reduced* if all its cyclic permutations are reduced.

A *cyclic word* is defined to be the set of all cyclic permutations of a cyclically reduced word. By (v) we denote the cyclic word associated with a cyclically reduced word v. Also by $(u) \equiv (v)$ we mean the *visual equality* of two cyclic words (u) and (v). In fact, $(u) \equiv (v)$ if and only if v is visually a cyclic shift of u.

Definition 3.3. (1) Let (v) be a cyclic word in $\{a, b\}$. Decompose (v) into

$$(v) \equiv (v_1 v_2 \cdots v_t),$$

where all letters in v_i have positive (resp., negative) exponents, and all letters in v_{i+1} have negative (resp., positive) exponents (taking subindices modulo t). Then the *cyclic* sequence of positive integers $CS(v) := ((|v_1|, |v_2|, \ldots, |v_t|))$ is called the *cyclic S-sequence of* (v). Here the double parentheses denote that the sequence is considered modulo cyclic permutations.

(2) A reduced word v in $\{a, b\}$ is said to be *alternating* if $a^{\pm 1}$ and $b^{\pm 1}$ appear in v alternately, i.e., neither $a^{\pm 2}$ nor $b^{\pm 2}$ appears in v. A cyclic word (v) is said to be *alternating* if all cyclic permutations of v are alternating. In the latter case, we also say that v is *cyclically alternating*.

Definition 3.4. For a rational number s with $0 < s \leq 1$, let u_s be defined as in Lemma 3.1. Then the symbol $CS(s)$ denotes the cyclic S-sequence $CS(u_s)$ of (u_s), which is called the *cyclic S-sequence of slope s*).

We recall the following basic property of $CS(s)$.

Lemma 3.5 ([6, Proposition 4.3])**.** *Suppose that s is a rational number with $0 < s \leq 1$, and write s as a continued fraction:*

$$s = [m_1, m_2, \ldots, m_k] := \cfrac{1}{m_1 + \cfrac{1}{m_2 + \ddots + \cfrac{1}{m_k}}},$$

where $k \geq 1$, $(m_1, \ldots, m_k) \in (\mathbb{Z}_+)^k$ and $m_k \geq 2$ unless $k = 1$. Then the following hold.

(1) *Suppose $k = 1$, i.e., $s = 1/m_1$. Then $CS(s) = ((m_1, m_1))$.*
(2) *Suppose $k \geq 2$. Then each term of $CS(s)$ is either m_1 or $m_1 + 1$.*

Corollary 3.6. *Suppose that n is an integer greater than 1. If s is a rational number with $1/n \leq s \leq 1$, then every term of $CS(s)$ less than or equal to n.*

Proof. If $s = 1/n$, then $CS(s) = ((n, n))$ by Lemma 3.5(1), and hence the assertion clearly holds. So let $1/n < s \leq 1$. If $s = [m_1, \ldots, m_k]$ is a continued fraction as in the statement of Lemma 3.5, then $m_1 \leq n - 1$. Hence by Lemma 3.5, the assertion holds. □

3.2. Van Kampen diagrams and annular diagrams

Let us begin with necessary definitions and notation following [12]. A *map* M is a finite 2-dimensional cell complex embedded in $\mathbb{R}^2$. To be precise, M is a finite collection of vertices (0-cells), edges (1-cells), and faces (2-cells) in $\mathbb{R}^2$ satisfying the following conditions.

(i) A vertex is a point in $\mathbb{R}^2$.

(ii) An edge e is homeomorphic to an open interval such that $\bar{e} = e \cup \{a\} \cup \{b\}$, where a and b are vertices of M which are possibly identical.

(iii) For each face D of M, there is a continuous map f from the 2-ball B^2 to $\mathbb{R}^2$ such that

  (a) the restriction of f to the interior of B^2 is a homeomorphism onto D, and

  (b) the image of ∂B^2 is equal to $\cup_{i=1}^{t} \bar{e}_i$ for some set $\{e_1, \dots, e_t\}$ of edges of M.

