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EXISTENCE OF POSITIVE SOLUTIONS TO NONLOCAL BOUNDARY VALUE PROBLEMS WITH BOUNDARY PARAMETER

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Abstract. We establish the existence of positive solutions to nonlocal boundary value problems with integral boundary condition and non-negative real boundary parameter by mainly using the Schauder-Fixed point theorem.

1. Introduction

The theory of boundary-value problems with integral boundary conditions for differential equations arises in various areas of applied mathematics and physics like heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics. For the integral boundary value problems and comments on their importance, we refer the reader to the papers by M. Feng, D. Ji and W. Ge [2], Gallardo [3], Karakostas and Tsamatos [5], Lomtatidze and Malaguti [6] and the references therein. In this paper, we study the existence of positive solutions to the nonhomogeneous integral boundary value problem of the following,

$$\begin{cases} u''(t) + a(t)f(u(t)) = 0, & t \in (0,1), \\ u(0) = 0, \quad u(1) - \int_0^1 g(s)u(s)ds = b \end{cases} \tag{P_b}$$

satisfying $g \in L^1(0,1), g(t) \geq 0$ and $0 < \int_0^1 sg(s)ds < 1, a \in C([0,1],[0,\infty)), f \in C([0,\infty),[0,\infty))$. We consider the following assumptions:

$(A1)$ $a(t) \equiv 0$ does not hold on any subinterval of [0,1]
$(A2)$ $f_0 = lim_{u\to 0}\frac{f(u)}{u} = 0, f_\infty = lim_{u\to\infty}\frac{f(u)}{u} = \infty$

The main result of the this paper is as follows.

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Theorem 1.1. *Assume* $(A1)$ *and* $(A2)$ *hold. Then there exists a positive number* $\bar{b}$ *such that if* $0 < b \leq \bar{b}$ *then* (P_b) *has at least one positive solution and if* $b > \bar{b}$ *then* (P_b) *has no positive solution.*

The proof of above theorem is based upon the following well known fixed point theorem.

Theorem 1.2. *(Schauder-Fixed Point Theorem) Let* S *be a nonempty, closed, bounded and convex subset of a Banach space* X *and* A *be a continuous self-map on* S *such that* $cl_X(A(S))$ *is compact. Then* A *has a fixed point on* S *.*

2. Preliminaries

Lemma 2.1. *The problem*

$$\begin{cases} u''(t) = 0, & t \in (0,1), \\ u(0) = 0, & u(1) - \int_0^1 g(s)u(s)ds = 1 \end{cases} \tag{1}$$

has a unique solution $h(t) = \frac{1}{1-\sigma}t$, *when* $\sigma = \int_0^1 sg(s)ds$.

Proof. Since $h''(t) = 0$ and $h(0) = 0$, we have $h(t) = at$ for some $a \in \mathbb{R}\backslash\{0\}$. From $h(1) = 1 + \int_0^1 g(s)h(s)ds = a$,

$$h(t) = (1 + \int_0^1 g(s)h(s)ds)t \tag{2}$$

From $at = (1 + \int_0^1 g(s)asds)t$, we have $a = 1 + a\int_0^1 sg(s)ds$ and thus $a = \frac{1}{1-\int_0^1 sg(s)ds} = \frac{1}{1-\sigma}$. The proof is completed. $\square$

Lemma 2.2. *If* h *is a solution of (1) and* v *is a solution of*

$$\begin{cases} v''(t) + a(t)f(v(t) + bh(t)) = 0, & t \in (0,1), \\ v(0) = 0, \quad v(1) - \int_0^1 g(s)v(s)ds = 0 \end{cases} \tag{$\tilde{P_b}$}$$

Then $u = v + bh$ *is a solution of* (P_b)

Proof. If $u = v + bh$, then we have

$$u''(t) = v''(t) + bh''(t) = -a(t)f(v(t) + bh(t)) = -a(t)f(u(t)),$$

$u(0) = v(0) + bh(0) = 0$ and

$$\begin{aligned} u(1) = v(1) + bh(1) &= \int_0^1 g(s)v(s)ds + b(1 + \int_0^1 g(s)h(s)ds) \\ &= b + \int_0^1 g(s)[v(s) + bh(s)]ds = b + \int_0^1 g(s)u(s)ds. \end{aligned}$$

