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CERTAIN IDENTITIES ASSOCIATED WITH CHARACTER FORMULAS, CONTINUED FRACTION AND COMBINATORIAL PARTITION IDENTITIES

M. P. Chaudhary and Junesang Choi*

Abstract. Folsom [10] investigated character formulas and Chaudhary [7] expressed those formulas in terms of continued fraction identities. Andrews *et al.* [2] introduced and investigated combinatorial partition identities. By using and combining known formulas, we aim to present certain interrelationships among character formulas, combinatorial partition identities and continued partition identities.

1. Introduction and Preliminaries

Throughout this paper, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ denote the sets of positive integers, integers, and complex numbers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The following q-notations are recalled (see, *e.g.*, [13, Chapter 6]): The *q-shifted factorial* $(a;q)_n$ is defined by

$$(a;q)_n := \begin{cases} 1 & (n=0) \\ \displaystyle\prod_{k=0}^{n-1} \left(1 - a\, q^k\right) & (n \in \mathbb{N}), \end{cases} \tag{1.1}$$

where $a,\, q \in \mathbb{C}$ and it is assumed that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$. We also write

$$\begin{aligned} (a;q)_\infty &:= \prod_{k=0}^{\infty} \left(1 - a\, q^k\right) \\ &= \prod_{k=1}^{\infty} \left(1 - a\, q^{k-1}\right) \quad (a,\, q \in \mathbb{C};\ |q| < 1). \end{aligned} \tag{1.2}$$

It is noted that, when $a \neq 0$ and $|q| \geqq 1$, the infinite product in (1.2) diverges. So, whenever $(a;q)_\infty$ is involved in a given formula, the constraint $|q| < 1$ will be tacitly assumed.

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* Corresponding author.

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The following notations are also frequently used:

$$(a_1, a_2, \cdots, a_m; q)_n := (a_1; q)_n \, (a_2; q)_n \cdots (a_m; q)_n \tag{1.3}$$

and

$$(a_1, a_2, \cdots, a_m; q)_\infty := (a_1; q)_\infty \, (a_2; q)_\infty \cdots (a_m; q)_\infty . \tag{1.4}$$

Ramanujan defined the general theta function $f(a, b)$ as follows (see, for details, [3, p. 31, Eq.(18.1)] and [5]; see also [1]):

$$\begin{aligned} f(a, b) &= 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} (a^n + b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} = f(b, a) \quad (|ab| < 1). \end{aligned} \tag{1.5}$$

We find from (1.5) that

$$f(a, b) = a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} f(a(ab)^n, b(ab)^{-n}) = f(b, a) \quad (n \in \mathbb{Z}). \tag{1.6}$$

Ramanujan also rediscovered the Jacobi's famous triple-product identity (see [3, p. 35, Entry 19]):

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty , \tag{1.7}$$

which was first proved by Gauss.

Several q-series identities emerging from Jacobi's triple-product identity (1.7) are worthy of note here (see [3, pp. 36-37, Entry 22]):

$$\begin{aligned} \phi(q) &:= \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \\ &= \{(-q; q^2)_\infty\}^2 (q^2; q^2)_\infty = \frac{(-q; q^2)_\infty (q^2; q^2)_\infty}{(q; q^2)_\infty (-q^2; q^2)_\infty}; \end{aligned} \tag{1.8}$$

$$\psi(q) := f(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}; \tag{1.9}$$

$$\begin{aligned} f(-q) &:= f(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} \\ &= \sum_{n=0}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n q^{\frac{n(3n+1)}{2}} = (q; q)_\infty . \end{aligned} \tag{1.10}$$

Equation (1.10) is known as Euler's *Pentagonal Number Theorem.* The following q-series identity:

$$(-q; q)_\infty = \frac{1}{(q; q^2)_\infty} = \frac{1}{\chi(-q)} \tag{1.11}$$

provides the *analytic equivalence* of Euler's famous theorem: *The number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts.*

We also recall the Rogers-Ramanujan continued fraction of $R(q)$:

$$\begin{aligned} R(q) &:= q^{\frac{1}{5}} \frac{H(q)}{G(q)} = q^{\frac{1}{5}} \frac{f(-q,-q^4)}{f(-q^2,-q^3)} = q^{\frac{1}{5}} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \\ &= \frac{q^{\frac{1}{5}}}{1+} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \cdots \quad (|q|<1). \end{aligned} \tag{1.12}$$

