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FIXED POINT THEOREMS IN S-METRIC SPACES

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Abstract. In this paper, the notion of S-metric spaces will be introduced. We present some fixed point theorems for two maps on complete S-metric spaces and an illustrative example is given for the single-valued case. By using the similar method as in [4], a common fixed point theorem for two single-valued mappings is obtained in S-metric spaces.

1. Introduction

The Banach contraction principle is the most celebrated fixed point theorem and has been generalized in various directions. Fixed point problems for contractive mappings in metric spaces with a partial order have been studied by many authors (see [1]-[2]).

In the present paper, we introduce the notion of S-metric spaces and give some properties of them. Implicit relations on S-metric spaces have been used in many articles (see [3]-[7]). Fixed point theorems for two mappings on complete S-metric spaces will be proved. In addition, we give an illustrative example for the single-valued case.

We begin with the following definition.

Definition 1. [4] Let X be a nonempty set. A function $S : X^3 \to [0, \infty)$ is said to be an S-metric on X, if for each $x, y, z, a \in X$,

(1) $S(x, y, z) \geq 0$,
(2) $S(x, y, z) = 0$ if and only if $x = y = z$,
(3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an *S-metric space.*

Example 1. [4] *We can easily check that the following examples are S-metric spaces.*

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(1) *Let $X = \mathbb{R}^n$ and $||\cdot||$ a norm on X. Then $S(x,y,z) = ||y+z-2x|| + ||y-z||$ is an S-metric on X.*
(2) *Let $X = \mathbb{R}^n$ and $||\cdot||$ a norm on X. Then $S(x,y,z) = ||x-z|| + ||y-z||$ is an S-metric on X.*
(3) *Let X be a nonempty set and d be an ordinary metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ is an S-metric on X.*

Lemma 1.1. [7] *Let (S,X) be an S-metric space. Then, we have $S(x,x,y) = S(y,y,x)$, $x,y \in X$.*

Definition 2. Let (X,S) be an S-metric space. For $r > 0$ and $x \in X$ we define the open ball $B_S(x,r)$ and closed ball $B_S[x,r]$ with center x and radius r as follows, respectively:

$$B_S(x,r) = \{y \in X : S(y,y,x) < r\},$$
$$B_S[x,r] = \{y \in X : S(y,y,x) \le r\}.$$

Example 2. *Let $X = \mathbb{R}$. Denote $S(x,y,z) = |y+z-2x| + |y-z|$ for all $x,y,z \in \mathbb{R}$. Thus*

$$\begin{aligned} B_S(1,2) &= \{y \in \mathbb{R} : S(y,y,1) < 2\} = \{y \in \mathbb{R} : |y-1| < 1\} \\ &= \{y \in \mathbb{R} : 0 < y < 2\} = (0,2). \end{aligned}$$

Definition 3. [6] Let (X,S) be an S-metric space and $A \subset X$.

(1) The set A is said to be an open subset of X, if for every $x \in A$ there exists $r > 0$ such that $B_S(x,r) \subset A$.
(2) The set A is said to be S-bounded if there exists $r > 0$ such that $S(x,x,y) < r$ for all $x,y \in A$.
(3) A sequence $\{x_n\}$ in X converges to x if $S(x_n,x_n,x) \to 0$ as $n \to \infty$, that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$, $S(x_n,x_n,x) < \varepsilon$. In this case, we denote by $\lim_{n\to\infty} x_n = x$ and we say that x is the limit of $\{x_n\}$ in X.
(4) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n,m \ge n_0$.
(5) The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.
(6) Let τ be the set of all $A \subset X$ with $x \in A$ and there exists $r > 0$ such that $B_S(x,r) \subset A$. Then τ is a topology on X (induced by the S-metric S).

Definition 4. Let (X,S) and (X',S') be two S-metric spaces. A function $f : (X,S) \to (X',S')$ is said to be continuous at a point $a \in X$ if for every sequence $\{x_n\}$ in X with $S(x_n,x_n,a) \to 0$, $S'(f(x_n),f(x_n),f(a)) \to 0$. We say that f is continuous on X if f is continuous at every point $a \in X$.