$\square$

Let $P = \{u \in C[0,1] | u(t) \geq 0$ for all $t \in [0,1]\}$. Define $A_b : P \to C[0,1]$ by

$$A_b w(t) = \int_0^1 H(t,s)a(s)f(w(s) + bh(s))ds,$$

where

$$H(t,s) = G(t,s) + \frac{t}{1-\sigma}\int_0^1 G(s,\tau)g(\tau)d\tau$$

$$G(t,s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Then $v = A_b w$ be the solution of

$$\begin{cases} v''(t) + a(t)f(w(t) + bh(t)) = 0, & t \in (0,1), \\ v(0) = 0, \quad v(1) - \int_0^1 g(s)v(s)ds = 0, \end{cases} \tag{3}$$

and we know that the fixed point of A_b is the solution of $(\tilde{P}_b)$.

Lemma 2.3. *If $A : [0,\infty) \times P \to C[0,1]$ is defined by $A(b,u) = A_b(u)$, then A is compact and continuous.*

Proof. For bounded subset M of $[0,\infty) \times P$, we claim that $A(M)$ is relatively compact. Firstly, since M is bounded in $[0,\infty) \times P$, there exists $B > 0$ such that $|b| + \|w\|_\infty \leq B$, for all $(b,w) \in M$. If we take $M_1 > 0$ by $M_1 = \max_{s \in [0, B + B\|h\|_\infty]} f(s)$, then for $(b,w) \in M$,

$$\begin{aligned} A(b,w)(t) = A_b w(t) &= \int_0^1 G(t,s)a(s)f(w(s) + bh(s))ds \\ &+ \frac{t}{1-\sigma}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(w(\tau) + bh(\tau))d\tau)ds] \\ &\leq \int_0^1 G(t,s)a(s)M_1 ds + \frac{t}{1-\sigma}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)M_1 d\tau)ds] \\ &\leq \int_0^1 G(s,s)a(s)M_1 ds + \frac{1}{1-\sigma}[\int_0^1 g(s)(\int_0^1 G(\tau,\tau)a(\tau)M_1 d\tau)ds] \\ &= M_1(\int_0^1 G(s,s)a(s)ds)(1 + \frac{1}{1-\sigma}\int_0^1 g(s)ds). \end{aligned}$$

That is, $\|A(b,w)\|_\infty \leq M_1(\int_0^1 s(1-s)a(s)ds)(1 + \frac{1}{1-\sigma}\int_0^1 g(s)ds)$ for all $(b,w) \in M$ and thus $A(M)$ is bounded. Secondly, for $(b,w) \in M$, we have

$$\begin{aligned} &|A(b,w)'(t)| = |(A_b w)'(t)| \\ &= |-\int_0^t sa(s)f(w(s) + bh(s))ds + \int_t^1 (1-s)a(s)f(w(s) + bh(s))ds \\ &\quad + \frac{1}{1-\sigma}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(w(\tau) + bh(\tau))d\tau)ds]\ | \\ &= M_1(\gamma(t) + M_2), \end{aligned} \tag{4}$$

where $M_2 = \frac{1}{1-\sigma}[\int_0^1 g(s)[\int_0^1 G(\tau, s)a(\tau)d\tau]ds]$ and

$$\gamma_1(t) = \int_0^t sa(s)ds + \int_t^1 (1-s)a(s)ds.$$

Now, we show that

$$\begin{aligned}\int_0^1 \gamma(t)dt &= \int_0^1 \int_0^t sa(s)dsdt + \int_0^1 \int_t^1 (1-s)a(s)dsdt \\ &= \int_0^1 \int_s^1 sa(s)dtds + \int_0^1 \int_0^s (1-s)a(s)dtds \\ &= \int_0^1 s(1-s)a(s)ds + \int_0^1 s(1-s)a(s)ds \\ &= 2\int_0^1 s(1-s)a(s)ds < \infty.\end{aligned}$$