The underlying space of M, i.e., the union of the cells in M, is also denoted by the same symbol M. The boundary (frontier), ∂M, of M in $\mathbb{R}^2$ is regarded as a 1-dimensional subcomplex of M. An edge may be traversed in either of two directions. If v is a vertex of a map M, $d_M(v)$, the *degree of* v, denotes the number of oriented edges in M having v as initial vertex. A vertex v of M is called an *interior vertex* if $v \notin \partial M$, and an edge e of M is called an *interior edge* if $e \not\subset \partial M$.

A *path* in M is a sequence of oriented edges $e_1, \dots, e_t$ such that the initial vertex of e_{i+1} is the terminal vertex of e_i for every $1 \le i \le t-1$. A *cycle* is a closed path, namely a path $e_1, \dots, e_t$ such that the initial vertex of e_1 is the terminal vertex of e_t. If D is a face of M, any cycle of minimal length which includes all the edges of the boundary, ∂D, of D going around once along the boundary of D is called a *boundary cycle* of D. To be precise it is defined as follows. Let $f : B^2 \to D$ be a continuous map satisfying condition (iii) above. We may assume that ∂B^2 has a cellular structure such that the restriction of f to each cell is a homeomorphism. Choose an arbitrary orientation of ∂B^2, and let $\hat{e}_1, \dots, \hat{e}_t$ be the oriented edges of ∂B^2, which are oriented in accordance with the orientation of ∂B^2 and which lie on ∂B^2 in this cyclic order with respect to the orientation of ∂B^2. Let e_i be the orientated edge $f(\hat{e}_i)$ of M. Then the cycle $e_1, \dots, e_t$ is a boundary cycle of D.

Let $F(X)$ be the free group with basis X. A subset R of $F(X)$ is said to be *symmetrized*, if all elements of R are cyclically reduced and, for each $w \in R$, all cyclic permutations of w and w^{-1} also belong to R.

Definition 3.7. Let R be a symmetrized subset of $F(X)$. An *R-diagram* is a pair (M, ϕ) of a map M and a function ϕ assigning to each oriented edge e of M, as a *label*, a reduced word $\phi(e)$ in X such that the following hold.

(i) If e is an oriented edge of M and e^{-1} is the oppositely oriented edge, then $\phi(e^{-1}) = \phi(e)^{-1}$.

(ii) For any boundary cycle δ of any face of M, $\phi(\delta)$ is a cyclically reduced word representing an element of R. (If $\alpha = e_1, \dots, e_t$ is a path in M, we define $\phi(\alpha) \equiv \phi(e_1) \cdots \phi(e_t)$.)

We denote an R-diagram (M, ϕ) simply by M.

Definition 3.8. Let a group G be presented by $G = \langle X \mid R \rangle$ with R being symmetrized.

(1) A connected and simply connected R-diagram is called a *van Kampen diagram* over $G = \langle X \mid R \rangle$.

(2) An R-diagram M is called an *annular diagram* over $G = \langle X \mid R \rangle$, if $\mathbb{R}^2 - M$ has exactly two connected components.

Suppose that R is a symmetrized subset of $F(X)$. A nonempty word b is called a *piece* if there exist distinct $w_1, w_2 \in R$ such that $w_1 \equiv bc_1$ and $w_2 \equiv bc_2$. Let D_1 and D_2 be faces (not necessarily distinct) of M with an edge $e \subseteq \partial D_1 \cap \partial D_2$. Let $e\delta_1$ and $\delta_2 e^{-1}$ be boundary cycles of D_1 and D_2, respectively. Let $\phi(\delta_1) = f_1$ and $\phi(\delta_2) = f_2$. An R-diagram M is said to be *reduced* if one never has $f_2 = f_1^{-1}$. It should be noted that if M is reduced then $\phi(e)$ is a piece for every interior edge e of M.

We recall the following lemma which is a well-known classical result in combinatorial group theory (see [12]).