Here $G(q)$ and $H(q)$ are widely investigated Roger-Ramanujan identities defined by

$$\begin{aligned} G(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^5)}{f(-q,-q^4)} \\ &= \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} = \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty}{(q;q)_\infty}; \end{aligned} \tag{1.13}$$

$$\begin{aligned} H(q) &:= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{f(-q^5)}{f(-q^2,-q^3)} \\ &= \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = \frac{(q;q^5)_\infty (q^4;q^5)_\infty (q^5;q^5)_\infty}{(q;q)_\infty}; \end{aligned} \tag{1.14}$$

and the functions $f(a,b)$ and $f(-q)$ are given in (1.5) and (1.10), respectively. For a detailed historical account of (and for various investigated developments stemming from) the Rogers-Ramanujan continued fraction (1.12) and identities (1.13) and (1.14), the interested reader may refer to the monumental work [3, p. 77 et seq.] (see also [1, 4]).

The following continued fraction was recalled in [6, p. 5, Eq. (2.8)] from an earlier work cited therein: For $|q|<1$,

$$\begin{aligned} (q^2;q^2)_\infty (-q;q)_\infty &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \\ &= \frac{1}{1-} \frac{q}{1+} \frac{q(1-q)}{1-} \frac{q^3}{1+} \frac{q^2(1-q^2)}{1-} \frac{q^5}{1+} \frac{q^3(1-q^3)}{1-\cdots}; \end{aligned} \tag{1.15}$$

$$\frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} = \frac{1}{1+} \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \frac{q^4}{1+} \frac{q^5}{1+} \frac{q^6}{1+} \cdots; \tag{1.16}$$

$$C(q) := \frac{(q^2;q^5)_\infty (q^3;q^5)_\infty}{(q;q^5)_\infty (q^4;q^5)_\infty} = 1 + \frac{q}{1+} \frac{q^2}{1+} \frac{q^3}{1+} \frac{q^4}{1+} \frac{q^5}{1+} \frac{q^6}{1+} \cdots. \tag{1.17}$$

Andrews *et al.* [2] investigated new double summation hypergeometric q-series representations for several families of partitions and further explored the role of double series in combinatorial partition identities by introducing the following general family:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn} r(l,u,v,w;n), \tag{1.18}$$

where

$$r(l,u,v,w:n) := \sum_{j=0}^{[\frac{n}{u}]} (-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}(q^{uv};q^{uv})_j}. \tag{1.19}$$

The following interesting special cases of (1.18) are recalled (see [2, p. 106, Theorem 3]; see also [12]):

$$R(2,1,1,1,2,2) = (-q;q^2)_\infty; \tag{1.20}$$

$$R(2,2,1,1,2,2) = (-q^2;q^2)_\infty; \tag{1.21}$$

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{1.22}$$

Here, in this paper, we aim to present certain interrelations between character formulas, combinatorial partition identities and continued partition identities associated with the identities in (1.15)-(1.17) and (1.20)-(1.22).

2. A Set of Preliminary Results

Here we recall the following q-product identities (see [7]) for the verification of the main results in Section 3.

$$\begin{aligned} f(-q) = &- 4q \cdot \widehat{\Theta}_{12}^{-1}\left(q^{\frac{1}{2}} \cdot \mathrm{tr}_{L(\bigwedge_{(1)};\ 13)} q^{L_0} + q^{-\frac{1}{2}} \cdot \mathrm{tr}_{L(\bigwedge_{(5)};\ 13)} q^{L_0}\right) + 4q\widehat{\beta}_{12,1}(\tau) \\ &+ \frac{(q^2;q^2)_\infty (q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty \{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^2}{(q^4,q^{16};q^{20})_\infty \{(q^{16};q^{16})_\infty\}^2} \\ &\times \left\{\frac{1}{1-}\ \frac{q^8}{1+}\ \frac{q^8(1-q^8)}{1-}\ \frac{q^{24}}{1+}\ \frac{q^{16}(1-q^{16})}{1-}\ \frac{q^{40}}{1+}\ \frac{q^{24}(1-q^{24})}{1-} \cdots\right\}^2 \\ &\times \left\{\frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-} \cdots\right\}^3 \\ &\times \left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+} \cdots\right\}; \end{aligned} \tag{2.1}$$

$$\begin{aligned}\phi(q) =& - 2q\cdot\widehat{\Theta}_{12}^{-1}\big(q^{\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(1)};\ 13)}q^{L_0} + q^{-\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(5)};\ 13)}q^{L_0}\big) +\ 2q\widehat{\beta}_{12,1}(\tau)\\ &+ \frac{(q^2;q^2)_\infty(q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty\{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^2}{(q^4,q^{16};q^{20})_\infty\{(q^{16};q^{16})_\infty\}^2}\\ &\times\left\{\frac{1}{1-}\ \frac{q^8}{1+}\ \frac{q^8(1-q^8)}{1-}\ \frac{q^{24}}{1+}\ \frac{q^{16}(1-q^{16})}{1-}\ \frac{q^{40}}{1+}\ \frac{q^{24}(1-q^{24})}{1-}\cdots\right\}^2\\ &\times\left\{\frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-}\cdots\right\}^3\\ &\times\left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\};\end{aligned}\tag{2.2}$$