This means

$$\int_0^1 M_1(\gamma(t) + M_2)dt < \infty. \tag{5}$$

From (4) and (5), for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|u - v| < \delta$ then

$$\begin{aligned}|A(b, w)(u) - A(b, w)(v)| &= |\int_u^v A(b, w)'(t)dt| \\ &\le \int_u^v |A(b, w)'(t)|dt \\ &\le \int_u^v M_1(\gamma(t) + M_2)dt < \varepsilon,\end{aligned}$$

for any $(b, w) \in M$. Consequently, $A(M)$ is equicontinuous on $[0, 1]$. By Arzela-Ascoli theorem, $A(M)$ is relatively compact and thus A is compact operator. Now, we claim $A : [0, \infty) \times P \to C[0, 1]$ is continuous. We note that

$$\begin{aligned}H(t, s) &= G(t, s) + \frac{t}{1-\sigma}\int_0^1 G(s, \tau)g(\tau)d\tau \\ &\le s(1-s) + \frac{t}{1-\sigma}\int_0^1 s(1-s)g(\tau)d\tau \\ &= s(1-s)[1 + \frac{1}{1-\sigma}\int_0^1 g(\tau)d\tau].\end{aligned}$$

If (b_n, w_n) is the sequence of $[0, \infty) \times P$ such that $(b_n, w_n) \to (b, w)$ in $[0, \infty) \times P$, then (b_n, w_n) is bounded sequence so that there exist $\bar{B} > 0$ such that $\sup\{\|w_n\|_\infty + b_n\|h\|_\infty, \|w\|_\infty + b\|h\|_\infty\} < \bar{B}$. Since f is uniformly continuous on $[0, \bar{B}]$, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $u,\ v \in [0, \bar{B}]$ and $|u - v| < \delta$, then

$$|f(u) - f(v)| < \frac{\varepsilon}{L}, \tag{6}$$

where $L = (\int_0^1 s(1-s)a(s)ds)(1 + \frac{1}{1-\sigma}\int_0^1 g(\tau)d\tau)$. Since $\|w_n - w\|_\infty \to 0$ and $|b_n - b| \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that $\|w_n - w\|_\infty < \frac{\delta}{2}$ and $|b_n - b| < \frac{\delta}{2\|h\|_\infty}$ for $n \geq N$. Then by (6), $|f(w_n(t)+b_nh(t)) - f(w(t)+bh(t))| < \frac{\varepsilon}{L}$ for all $t \in [0,1]$. Then $\|f(w_n + b_nh) - f(w+bh)\|_\infty < \frac{\varepsilon}{L}$. Thus,

$$\begin{aligned}
&|A(b_n, w_n)(t) - A(b,w)(t)| \\
&\quad = |\int_0^1 H(t,s)a(s)(f(w_n(s) + b_nh(s)) - f(w(s) + bh(s)))ds| \\
&\quad \leq (\int_0^1 H(t,s)a(s)ds)\|f(w_n + b_nh) - f(w+bh)\|_\infty \\
&\quad \leq (\int_0^1 s(1-s)a(s)ds)(1 + \frac{1}{1-\sigma}\int_0^1 g(\tau)d\tau)\|f(w_n + b_nh) - f(w+bh)\|_\infty \\
&\quad = L\|f(w_n + b_nh) - f(w+bh)\|_\infty < \varepsilon,
\end{aligned}$$

for all $t \in [0,1]$. Thus, $\|A(b_n, w_n) - A(b,w)\|_\infty < \varepsilon$. So, we verified that A is continuous. $\square$

3. Main Result

For the proof of theorem 1.1, we need the following theorems.

Theorem 3.1. *Assume (A2) holds. Then there exists a positive number $b_1 > 0$ such that (P_b) has a positive solution for $0 < b < b_1$.*

Proof. Let $\widetilde{f}(x) = \sup_{0 \leq s \leq x} f(s)$, then $\widetilde{f}$ is monotone increasing and $f(s) \leq \widetilde{f}(s)$ for all $s \geq 0$. From $(A2)$, we know $lim_{u \to 0^+} \frac{\widetilde{f}(u)}{u} = 0$. Then there exists $b_1 > 0$ such that

$$\widetilde{f}(b_1 + b_1\|h\|_\infty)\|p\|_\infty \leq b_1,$$

where $p(t) = \int_0^1 H(t,s)a(s)ds$ is the solution of

$$\begin{cases} u'' + a(t) = 0, & t \in (0,1), \\ u(0) = 0, & u(1) - \int_0^1 g(s)u(s)ds = 0. \end{cases}$$