Lemma 1.2. [4] *Let (X,S) be an S-metric space. If $r > 0$ and $x \in X$, then the ball $B_S(x,r)$ is an open subset of X.*

Lemma 1.3. [6] *The limit of $\{x_n\}$ in S-metric space (X, S) is unique.*

Lemma 1.4. [4] *Let (X, S) be an S-metric space. Then the convergent sequence $\{x_n\}$ in X is Cauchy.*

Lemma 1.5. [6] *Let (X, S) be an S- metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

Lemma 1.6. *Let (X, S) be an S- metric space and suppose that $\{x_n\}$ and $\{y_n\}$ are S-convergent to x, y, respectively. Then we have*

$$\limsup_{n\to\infty} S(x_n, z, y_n) \le S(z, z, x) + S(x, x, y).$$

In particular, if $x = y$, then we have $\limsup_{n\to\infty} S(x_n, z, y_n) \le S(z, z, x)$.

Proof. Let $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Then for each $\varepsilon > 0$ there exist $n_1, n_2 \in \mathbb{N}$ such that for all $n \ge n_1$,

$$S(x_n, x_n, x) < \frac{\varepsilon}{2}$$

and for all $n \ge n_2$,

$$S(y_n, y_n, y) < \frac{\varepsilon}{4}.$$

If set $n_0 = \max\{n_1, n_2\}$, then for every $n \ge n_0$ by condition (3) of S-metric, we have

$$\begin{aligned} S(x_n, z, y_n) &\le S(x_n, x_n, x) + S(z, z, x) + S(y_n, y_n, x) \\ &\le S(x_n, x_n, x) + S(z, z, x) + 2S(y_n, y_n, y) + S(x, x, y). \end{aligned}$$

Taking the upper limit as $n \to \infty$ in the above inequality, we obtain the first desired result. The second result is obvious. □

2. Main Results

Let Φ denote the class of all functions $\phi : R^+ \to R^+$ such that ϕ is nondecreasing, continuous and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for all $t > 0$. It is clear that $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$ and hence, we have $\phi(t) < t$ for all $t > 0$.

Theorem 2.1. *Let (X, S) be a complete S-metric space and $A, B : X \to X$ be mappings satisfying the following conditions:*

(1) *$A(X) \subseteq B(X)$ and either $A(X)$ or $B(X)$ is a closed subset of X,*

(2) *The pair (A, B) is weakly compatible,*

(3)

$$\begin{aligned}&S(Ax, Ay, Az)\\ &\leq \phi\left(\max\left\{S(Bx, By, Bz), k_1 S(Bz, Ax, Az), k_2 S(Bz, Ay, Az)\right\}\right)\end{aligned}$$

for all $x, y, z \in X$ *and* $0 < k_1, k_2 < 1$, *where* $\phi \in \Phi$.

Then the maps A *and* B *have a unique common fixed point. If* B *is continuous at the fixed point* p, *then* A *is also continuous at* p.

Proof. Let $x_0 \in X$. Define the sequence $y_n = Ax_n = Bx_{n+1}$, $n = 0, 1, 2, \cdots$ and let $d_{n+1} = S(y_n, y_n, y_{n+1})$. Then we have

$$\begin{aligned}d_{n+1} &= S(y_n, y_n, y_{n+1})\\ &= S(Ax_n, Ax_n, Ax_{n+1})\\ &\leq \phi(\max\{S(Bx_n, Bx_n, Bx_{n+1}), k_1 S(Bx_{n+1}, Ax_n, Ax_{n+1}),\\ &\qquad k_2 S(Bx_{n+1}, Ax_n, Ax_{n+1})\})\\ &= \phi(\max\{S(y_{n-1}, y_{n-1}, y_n), k_1 S(y_n, y_n, y_{n+1}), k_2 S(y_n, y_n, y_{n+1})\})\\ &\leq \phi(\max\{d_n, k_1 d_{n+1}, k_2 d_{n+1}\}).\end{aligned}$$