Lemma 3.9 (van Kampen). *Suppose $G = \langle X \mid R \rangle$ with R being symmetrized. Let v be a cyclically reduced word in X. Then $v = 1$ in G if and only if there exists a reduced van Kampen diagram M over $G = \langle X \mid R \rangle$ with a boundary label v.*

Let M be an annular diagram over $G = \langle X \mid R \rangle$, and let K and H be, respectively, the unbounded and bounded components of $\mathbb{R}^2 - M$. We call $\partial K (\subset \partial M)$ the *outer boundary* of M, while $\partial H (\subset \partial M)$ is called the *inner boundary* of M. Clearly, the *boundary* of M, ∂M, is the union of the outer boundary and the inner boundary. A cycle of minimal length which contains all the edges in the outer (inner, resp.) boundary of M going around once along the boundary of K (H, resp.) is an *outer* (*inner,* resp.) *boundary cycle* of M. An *outer* (*inner,* resp.) *boundary label of* M is defined to be a word $\phi(\alpha)$ in X for α an outer (inner, resp.) boundary cycle of M. The annular diagram M is said to be *nontrivial* if it contains a 2-cell.

We recall another well-known classical result in combinatorial group theory.

Lemma 3.10 ([12, Lemmas V.5.1 and V.5.2]). *Suppose $G = \langle X \mid R \rangle$ with R being symmetrized. Let u, v be two cyclically reduced words in X which are not trivial in G and which are not conjugate in $F(X)$. Then u and v represent conjugate elements in G if and only if there exists a reduced nontrivial annular diagram M over $G = \langle X \mid R \rangle$ such that u is an outer boundary label and v^{-1} is an inner boundary label of M.*

4. Proof of Theorem 2.3

4.1. Proof of Theorem 2.3(1)

Suppose on the contrary that there exists a rational number $s \in [1/n, 1] \cup \{0\}$ such that for which α_s is null-homotopic in $\boldsymbol{S}(r; n)$. Then u_s equals the identity in $G(0; n) = \langle a, b \,|\, (ab)^n \rangle$. Since u_0 is a non-trivial torsion element in $G(0; n)$ by [12, Theorem IV.5.2], we may assume $s \in [1/n, 1]$. By Lemma 3.9, there is a reduced connected and simply-connected diagram M over $G(0; n) = \langle a, b \,|\, (ab)^n \rangle$ with $(\phi(\partial M)) = (u_s)$.

Claim. *There is no interior edge in M.*

Proof of Claim. Suppose on the contrary that there are two 2-cells D_1 and D_2 in M such that D_1 and D_2 have a common edge e. Since M is reduced, $\phi(e)$ is a piece for the symmetrized subset R of $F(a, b)$ generated by $\{(ab)^n\}$. But this is a contradiction, since there is no piece for this R. □

Choose an extremal disk, say J, of M. Here, recall that an *extremal disk* of a map M is a submap of M which is topologically a disk and which has a boundary cycle $e_1, \dots, e_t$ such that the edges $e_1, \dots, e_t$ occur in order in some boundary cycle of the whole map M. Then by Claim, J consists of only one 2-cell. This implies that $CS(\phi(\partial J)) = ((2n))$, so that $CS(\phi(\partial M)) = CS(u_s) = CS(s)$ contains a term greater than or equal to $2n$, which is a contradiction to Corollary 3.6. □

Remark 4.1. Theorem 2.3(1) can be also proved by using Newman's Spelling Theorem [13, Theorem 3] (cf. [12, Theorem IV.5.5]), which is a powerful theorem for the word problem for one relator groups with torsion. This implies that if a cyclically reduced word v represents the trivial element in $G(0; n) \cong \langle a, b \,|\, (ab)^n \rangle$, then the cyclic word (v) contains a subword of the cyclic word $((ab)^{\pm n})$ of length greater than $(n-1)/n = 1 - 1/n$ times the length of $(ab)^n$, so (v) contains a subword w of $((ab)^{\pm n})$ such that $|w| > 2n(1 - 1/n) = 2n - 2 \geq n$. Hence if $u_s = 1$ in $G(0; n)$ for some $s \in [1/n, 1]$, then $CS(u_s) = CS(s)$ contains a term bigger than n, which is a contradiction to Corollary 3.6.