Define a closed bounded convex subset in $C[0,1]$ by

$$D_{b_1} = \{u \in C[0,1] \mid 0 \leq u(t) \leq b_1, \quad t \in [0,1]\}.$$

For $0 < b < b_1$, we claim that $A_b(D_{b_1}) \subset D_{b_1}$. Indeed, for $w \in D_{b_1}$, let $v = A_b w$, then v is the solution of (3). From the choice of b_1, we have

$$\begin{aligned}
0 \le v(t) &= \int_0^1 G(t,s)a(s)f(w(s)+bh(s))ds \\
&+ \frac{t}{1-\int_0^1 sg(s)ds}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(w(\tau)+bh(\tau))d\tau)ds] \\
&\le \int_0^1 G(t,s)a(s)\widetilde{f}(w(s)+bh(s))ds \\
&+ \frac{t}{1-\int_0^1 sg(s)ds}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)\widetilde{f}(w(\tau)+bh(\tau))d\tau)ds] \\
&\le \int_0^1 G(t,s)a(s)\widetilde{f}(b_1+b_1\|h\|_\infty)ds \\
&+ \frac{t}{1-\int_0^1 sg(s)ds}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)\widetilde{f}(b_1+b_1\|h\|_\infty)d\tau)ds] \\
&\le \widetilde{f}(b_1+b_1\|h\|_\infty)[\int_0^1 G(t,s)a(s)ds \\
&+ \frac{t}{1-\int_0^1 sg(s)ds}\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)d\tau)ds] \\
&\le \widetilde{f}(b_1+b_1\|h\|_\infty)\|p\|_\infty \le b_1.
\end{aligned}$$

Thus $A_b w = v \in D_{b_1}$. By Lemma 2.3, $\overline{A_b(D_{b_1})}$ is compact. Thus by Theorem 1.2, A_b has a fixed point v in D_{b_1} and by Lemma 2.2 $u = v + bh$ is a positive solution of (P_b). $\square$

Theorem 3.2. *Assume* $(A1)$ *and* $(A2)$. *There exists* $B > 0$ *such that* $\|u\|_\infty < B$ *for all possible positive solutions* u *of* (P_b).

Proof. Let u be a positive solution to (P_b). Then $v = u - bh$ satisfies $(\tilde{P}_b)$. Let J be a set of positive measure $J \subset (\delta, 1-\delta)$ for some positive $\delta < \frac{1}{2}$ such that $a(t) > 0$ for $t \in J$. This J can be taken by condition $(A1)$. Let $\gamma = \min\{a^*, 1-b^*\}$ where $a^* = \inf J$ and $b^* = \sup J$. Firstly, we will show that

$$\inf_{t\in J} v(t) \ge \gamma\|v\|_\infty. \tag{7}$$

Infact, since v is the solution of the equation $(\tilde{P}_b)$, we have

$$\begin{aligned}
v(t) \le &\int_0^1 G(s,s)a(s)f(v(s)+bh(s))ds \\
&+ \frac{1}{1-\sigma}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(v(\tau)+bh(\tau))d\tau)ds]
\end{aligned}$$

for all $t \in [0,1]$. Thus,

$$\|v\|_\infty \leq \int_0^1 G(s,s)a(s)f(v(s)+bh(s))ds + \frac{1}{1-\sigma}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(v(\tau)+bh(\tau))d\tau)ds]. \tag{8}$$

For $t \in J$,

$$\begin{aligned} v(t) &= \int_0^1 G(t,s)(s)f(v(s)+bh(s))ds + \frac{t}{1-\sigma}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(v(\tau)+bh(\tau))d\tau)ds] \\ &= \int_0^t s(1-t)a(s)f(v(s)+bh(s))ds + \int_t^1 t(1-s)a(s)f(v(s)+bh(s))ds \\ &+ \frac{t}{1-\sigma}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(v(\tau)+bh(\tau))d\tau)ds] \\ &\geq \int_0^t s(1-b^*)a(s)f(v(s)+bh(s))ds + \int_t^1 a^*(1-s)a(s)f(v(s)+bh(s))ds \\ &+ \frac{a^*}{1-\sigma}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(v(\tau)+bh(\tau))d\tau)ds] \\ &\geq \min\{a^*,1-b^*\}[\int_0^t sa(s)f(v(s)+bh(s))ds + \int_t^1 (1-s)a(s)f(v(s)+bh(s))ds \\ &+ \frac{1}{1-\sigma}[\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(v(\tau)+bh(\tau))d\tau)ds]] \\ &\geq \min\{a^*,1-b^*\}[\int_0^t s(1-s)a(s)f(v(s)+bh(s))ds + \int_t^1 s(1-s)a(s)f(v(s)+bh(s))ds \\ &+ \frac{1}{1-\sigma}(\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(v(\tau)+bh(\tau))d\tau)ds)] \\ &= \min\{a^*,1-b^*\}[\int_0^1 G(s,s)a(s)f(v(s)+bh(s))ds \\ &+ \frac{1}{1-\sigma}(\int_0^1 g(s)(\int_0^1 G(\tau,s)a(\tau)f(v(\tau)+bh(\tau))d\tau)ds)] \\ &\geq \min\{a^*,1-b^*\}\|v\|_\infty = \gamma\|v\|_\infty. \end{aligned}$$