Thus $d_{n+1} \leq \phi(d_n), n = 1, 2, 3, \cdots$. Hence we have,

$$\begin{aligned}S(y_n, y_n, y_{n+1}) &\leq \phi(S(y_{n-1}, y_{n-1}, y_n))\\ &\leq \phi^2(S(y_{n-2}, y_{n-2}, y_{n-1}))\\ &\vdots\\ &\leq \phi^n(S(y_0, y_0, y_1)).\end{aligned}$$

Hence for every $m > n$ by condition (3) of S-metric, we have

$$\begin{aligned}&S(y_n, y_n, y_m)\\ \leq\ & 2S(y_n, y_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, y_{n+2})\\ \leq\ & 2[S(y_n, y_n, y_{n+1}) + S(y_{n+1}, y_{n+1}, y_{n+2})] + S(y_{n+2}, y_{n+2}, y_{n+3})\\ &\vdots\\ \leq\ & 2\sum_{i=n}^{m-2} S(y_i, y_i, y_{i+1}) + S(y_{m-1}, y_{m-1}, y_m)\\ \leq\ & 2[\phi^n(S(y_0, y_0, y_1)) + \phi^{n+1}(S(y_0, y_0, y_1)) + \cdots + \phi^{m-1}(S(y_0, y_0, y_1))]\\ =\ & 2\sum_{i=n}^{m-1} \phi^i(S(y_0, y_0, y_1)).\end{aligned}$$

Since $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ *for all* $t > 0$, $S(y_n, y_n, y_m) \to 0$ as $n \to \infty$. Therefore, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for each $n, m \geq n_0$

$$S(y_n, y_n, y_m) < \epsilon.$$

This show that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $p \in X$ such that $\lim_{n\to\infty} y_n = p$ and

$$p = \lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_{n+1}.$$

Let $B(X)$ be a closed subset of X. Then there exists $v \in X$ such that $Bv = p$. We prove that $Av = p$. Since,

$$\begin{aligned} & S(Av, Av, Ax_n) \\ \leq\ & \phi(\max\{S(Bv, Bv, Bx_n), k_1 S(Bx_n, Av, Ax_n), k_2 S(Bx_n, Av, Ax_n)\}) \\ =\ & \phi(\max\{S(p, p, y_{n-1}), k_1 S(y_{n-1}, Av, y_n), k_2 S(y_{n-1}, Av, y_n)\}), \end{aligned}$$

taking the upper limit as $n \to \infty$ in the above inequality, by Lemma 1.6 we obtain

$$\begin{aligned} & S(Av, Av, p) \\ \leq\ & \phi(\max\{0, k_1 \limsup_{n\longrightarrow\infty} S(y_{n-1}, Av, y_n), \\ & k_2 \limsup_{n\longrightarrow\infty} S(y_{n-1}, Av, y_n)\}) \\ \leq\ & \phi(\max\{0, k_1 S(Av, Av, p), k_2 S(Av, Av, p)\}) \\ \leq\ & \max\{k_1, k_2\} S(Av, Av, p). \end{aligned}$$

This implies that $1 \leq \max\{k_1, k_2\}$, which is a contradiction. Hence, from $\phi(t) < t$ for all $t > 0$, we have $Av = Bv = p$.

By the weak compatiblility of the pair (A, B), we have $ABv = BAv$, and hence $Ap = Bp$. Next, we have to prove that $Ap = p$. Suppose that $Ap \neq p$. Then

$$\begin{aligned} & S(Ap, Ap, Ax_n) \\ \leq\ & \phi\left(\max\left\{\ S(Bp, Bp, Bx_n), k_1 S(Bx_{x_n}, Ap, Ax_n), k_2 S(Bx_{x_n}, Ap, Ax_n)\ \right\}\right) \\ =\ & \phi\left(\max\left\{\ S(Bp, Bp, y_{n-1}), k_1 S(y_{n-1}, Ap, y_n), k_2 S(y_{n-1}, Ap, y_n)\ \right\}\right). \end{aligned}$$

