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ON SUFFICIENCY AND DUALITY FOR FRACTIONAL ROBUST OPTIMIZATION PROBLEMS INVOLVING (V, ρ)-INVEX FUNCTION

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Abstract. In this paper, we prove a sufficient optimality theorems for the problem (FP) under (V, ρ)-invexity assumption. And we give Mond-Weir type dual problem and proved weak and strong duality theorem under (V, ρ)-invexity.

1. Introduction

Consider a fractional robust optimization problem:

$$\text{(FP)} \quad \inf_{x \in \mathbb{R}^n} \left\{ \frac{f(x)}{g(x)} : h_j(x, v_j) \leq 0, \ \forall v_j \in V_j, \ j = 1, \cdots, m \right\},$$

where v_j are uncertain parameters and $v_j \in V_j, \ j = 1, \cdots, m$ for some convex compact sets $V_j \subset \mathbb{R}^q, \ j = 1, \cdots, m$ and $f : \mathbb{R}^n \to \mathbb{R}, \ g : \mathbb{R}^n \to \mathbb{R}$ and $h_j : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}, \ j = 1, \cdots, m$ are continuously differentiable functions. Assume that $f(x) \geq 0$ and $g(x) > 0$.

Let $F := \{x \in \mathbb{R}^n : h_j(x, v_j) \leq 0, \ \forall v_j \in V_j, \ j = 1, \cdots, m\}$ be the robust feasible set of (FP). Then we say that x^* is a robust solution of (FP) if $x^* \in F$ and $\frac{f(x)}{g(x)} \geqq \frac{f(x^*)}{g(x^*)}$ for any $x \in F$.

We introduce the following definition due to Kuk et al. [7].

Definition 1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be (V, ρ)-invex at $u \in \mathbb{R}^n$ with respect to the function η and $\theta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ if there exists $\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+ \setminus \{0\}$ and $\rho \in \mathbb{R}$ such that for any $x \in \mathbb{R}^n$

$$\alpha(x, u)[f(x) - f(u)] \geqq \nabla f(u)^T \eta(x, u) + \rho \|\theta(x, u)\|^2.$$

Definition 2. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be η-invex at $u \in \mathbb{R}^n$ such that for any $x \in \mathbb{R}^n$

$$f(x) - f(u) \geqq \nabla f(u)^T \eta(x, u).$$

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Robust optimization provides a tool for handling the uncertainty related with the optimization problems ([1]- [5]). Recently, Kim and Kim [6] gave necessary optimality theorems for (FP). Moreover, they give Mond-Weir type dual problem and proved weak and strong duality theorem under convexity.

In this paper, we give a sufficient optimality theorems for the problem (FP) under (V, ρ)-invexity assumption. And we give Mond-Weir type dual problem and proved weak and strong duality theorem under (V, ρ)-invexity.

2. Optimality and duality theorems

In this section, we give necessary optimality conditions for the fractional robust optimization problem (FP).

Let $\bar{x} \in F$ and let us decompose $J := \{1, \cdots, m\}$ into two index sets $J = J_1(\bar{x}) \cup J_2(\bar{x})$ where $J_1(\bar{x}) = \{j \in J \mid \exists v_j \in V_j \text{ s.t. } h_j(\bar{x}, v_j) = 0\}$ and $J_2(\bar{x}) = J \setminus J_1(\bar{x})$. Let $V_j^0 = \{v_j \in V_j \mid h_j(\bar{x}, v_j) = 0\}$ for $j \in J_1(\bar{x})$. For a continuously differentiable function $h : \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}$, we use $\nabla_1 h$ to denote the derivative of h with respect to the first variable.

Now we say that an Extended Mangasarian-Fromovitz constraint qualification (EMFCQ) holds at $\bar{x}$ for (FP) if there exists $d \in \mathbb{R}^n$ such that for any $j \in J_1(\bar{x})$ and any $v_j \in V_j^0$,

$$\nabla_1 h_j(\bar{x}, v_j)^T d < 0.$$

Now we give a necessary optimality theorem for a solution of (FP).