4.2. Proof of Theorem 2.3(2)

Suppose on the contrary that there exist two distinct rational numbers s and s' in $[1/n, 1]$ for which the simple loops α_s and $\alpha_{s'}$ are homotopic in $\boldsymbol{S}(0; n)$. Then u_s and $u_{s'}^{\pm 1}$ are conjugate in $G(0; n)$. By Lemma 3.10, there is a reduced nontrivial annular diagram M over $G(0; n) = \langle a, b \,|\, (ab)^n \rangle$ with $(\phi(\alpha)) \equiv (u_s)$ and $(\phi(\beta)) \equiv (u_{s'}^{\pm 1})$, where α and β are, respectively, the outer and inner boundary cycles of M. Let the outer and inner boundaries of M be denoted by σ and τ, respectively.

Claim 1. *σ and τ are simple, i.e., they are homeomorphic to the circle.*

Proof of Claim 1. Suppose on the contrary that σ or τ is not simple. Then there is an extremal disk, say J, of M. As in the proof of Theorem 2.3(1), J consists of only one 2-cell. Then $CS(u_s) = CS(s)$ or $CS(u_{s'}) = CS(s')$ contains a term greater than or equal to $2n$, which is a contradiction to Corollary 3.6. □

Claim 2. *σ and τ do not have a common edge.*

Proof of Claim 2. Suppose on the contrary that σ and τ have a common edge e as in Figure 3(a). Since σ and τ are simple by Claim 1, and since there is no interior edge in M as in the proof of Theorem 2.3(1), there is a vertex $v \in \sigma \cap \tau$ such that $d_M(v) = 3$. But since both (u_s) and $(u_{s'})$ are alternating and since $((ab)^n)$ is alternating, this is a contradiction. □

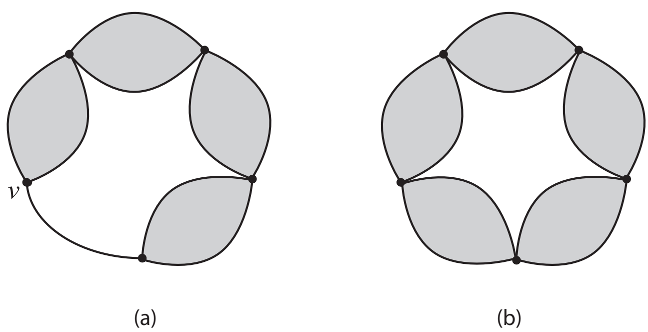

FIGURE 3. (a) For the proof of Claim 2; (b) A possible shape of M

By Claims 1–2 together with the fact that there is no interior edge in M as in the proof of Theorem 2.3(1), we see that Figure 3(b) illustrates the only possible shape of M. In particular, $\sigma \cap \tau$ consists of finitely many vertices, M consists of a single layer, and the number of faces of M is equal to the number of degree 4 vertices of M. Here the number of faces is variable.

Notation 4.2. Suppose that M is a connected annular map as in Figure 3(b). Choose a vertex, say v_0, lying in both the outer and inner boundaries of M, and let α and β be, respectively, the outer and inner boundary cycles of M starting from v_0, where α is read clockwise and β is read counterclockwise. Let $D_1, \ldots, D_t$ be the 2-cells of M such that α goes through their boundaries in this order. By the symbol $\partial D_i^{\pm}$, we denote an oriented edge path contained in ∂D_i such that

$$\alpha = \partial D_1^+ \cdots \partial D_t^+,$$
$$\beta^{-1} = \partial D_1^- \cdots \partial D_t^-.$$