Taking the upper limit as $n \to \infty$ in the above inequality, we obtain

$$\begin{aligned} & S(Ap, Ap, p) \\ \leq\ & \phi(\max\{S(Ap, Ap, p), k_1 \limsup_{n\longrightarrow\infty} S(y_{n-1}, Ap, y_n), \\ & k_2 \limsup_{n\longrightarrow\infty} S(y_{n-1}, Ap, y_n)\}) \\ \leq\ & \phi(\max\{S(Ap, Ap, p), k_1 S(Ap, Ap, p), k_2 S(Ap, Ap, p)\}) \\ \leq\ & \max\{k_1, k_2\} S(Ap, Ap, p). \end{aligned}$$

Since $\phi(t) < t$ for all $t > 0$, we have $Bp = Ap = p$. Thus p is a common fixed point of A and B.

Suppose $p\,'$ is another common fixed point of A and B. Then, we have

$$\begin{aligned} S(p, p, p\,') &= S(Ap, Ap, Ap\,') \\ &\leq \phi(\max\{S(p, p, p\,'), k_1 S(p\,', p, p\,'), k_2 S(p\,', p, p\,')\}). \end{aligned}$$

If $S(p,p,p\,') \leq \phi(S(p,p,p\,'))$, then $S(p,p,p\,') \leq \phi(S(p,p,p\,')) < S(p,p,p\,')$ which is a contraction. Hence, we have $p = p\,'$. If $S(p,p,p\,') < kS(p\,',p,p\,')$, then

$$\begin{aligned} S(p,p,p\,') &< kS(p\,',p,p\,') \\ &\leq k(2S(p\,',p\,',p\,') + S(p,p,p\,')) = kS(p,p,p\,'), \end{aligned}$$

where $k = \max\{k_1, k_2\}$. This is also a contraction. Hence, we have $p = p\,'$. Thus, p is the unique common fixed point of A and B.

Next, we shall prove the continuity of the mapping in S-metric spaces.

Let $\{z_n\}$ be any sequence in X such that $\{z_n\}$ is convergent to p. Then we have

$$\begin{aligned} & S(Ap, Ap, Az_n) \\ \leq\ & \phi(\max\{S(Bp, Bp, Bz_n), k_1 S(Bz_n, Ap, Az_n), k_2 S(Bz_n, Ap, Az_n)\}). \end{aligned}$$

Taking the upper limit as $n \to \infty$ in the above inequality, from the continuity of B at a point p we get

$$\begin{aligned} & \limsup_{n\longrightarrow\infty} S(p,p,Az_n) \\ = \ & \limsup_{n\longrightarrow\infty} S(Ap,Ap,Az_n) \\ \leq\ & \phi(\max\{ \limsup_{n\longrightarrow\infty} S(Bp,Bp,Bz_n), k_1 \limsup_{n\longrightarrow\infty} S(Bz_n,Ap,Az_n), \\ & k_2 \limsup_{n\longrightarrow\infty} S(Bz_n,Ap,Az_n)\}) \\ \leq\ & \phi(\max\{0, 0+0+k_1 \limsup_{n\longrightarrow\infty} S(p,p,Az_n), k_2 \limsup_{n\longrightarrow\infty} S(p,p,Az_n)\}) \\ \leq\ & \max\{k_1,k_2\} S(p,p,Az_n). \end{aligned}$$

Since

$$\begin{aligned} & k_1 \limsup_{n\longrightarrow\infty} S(Bz_n, Ap, Az_n) \\ \leq\ & k_1 \limsup_{n\longrightarrow\infty} S(Bz_n, Bz_n, Bp) \\ & + k_1 \limsup_{n\longrightarrow\infty} S(Ap, Ap, Bp) + k_1 \limsup_{n\longrightarrow\infty} S(Az_n, Az_n, Bp) \end{aligned}$$

and

$$\begin{aligned} & k_2 \limsup_{n\longrightarrow\infty} S(Bz_n, Ap, Az_n) \\ \leq\ & k_2 \limsup_{n\longrightarrow\infty} S(Bz_n, Bz_n, Bp) \\ & + k_2 \limsup_{n\longrightarrow\infty} S(Ap, Ap, Bp) + k_2 \limsup_{n\longrightarrow\infty} S(Az_n, Az_n, Bp), \end{aligned}$$

we have

$$\limsup_{n\longrightarrow\infty} S(p,p,Az_n) \leq \max\{k_1,k_2\} \limsup_{n\longrightarrow\infty} S(p,p,Az_n).$$

This implies that $\limsup_{n\longrightarrow\infty} S(p,p,Az_n)=0$. Then, we deduce that A is continuous at p. □

If $B=I$ identity map in Theorem 2.1, then we have the following.