Theorem 2.1. [6] *Let $\bar{x} \in F$ be a robust solution of (FP). Suppose that $h_j(\bar{x}, \cdot)$ is concave on V_j, $j = 1, \cdots, m$. Then there exist $\lambda \geq 0$, $\mu_j \geq 0$, $j = 1, \cdots, m$, not all zero, $\bar{v}_j \in V_j$, $j = 1, \cdots, m$ such that*

$$\begin{aligned} &\lambda \left[\nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})} \nabla g(\bar{x}) \right] + \sum_{j=1}^{m} \mu_j \nabla_1 h_j(\bar{x}, \bar{v}_j) = 0, \\ &\mu_j h_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \cdots, m. \end{aligned}$$

Moreover, if we assume that the Extended Mangasarian-Fromovitz constraint qualification then we have (EMFCQ) holds, then

$$\begin{aligned} &\nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})} \nabla g(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 h_j(\bar{x}, \bar{v}_j) = 0, \\ &\mu_j h_j(\bar{x}, \bar{v}_j) = 0, \ j = 1, \cdots, m. \end{aligned}$$

Now we give a sufficient optimality theorems for the fractional robust optimization problem (FP).

Theorem 2.2. *Let $\bar{x} \in F$ and assume that $h_j(\bar{x}, \cdot)$ is concave on V_j, $j = 1, \cdots, m$. Suppose that there exist $\mu_j \geqq 0$, $\bar{v}_j \in V_j$, $j = 1, \cdots, m$ such that*

$$\nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})}\nabla g(\bar{x}) + \sum_{j=1}^{m} \mu_j \nabla_1 h_j(\bar{x}, \bar{v}_j) = 0, \tag{1}$$

$$\sum_{j=1}^{m} \mu_j h_j(\bar{x}, \bar{v}_j) = 0.$$

Assume that $f(\cdot)$ and $-g(\cdot)$ are (V, ρ)-invex at $\bar{x}$ and $h_j(\cdot, \bar{v}_j)$, $j = 1, \cdots, m$ are η-invex at $\bar{x}$ with respect to the same η and $\rho\|\theta(x, \bar{x})\|^2 \geqq 0$. Then $\bar{x}$ is a robust solution of (FP).

Proof. Suppose that $\bar{x} \in F$ is not a robust solution of (FP). Then there exist a feasible solution $\widetilde{x}$ of (FP) such that

$$\frac{f(\widetilde{x})}{g(\widetilde{x})} < \frac{f(\bar{x})}{g(\bar{x})}.$$

Since

$$f(\widetilde{x}) - \frac{f(\bar{x})}{g(\bar{x})}g(\widetilde{x}) < 0 = f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})}g(\bar{x}).$$

Since $\alpha(x, u) > 0$,

$$\alpha(x, u)[f(\widetilde{x}) - f(\bar{x})] - \alpha(x, u)\frac{f(\bar{x})}{g(\bar{x})}[g(\widetilde{x}) - g(\bar{x})] < 0.$$

Since $f(\cdot)$ and $-g(\cdot)$ are (V, ρ)-invex at $\bar{x}$ with respect to the same η and ρ,

$$\nabla f(\bar{x})^T \eta(x, \bar{x}) + \rho\|\theta(\widetilde{x}, \bar{x})\|^2 - \frac{f(\bar{x})}{g(\bar{x})}[\nabla g(\bar{x})^T \eta(\widetilde{x}, \bar{x}) + \rho\|\theta(\widetilde{x}, \bar{x})\|^2] < 0.$$

Since $\rho\|\theta(\widetilde{x}, \bar{x})\|^2 \geq 0$,

$$\left[\nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})}\nabla g(\bar{x})\right]^T \eta(\widetilde{x}, \bar{x}) < 0,$$

and so, it follows from (1) that $\sum_{j=1}^{m} \mu_j \nabla_1 h_j(\bar{x}, \bar{v}_j)^T \eta(\widetilde{x}, \bar{x}) > 0$. Then, by the η-invexity of $h(\cdot, \bar{v}_j)$, we have

$$\sum_{j=1}^{m} \mu_j h_j(\widetilde{x}, \bar{v}_j) - \sum_{j=1}^{m} \mu_j h_j(\bar{x}, \bar{v}_j) > 0.$$

Since $\sum_{j=1}^{m} \mu_j h_j(\bar{x}, \bar{v}_j) = 0$, we have $\sum_{j=1}^{m} \mu_j h_j(\widetilde{x}, \bar{v}_j) > 0$, which is contradiction, since $\mu_j \geq 0$, $j = 1, \cdots, m$ and $\widetilde{x}$ is a feasible solution of (FP). Consequently, $\bar{x}$ is a robust solution of (FP). $\square$

We formulate a Mond-Weir type robust dual problem (FD) for (FP).

$$\begin{aligned} \text{(FD)} \quad & \text{maximize} \quad && p \\ & \text{subject to} && \nabla f(x) - p\nabla g(x) + \sum_{j=1}^{m} \mu_j \nabla_1 h_j(x, v_j) = 0, \qquad (2) \\ &&& f(x) - pg(x) \geq 0, \\ &&& \sum_{j=1}^{m} \mu_j h_j(x, v_j) \geq 0, \\ &&& v_j \in V_j, \ \mu_j \geq 0, \ j = 1, \cdots, m. \end{aligned}$$

Let $V = V_1 \times \cdots \times V_m$.