For the last inequality in the above, we used (8). Now, we need to prove

$$\inf_{t\in J}(v(t)+bh(t)) \geq \gamma\|v+bh\|_\infty. \tag{9}$$

In fact, since $h(t) = \frac{t}{1-\sigma}$, we have

$$\inf_{t\in J} h(t) = a^* \times \frac{1}{1-\sigma} = a^* \times \|h\|_\infty \geq \gamma\|h\|_\infty. \tag{10}$$

By using (7) and (10),

$$\begin{aligned}\inf_{t\in J}(v(t)+bh(t)) &\geq \inf_{t\in J} v(t)+\inf_{t\in J} bh(t)\\ &\geq \gamma\|v\|_\infty + b\gamma\|h\|_\infty\\ &= \gamma(\|v\|_\infty + \|bh\|_\infty)\\ &\geq \gamma\|v+bh\|_\infty.\end{aligned}$$

Let $\overline{f}(t) = \inf_{t\leq s} f(s)$, then $\overline{f}$ is monotone increasing, $\overline{f}(s) \leq f(s)$ for all $s \in [0,\infty)$ and from $f_\infty = \infty$,

$$\overline{f}_\infty = \lim_{u\to\infty} \frac{\overline{f}(u)}{u} = \infty. \tag{11}$$

Thus we have

$$\begin{aligned}\|v\|_\infty \geq v(a^*) &= \int_0^1 G(a^*,s)a(s)f(v(s)+bh(s))ds\\ &+ \frac{a^*}{1-\sigma}\int_0^1 g(s)[\int_0^1 G(\tau,s)a(\tau)f(v(\tau)+bh(\tau))d\tau]ds\\ &\geq \int_{a^*}^{b^*} G(a^*,s)a(s)f(v(s)+bh(s))ds\\ &+ \frac{a^*}{1-\sigma}\int_{a^*}^{b^*} g(s)[\int_{a^*}^{b^*} G(\tau,s)a(\tau)f(v(\tau)+bh(\tau))d\tau]ds\\ &= \int_{a^*}^{b^*} a^*(1-s)a(s)f(v(s)+bh(s))ds\\ &+ \frac{a^*}{1-\sigma}\int_{a^*}^{b^*} g(s)[\int_{a^*}^{s} \tau(1-s)a(\tau)f(v(\tau)+bh(\tau))d\tau\\ &+ \int_{s}^{b^*} s(1-\tau)a(\tau)f(v(\tau)+bh(\tau))d\tau]ds\\ &\geq a^*(1-b^*)\int_{a^*}^{b^*} a(s)f(v(s)+bh(s))ds\\ &+ \frac{a^*}{1-\sigma}\int_{a^*}^{b^*} g(s)[\int_{a^*}^{s} a^*(1-b^*)a(\tau)f(v(\tau)+bh(\tau))d\tau\\ &+ \int_{s}^{b^*} a^*(1-b^*)a(\tau)f(v(\tau)+bh(\tau))d\tau]ds\end{aligned}$$