$$\begin{aligned}\chi(-q) =& -q\cdot\widehat{\Theta}_{12}^{-1}\big(q^{\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(1)};\ 13)}q^{L_0} + q^{-\frac{1}{2}}\cdot\mathrm{tr}_{L(\bigwedge_{(5)};\ 13)}q^{L_0}\big)\\ &+\ q\widehat{\beta}_{12,1}(\tau) + \frac{(q^4;q^4)_\infty(q^6;q^6)_\infty}{\{(q^{12};q^{12})_\infty\}^2\{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})_\infty\}^2}\\ &\times\left\{\frac{1}{1-}\ \frac{q^3}{1+}\ \frac{q^3(1-q^3)}{1-}\ \frac{q^9}{1+}\ \frac{q^6(1-q^6)}{1-}\ \frac{q^{15}}{1+}\ \frac{q^9(1-q^9)}{1-}\cdots\right\}\\ &\times\left\{\frac{1}{1-}\ \frac{q^{10}}{1+}\ \frac{q^{10}(1-q^{10})}{1-}\ \frac{q^{30}}{1+}\ \frac{q^{20}(1-q^{20})}{1-}\ \frac{q^{50}}{1+}\ \frac{q^{30}(1-q^{30})}{1-}\cdots\right\}^2\\ &\times\left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\}^2\\ &\times\left\{1+\frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\}^2;\end{aligned}\tag{2.3}$$

$$\begin{aligned}\upsilon(q) =& -\ q\cdot\widehat{\Theta}_{12}^{-1}\big(\mathrm{tr}_{L(\bigwedge_{(-2)};\ 13)}q^{L_0} + \mathrm{tr}_{L(\bigwedge_{(2)};\ 13)}q^{L_0}\big) +\ q\widehat{\beta}_{12,-2}(\tau)\\ &+ \frac{(q^4;q^4)_\infty}{\{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})_\infty\}^2}\\ &\times\left\{\frac{1}{1-}\ \frac{q^{10}}{1+}\ \frac{q^{10}(1-q^{10})}{1-}\ \frac{q^{30}}{1+}\ \frac{q^{20}(1-q^{20})}{1-}\ \frac{q^{50}}{1+}\ \frac{q^{30}(1-q^{30})}{1-}\cdots\right\}^2\\ &\times\left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\}^2\\ &\times\left\{1+\frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\}^2;\end{aligned}\tag{2.4}$$

$$\begin{aligned}\rho(q) =& -\ \frac{1}{2}\cdot\widehat{\Theta}_6^{-1}\big(\mathrm{tr}_{L(\wedge_{(-1)};\ 7)}q^{L_0}+\mathrm{tr}_{L(\wedge_{(1)};\ 7)}q^{L_0}\big)+\frac{1}{2}\widehat{\beta}_{6,-1}(\tau)+\frac{3}{2}\frac{1}{(q^2;q^2)_\infty}\\ &\times\left\{\frac{1}{1-}\ \frac{q^3}{1+}\ \frac{q^3(1-q^3)}{1-}\ \frac{q^9}{1+}\ \frac{q^6(1-q^6)}{1-}\ \frac{q^{15}}{1+}\ \frac{q^9(1-q^9)}{1-}\cdots\right\}^2;\end{aligned} \tag{2.5}$$

$$\begin{aligned}\sigma(-q) =& q^2\cdot\widehat{\Theta}_{36}^{-1}\big(q^{\frac{3}{2}}\cdot\mathrm{tr}_{L(\wedge_{(3)};\ 37)}q^{L_0}+q^{-\frac{3}{2}}\cdot\mathrm{tr}_{L(\wedge_{(15)};\ 37)}q^{L_0}\big)-\ q^2\widehat{\beta}_{36,3}(\tau)\\ &+\frac{\{(q^2,q^{10};q^{12})_\infty\}^2(q^6;q^{12})_\infty}{(q;q^2)_\infty}\\ &\times\left\{\frac{1}{1-}\ \frac{q^6}{1+}\ \frac{q^6(1-q^6)}{1-}\ \frac{q^{18}}{1+}\ \frac{q^{12}(1-q^{12})}{1-}\ \frac{q^{30}}{1+}\ \frac{q^{18}(1-q^{18})}{1-}\cdots\right\};\end{aligned} \tag{2.6}$$