Then every 2-cell D of M satisfies that $\phi(\partial D^+)$ is a subword of the cyclic word $(\phi(\alpha)) = (u_s)$ and that $\phi(\partial D^-)$ is a subword of the cyclic word $(\phi(\beta^{-1})) = (u_{s'}^{\pm 1})$. Since $s, s' \in [1/n, 1]$, every term of both $CS(s)$ and $CS(s')$ is less than or equal to n by Corollary 3.6. Furthermore since $s \neq s'$, at least one of $CS(s)$ and $CS(s')$ has a term less than n. Without loss of generality, we assume that $CS(s)$ has a term less than n. This yields that there is a 2-cell D of M such that $\phi(\partial D^+)$ has length less than n. But then $\phi(\partial D^-)$ has length bigger than n, since $(\phi(\partial D^+)\phi(\partial D^-)^{-1}) = ((ab)^{\pm n})$. This implies that $CS(s')$ contains a term bigger than n, which is a contradiction to Corollary 3.6. □

4.3. Proof of Theorem 2.3(3)

Suppose on the contrary that there exists a rational number s in $[1/n, 1]$ for which the simple loop α_s is peripheral in $\boldsymbol{S}(0; n)$. Then u_s is conjugate to $a^{\pm t}$ or $b^{\pm t}$ in $G(1/p; n)$ for some integer $t \geq 1$. We assume that u_s is conjugate to $a^{\pm t}$ in $G(1/p; n)$. (The case when u_s is conjugate to $b^{\pm t}$ in $G(1/p; n)$ is treated similarly.) By Lemma 3.10, there is a reduced nontrivial annular diagram M over $G(0; n) = \langle a, b \,|\, (ab)^n \rangle$ with $(\phi(\alpha)) \equiv (u_s)$ and $(\phi(\beta)) \equiv (a^{\pm t})$, where α and β are, respectively, the outer and inner boundary cycles of M.

Let the outer and inner boundaries of M be denoted by σ and τ, respectively. As in Claim 1 in the proof of Theorem 2.3(2), σ and τ are simple. However Claim 2 in the proof of Theorem 2.3(2) does not hold, because (u_s) and $((ab)^n)$ are alternating while $(a^{\pm t})$ is not. So σ and τ might have a common edge, and hence M can be shaped as in Figure 3(a) and (b). In either case, every 2-cell D satisfies that $\phi(\partial D^+)$ is a subword of the cyclic word (u_s) and that $\phi(\partial D^-)$ is a subword of the cyclic word $(a^{\pm t})$. Here, the only possibility is that $\phi(\partial D^-)$ has length 1, since $\phi(\partial D^-)$ is also a subword of the cyclic word $((ab)^{\pm n})$. But then $\phi(\partial D^+)$ has length $2n-1$, which implies that $CS(\phi(\alpha)) = CS(s)$ contains a term bigger than or equal to $2n - 1 > n$. This is a contradiction to Corollary 3.6. □

4.4. Proof of Theorem 2.3(4)

Suppose that there exists a rational number s in $[1/n, 1]$ for which the simple loop α_s is torsion in $\boldsymbol{S}(0; n)$. Then $u_s^t = 1$ in $G(0; n) = \langle a, b \,|\, (ab)^n \rangle$ for some integer $t \geq 1$. By Lemma 3.9, there is a reduced connected and simply-connected diagram M over $G(0; n) = \langle a, b \,|\, (ab)^n \rangle$ with $(\phi(\partial M)) = (u_s^t)$. Choose an extremal disk, say J, of M Since there is no interior edge in M as in the proof of Theorem 2.3(1), J consists of only one 2-cell. But then $CS(\phi(\partial M)) = CS(u_s^t)$ contains a term greater than or equal to $2n$, which is a contradiction because every term of $CS(u_s)$, so of $CS(u_s^t)$, is less than or equal to n. □

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Donghi Lee
Department of Mathematics
Pusan National University
San-30 Jangjeon-Dong, Geumjung-Gu, Pusan, 609-735, Republic of Korea
E-mail address: `donghi@pusan.ac.kr`

Makoto Sakuma
Department of Mathematics
Graduate School of Science
Hiroshima University
Higashi-Hiroshima, 739-8526, Japan
E-mail address: `sakuma@math.sci.hiroshima-u.ac.jp`