Corollary 2.2. *Let (X,S) be a complete S-metric space and $A:X\to X$ be a mapping satisfying the following inequality*

$$S(Ax,Ay,Az)\le \phi(\max\{S(x,y,z),k_1,S(z,Ax,Az),k_2,S(z,Ay,Az)\}),$$

for all $x,y,z\in X$, where $\phi\in\Phi$. Then the mapping A has a unique common fixed point $p\in X$. And, the mapping A is continuous at p.

Example 3. *Let $X=\mathbb{R}$ and (X,S) be a complete S-metric space. For any $x,y,z\in X$, define $S(x,y,z)=|x-z|+|y-z|$ and mappings $A,B:X\to X$ by $Ax=1$ and*

$$Bx=\begin{cases}1, & x\in\mathbb{Q}\\ 0, & x\notin\mathbb{Q}.\end{cases}$$

Then, it is easy to see that

$$S(Ax,Ay,Az)\le\phi(\max\{S(Bx,By,Bz),k_1\,S(Bz,Ax,Az),k_2\,S(Bz,Ay,Az)\})$$

for all $x,y,z\in X$ and $0<k_1,k_2<1$. Therefore, all the conditions of Theorem 2.1 hold and $A1=B1=1$.

Theorem 2.3. *Let (X,S) be a complete S-metric space and $A,B:X\to X$ be continuous and B be commutative with A. If for every $n\in\mathbb{N}$, the following conditions are satisfying*

(1) *$A^n(X)\subseteq B^n(X)$ and either $A^n(X)$ or $B^n(X)$ is a closed subset of X,*

(2) *the pair (A^n,B^n) is weakly compatible,*

(3)

$$\begin{aligned}&S(A^nx,A^ny,A^nz)\\ \le\ &\phi(\max\{S(B^nx,B^ny,B^nz),k_1S(B^nz,A^nx,A^nz),k_2S(B^nz,A^ny,A^nz)\})\end{aligned}$$

for all $x,y,z\in X$ and $0<k_1,k_2<1$, where $\phi\in\Phi$,

then A and B have a unique common fixed point $p\in X$. Further, if B is continuous at p, then A is also continuous at p.

Proof. The proof is similar to the proof of Theorem 2.1. □

References

[1] R.P. Agarwal, M. A. El-Gebeily, D. O'regan, *Generalized contractions in partially ordered metric spaces*, Appl. Anal., 87(2008), 109-116.

[2] Lj. Ćirić, D. Mihet, R. Saadati, *Monotone generalized contractions in partiality ordered probabilistic metric spaces*, Topology Appl., 17(2009), 2838-2844.

[3] S. Sedghi, N. Shobe, H. Zhou, A common fixed point theorem in D^*-metric spaces. Fixed Point Theory Appl., 2007(2007), Article ID 7906, 13 pages

[4] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorems in S-metric spaces, Mat. Vanik, 64(2012), 258 - 266.
[5] J.K. Kim, S. Sedghi, N. Shobkolaei, Common Fixed Point Theorems for the R-weakly Commuting Mappings in S-metric Spaces. J.Comput. Anal. Appl., 19(2015), 751-759.
[6] S. Sedghi, NV. Dung, Fixed point theorems on S-metric spaces. Mat.Vensnik 66(2014), 113-124.
[7] S. Sedghi, I. Altun, N. Shobe, M.A. Salahshour, Some Properties of S-metric Spaces and Fixed Point Results, Kyungpook Math. J., 54(2014), 113-122.
[8] S. Sedghi, N. Shobe, T. Dosenovic, fixed point results in S-metric spaces, Nonlinear Functional Anal. and Appl., 20(2015), 55-67.

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