$$
\begin{aligned}
&= a^*(1-b^*)\int_{a^*}^{b^*} a(s)f(v(s)+bh(s))ds \\
&+ \frac{a^*}{1-\sigma}\int_{a^*}^{b^*} g(s)[a^*(1-b^*)\int_{a^*}^{b^*} a(\tau)f(v(\tau)+bh(\tau))d\tau]ds \\
&= a^*(1-b^*)(1+\frac{a^*}{1-\sigma}\int_{a^*}^{b^*} g(s)ds)(\int_{a^*}^{b^*} a(s)f(v(s)+bh(s)ds)) \\
&\geq a^*(1-b^*)(1+\frac{a^*}{1-\sigma}\int_{a^*}^{b^*} g(s)ds)(\int_{a^*}^{b^*} a(s)\overline{f}(v(s)+bh(s)ds)) \\
&\geq a^*(1-b^*)(1+\frac{a^*}{1-\sigma}\int_{a^*}^{b^*} g(s)ds)(\int_{a^*}^{b^*} a(s)ds)\overline{f}(\gamma\|v+bh\|_\infty) \\
&\geq \gamma^2(1+\frac{a^*}{1-\sigma}\int_{a^*}^{b^*} g(s)ds)(\int_{a^*}^{b^*} a(s)ds)\overline{f}(\gamma\|v+bh\|_\infty).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\frac{\overline{f}(\gamma\|u\|_\infty)}{\gamma\|u\|_\infty} \quad &= \quad \frac{\overline{f}(\gamma\|v+bh\|_\infty)}{\gamma\|v+bh\|_\infty} \leq \frac{\overline{f}(\gamma\|v+bh\|_\infty)}{\gamma\|v\|_\infty} \\
&\leq \quad [\gamma^3(1+\frac{a^*}{1-\sigma}\int_{a^*}^{b^*} g(s)ds)\cdot(\int_{a^*}^{b^*} a(s)ds)]^{-1}. \qquad (12)
\end{aligned}
$$

If $\|u\|_\infty \to \infty$, then (12) contradicts to $\overline{f}_\infty = \infty$ and the proof is complete. □

Theorem 3.3. *There is $\tilde{b} > 0$ such that (P_b) has no positive solution for $b > \tilde{b}$.*

Proof. Suppose on the contrary that there there exists a sequence $\{b_n\}$ such that $b_n \to \infty$ and (P_{b_n}) has a positive solution u_n. Since $u_n(1) = \int_0^1 g(s)u_n(s)ds + b_n \geq b_n$, $\|u_n\|_\infty \to \infty$, and it contradicts to Theorem 3.2. □

Now we prove our main Theorem1.1.

PROOF OF THEOREM 1.1

Let $\Lambda = \{b > 0 | (P_b)$ has a positive solution$\}$. By Theorem 3.1, Λ is nonempty set. Let $\overline{b} = \sup\Lambda$, then by Theorem 3.3, $0 < \overline{b} < \infty$. First, we show that (P_b) has a positive solution for $b = \overline{b}$. Indeed, there is a sequence $\{b_n\}$ in Λ such that $b_n \to \overline{b}$, and let u_n be a positive solution of (P_{b_n}). Then $v_n = u_n - b_n h$ are the solutions of $(\widetilde{P}_{b_n})$ and by Theorem 3.2 and Theorem 3.3, $\{(b_n, v_n)\}$ is bounded in $[0,\infty)\times P$. By compactness of A, $\{v_n\}$ has a convergent subsequence converging to, say $\bar{v}$ and by continuity of A, we see that $\bar{v}$ is a solution of $(\widetilde{P}_{\overline{b}})$. Let $\overline{u} = \overline{v} + \overline{b}h$, then $\overline{u}$ is a positive solution of $(P_{\overline{b}})$. Now, we show that (P_b) has a positive solution for $b \in (0, \overline{b})$. Let $b \in (0, \overline{b})$ and $\overline{u}$ be a positive solution of $(P_{\bar{b}})$. If we define $F : [0,1] \times [0,\infty) \to [0,\infty)$ by

$$F(t,u) = \begin{cases} f(\overline{u}(t)), & if \quad u > \overline{u}(t), \\ f(u), & if \quad 0 \le u \le \overline{u}(t), \\ 0, & if \quad u < 0. \end{cases}$$