Theorem 2.3. *(Weak Duality) Let $x \in F$ and $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p}) \in \mathbb{R}^n \times V \times \mathbb{R}^m \times \mathbb{R}$ be feasible for (FD). Suppose that $f(\cdot)$ and $-g(\cdot)$ is (V, ρ)-invex at $\bar{x}$ and $h_j(\cdot, \bar{v}_j)$, $j = 1, \cdots, m$ are η-invex at $\bar{x}$ with respect to the same η and $\rho\|\theta(x, \bar{x})\|^2 \geqq 0$, then*

$$\frac{f(x)}{g(x)} \geq \bar{p}.$$

Proof. Let x be any feasible for (FP) and let $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p})$ be any feasible for (FD). Suppose that

$$\frac{f(x)}{g(x)} - \bar{p} < 0, \text{ that is, } \quad f(x) - \bar{p}g(x) < 0.$$

Since $f(\bar{x}) - \bar{p}g(\bar{x}) \geq 0$, $f(x) - \bar{p}g(x) < f(\bar{x}) - \bar{p}g(\bar{x})$. Since $\alpha(x, u) > 0$,

$$\alpha(x, u)[f(x) - f(u)] - \bar{p}\alpha(x, u)[g(x) - g(\bar{x})] < 0.$$

By the (V, ρ)-invexity of $f(\cdot) - \bar{p}g(\cdot)$ at $\bar{x}$,

$$[\nabla f(\bar{x}) - \bar{p}\nabla g(\bar{x})]^T \eta(x, \bar{x}) + \rho\|\theta(x, \bar{x})\|^2 < 0.$$

Since $\rho\|\theta(x, \bar{x})\|^2 \geqq 0$,

$$[\nabla f(\bar{x}) - \bar{p}\nabla g(\bar{x})]^T \eta(x, \bar{x}) < 0. \qquad (3)$$

Since $\sum_{j=1}^{m} \bar{\mu}_j h_j(\bar{x}, \bar{v}_j) \geqq \sum_{j=1}^{m} \bar{\mu}_j h_j(x, \bar{v}_j)$, by the η-invexity $h_j(\cdot, \bar{v}_j)$ at $\bar{x}$,

$$\left[\sum_{j=1}^{m} \bar{\mu}_j \nabla_1 h_j(\bar{x}, \bar{v}_j)\right]^T \eta(x, \bar{x}) \leq 0. \qquad (4)$$

From (3) and (4),

$$\left[\nabla f(\bar{x}) - \bar{p}\nabla g(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 h_j(\bar{x}, \bar{v}_j)\right]^T \eta(x, \bar{x}) < 0,$$

which contradicts (2). □

Theorem 2.4. *(Strong Duality) Let $\bar{x}$ be a robust solution of (FP). Assume that the Extended Mangasarian-Fromovitz constraint qualification holds at $\bar{x}$. Then, there exist $(\bar{v}, \bar{\mu}, \bar{p})$ such that $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p})$ is feasible for (FD). Moreover, if the weak duality holds, then $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p})$ is a robust solution of (FD).*

Proof. By Theorem 2.1, there exist $\bar{\mu}_j \geq 0,\ j = 1, \cdots, m$, $\bar{v}_j \in V_j,\ j = 1, \cdots, m$ such that

$$\nabla f(\bar{x}) - \frac{f(\bar{x})}{g(\bar{x})}\nabla g(\bar{x}) + \sum_{j=1}^{m} \bar{\mu}_j \nabla_1 h_j(\bar{x}, \bar{v}_j) = 0,$$

$$\bar{\mu}_j h_j(\bar{x}, \bar{v}_j) = 0,\ j = 1, \cdots, m.$$

Let $\bar{p} = \frac{f(\bar{x})}{g(\bar{x})}$. Then $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p})$ is a feasible for (FD). By Theorem 2.2, $\frac{f(\bar{x})}{g(\bar{x})} \geq \widetilde{p}$, for any feasible solution $(\widetilde{x}, \widetilde{u}, \widetilde{v}, \widetilde{\mu}, \widetilde{p})$ for (FD). Since $\frac{f(\bar{x})}{g(\bar{x})} = \bar{p}$, $\bar{p} \geq \widetilde{p}$. Hence $(\bar{x}, \bar{v}, \bar{\mu}, \bar{p})$ is a solution of (FD). □

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