Then for the problem

$$\begin{cases} v''(t) + a(t)F(t, v(t) + bh(t)) = 0, \quad t \in (0,1), \\ v(0) = 0, \ v(1) - \int_0^1 g(s)v(s)ds = 0, \end{cases} \tag{13}$$

define $T : P \to C[0,1]$ by $Tv(t) = \int_0^1 H(t,s)a(s)F(s, v(s) + bh(s))ds$. From the definition of F, if $K = \sup_{u\in[0,\|\overline{u}\|_\infty]} f(u)$, then $|F(t,u(t))| \le K$ for all $u \in C[0,1]$ and $t \in [0,1]$. Thus for $v \in C[0,1]$,

$$\begin{aligned} |Tv(t)| &= |\int_0^1 H(t,s)a(s)F(s, v(s) + bh(s))ds| \\ &= |\int_0^1 [G(t,s) + \frac{t}{1-\sigma}\int_0^1 G(s,\tau)g(\tau)d\tau]a(s)F(s, v(s) + bh(s))ds| \\ &\le \int_0^1 s(1-s)[1 + \frac{1}{1-\sigma}\int_0^1 g(\tau)d\tau]a(s)|F(s, v(s) + bh(s))|ds \\ &\le K\int_0^1 s(1-s)[1 + \frac{1}{1-\sigma}\int_0^1 g(\tau)d\tau]a(s)ds, \end{aligned}$$

for all $t \in [0,1]$. Thus $\|Tv\|_\infty \le M$, where

$$M = K\int_0^1 s(1-s)[1 + \frac{1}{1-\sigma}\int_0^1 g(\tau)d\tau]a(s)ds.$$

Thus $T(B_M(0)) \subset B_M(0)$, when $B_M(0) = \{v \in C[0,1] \mid \|v\|_\infty \le M\}$ and $\overline{T(B_M(0))}$ is compact by the same argument in the proof of Lemma 2.3. By Theorem1.2, T has a fixed point v_b in $B_M(0) \cap P$. Then $u_b = v_b + bh$ is a positive solution of

$$\begin{cases} u''(t) + a(t)F(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \ u(1) - \int_0^1 g(s)u(s)ds = b. \end{cases} \tag{14}$$

We need to show that $u_b(t) \le \overline{u}(t)$ for all $t \in [0,1]$. Then from the definition of F, we can say that u_b is a positive solution of (P_b). Let $\Omega_0 = \{t \in (0,1] \mid u_b(t) > \overline{u}(t)\}$ and suppose that $\Omega_0 \ne \varnothing$. Firstly, we consider the case that $u_b(1) \le \overline{u}(1)$, then there exists $(a_1, b_1) \subset \Omega_0$ such that $w(a_1) = w(b_1) = 0$ and $w(t) > 0$ for $t \in (a_1, b_1)$, where $w(t) = u_b(t) - \overline{u}(t)$. Then, by the definition of F, for $t \in (a_1, b_1)$

$$\begin{aligned} w''(t) &= u_b''(t) - \overline{u}''(t) \\ &= -a(t)F(t, u_b(t)) - (-a(t)f(\overline{u}(t))) = 0. \end{aligned}$$

Thus $w \equiv 0$ in (a_1, b_1), but it's a contradiction to the fact that (a_1, b_1) is the subset of Ω_0. For the second case that $u_b(1) > \overline{u}(1)$, that is $w(1) > 0$. We claim that $\int_0^1 g(s)w(s)ds > 0$. In fact, from the assumption of

$$\begin{aligned} w(1) - \int_0^1 g(s)w(s)ds &= u_b(1) - \overline{u}(1) - \int_0^1 g(s)[u_b(s) - \overline{u}(s)]ds \\ &= [u_b(1) - \int_0^1 g(s)u_b(s)ds] - [\overline{u}(1) - \int_0^1 g(s)\overline{u}(s)ds] \\ &= b - \overline{b} < 0. \end{aligned}$$

From $w(1) > 0$, we have that $\int_0^1 g(s)w(s)ds > 0$. Now, we only need to deal with the following three cases. First, for the case that $w > 0$ in (0,1), we have

$$\begin{aligned} w''(t) &= u_b''(t) - \overline{u}''(t) = 0, \ t \in (0,1) \\ w(0) &= u_b(0) - \overline{u}(0) = 0 - 0 = 0 \\ w(1) - \int_0^1 g(s)w(s)ds &= [u_b(1) - \int_0^1 g(s)u_b(s)ds] - [\overline{u}(1) - \int_0^1 g(s)\overline{u}(s)ds] \\ &= b - \overline{b} < 0. \end{aligned}$$

Since $w''(t) = 0$ for $t \in (0,1), w$ is a linear function and we have $w(t) = \alpha t$ for some $\alpha \in \mathbb{R}$. From

$$w(1) = b - \overline{b} + \int_0^1 g(s)w(s)ds, \tag{15}$$

by putting $w(s) = \alpha s$ into (15), $\alpha \cdot 1 = b - \overline{b} + \int_0^1 g(s)(\alpha s)ds$, then $\alpha(1 - \int_0^1 sg(s)ds) = b - \overline{b}$. It claims that $\alpha = \frac{b-\overline{b}}{1-\sigma}$.

Thus,

$$w(t) = \frac{b - \overline{b}}{1 - \sigma} t.$$

Since $1 - \sigma > 0, b - \overline{b} < 0$, we have a contradiction that $w(t) < 0$ for $t \in (0,1)$. For the second case that there exists $\theta \in (0,1)$ such that $w(\theta) = 0$, $w(t) > 0$ for $t \in (\theta, 1]$ and $w(t) \leq 0$ in $[0, \theta]$. Thus, $w''(t) = 0$ on $[\theta, 1]$, and w is a linear function on $[\theta, 1]$. Since $w(1) = \int_0^1 g(s)w(s)ds + b - \overline{b} \leq \int_\theta^1 g(s)w(s)ds + b - \overline{b}$,

$$w(1) - \int_\theta^1 g(s)w(s)ds \leq b - \overline{b} < 0.$$

So let $r = w(1) - \int_\theta^1 g(s)w(s)ds$, then $r < 0$. From the fact that w is linear on $[\theta, 1]$, for any $s \in (\theta, 1]$,

$$w'(s) = w'(\theta). \tag{16}$$

Integrate (16) from t to 1 for $t \in [\theta, 1]$, then

$$w(1) - w(t) = w'(\theta)(1 - t). \tag{17}$$

Putting by $t=\theta$, we have

$$w'(\theta)=\frac{1}{1-\theta}w(1)=\frac{1}{1-\theta}[\int_\theta^1 g(s)w(s)ds+r]. \tag{18}$$

Substitute (18) into (17), we have

$$\begin{aligned} w(t)&=w(1)-w'(\theta)(1-t)\\ &=\int_\theta^1 g(s)w(s)ds+r-\frac{1}{1-\theta}[\int_\theta^1 g(s)w(s)ds+r](1-t)\\ &=(1-\frac{1-t}{1-\theta})\int_\theta^1 g(s)w(s)ds+(1-\frac{1-t}{1-\theta})r\\ &=(\frac{t-\theta}{1-\theta})[\int_\theta^1 g(s)w(s)ds+r]. \end{aligned}$$

Multiplying $g(t)$ to both sides,

$$g(t)w(t)=\frac{t-\theta}{1-\theta}g(t)[\int_\theta^1 g(s)w(s)ds+r]. \tag{19}$$

Integrate (19) from θ to 1, then

$$\int_\theta^1 g(t)w(t)dt=[\int_\theta^1 \frac{t-\theta}{1-\theta}g(t)dt]\cdot[\int_\theta^1 g(s)w(s)ds+r],$$

and

$$[1-\int_\theta^1 \frac{t-\theta}{1-\theta}g(t)dt]\int_\theta^1 g(t)w(t)dt=r\int_\theta^1 \frac{t-\theta}{1-\theta}g(t)dt. \tag{20}$$

Since $\frac{t-\theta}{1-\theta}\le t$, we have

$$0<\int_\theta^1 \frac{t-\theta}{1-\theta}g(t)dt<\int_\theta^1 tg(t)dt<\int_0^1 tg(t)dt=\sigma<1.$$

Then the left hand side of (20) is positive but the right hand side of (20) is negative and it's contradiction. For the last case that there exists $(a_2,b_2)\subsetneq(0,1)$ such that $w(a_2)=0=w(b_2)$ and $w(t)>0$ for $t\in(a_2,b_2)$, then this implies that $w''(t)=0$ in (a_2,b_2) and thus $w(t)\equiv 0$ in (a_2,b_2). Then we get a desired contradiction again. From these all cases, we can conclude that $\Omega_0=\varnothing$. Thus

$$0\le u_b(t)\le\overline{u}(t)$$

for all $t\in[0,1]$. From the definition of F, u_b is the positive solution of (P_b) for $b\in[0,\bar{b})$ and we complete the proof.

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