$$\begin{aligned}A(q^2) =& q\cdot\widehat{\Theta}_8^{-1}\mathrm{tr}_{L(\wedge_{(2)};\ 9)}q^{L_0}-q\widehat{\eta}_{8,2}(\tau)-q(-q^2;q^2)_\infty(-q^4;q^4)_\infty\\ &\times\left\{\frac{1}{1-}\ \frac{q^4}{1+}\ \frac{q^4(1-q^4)}{1-}\ \frac{q^{12}}{1+}\ \frac{q^8(1-q^8)}{1-}\ \frac{q^{20}}{1+}\ \frac{q^{12}(1-q^{12})}{1-}\cdots\right\};\end{aligned} \tag{2.7}$$

$$\begin{aligned}\mu(q^4) =& -\ 2q\cdot\widehat{\Theta}_4^{-1}\mathrm{tr}_{L(\wedge_{(0)};\ 5)}q^{L_0}+2q\widehat{\eta}_{4,0}(\tau)+\frac{12\,(q^8;q^8)_\infty}{(q;q)_\infty(q^2;q^4)_\infty}\\ &\times\left\{\frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-}\cdots\right\};\end{aligned} \tag{2.8}$$

$$\begin{aligned}\phi(q^4) =& -\ 2q\cdot\widehat{\Theta}_{12}^{-1}\mathrm{tr}_{L(\wedge_{(4)};\ 13)}q^{L_0}+2q\widehat{\eta}_{12,4}(\tau)\\ &+\frac{(q^2,q^4,q^6;q^8)_\infty(q^{12};q^{24})_\infty\{(q^3;q^6)_\infty\}^2}{(q;q)_\infty}\\ &\times\left\{\frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-}\cdots\right\}\\ &\times\left\{\frac{1}{1-}\ \frac{q^6}{1+}\ \frac{q^6(1-q^6)}{1-}\ \frac{q^{18}}{1+}\ \frac{q^{12}(1-q^{12})}{1-}\ \frac{q^{30}}{1+}\ \frac{q^{18}(1-q^{18})}{1-}\cdots\right\};\end{aligned} \tag{2.9}$$

$$\begin{aligned}\psi(q^4) =& -\ q^3\cdot\widehat{\Theta}_{12}^{-1}\mathrm{tr}_{L(\wedge_{(0)};\ 13)}q^{L_0}+q^3\widehat{\eta}_{12,0}(\tau)+\frac{(q^4,q^{12},q^{20},q^{24};q^{24})_\infty q^3}{(q^8,q^{16};q^{24})_\infty(q^3;q^3)_\infty}\\ &\times\left\{\frac{1}{1-}\ \frac{q}{1+}\ \frac{q(1-q)}{1-}\ \frac{q^3}{1+}\ \frac{q^2(1-q^2)}{1-}\ \frac{q^5}{1+}\ \frac{q^3(1-q^3)}{1-}\cdots\right\};\end{aligned} \tag{2.10}$$

$$\begin{aligned}\phi(q) =&2q\cdot\widehat{\Theta}_{10}^{-1}\mathrm{tr}_{L(\bigwedge_{(1)};\,11)}q^{L_0}-2q\widehat{\eta}_{10,1}(\tau)+\frac{j(-q^2;q^5)}{j(q^2;q^{10})}\\&\times\left\{\frac{1}{1-}\,\frac{q^5}{1+}\,\frac{q^5(1-q^5)}{1-}\,\frac{q^{15}}{1+}\,\frac{q^{10}(1-q^{10})}{1-}\,\frac{q^{25}}{1+}\,\frac{q^{15}(1-q^{15})}{1-}\cdots\right\};\end{aligned}\tag{2.11}$$

$$\begin{aligned}\psi(q) =&2q\cdot\widehat{\Theta}_{10}^{-1}\mathrm{tr}_{L(\bigwedge_{(3)};\,11)}q^{L_0}-2q\widehat{\eta}_{10,3}(\tau)-q\frac{j(-q;q^5)}{j(q^4;q^{10})}\\&\times\left\{\frac{1}{1-}\,\frac{q^5}{1+}\,\frac{q^5(1-q^5)}{1-}\,\frac{q^{15}}{1+}\,\frac{q^{10}(1-q^{10})}{1-}\,\frac{q^{25}}{1+}\,\frac{q^{15}(1-q^{15})}{1-}\cdots\right\};\end{aligned}\tag{2.12}$$

$$\begin{aligned}X(-q^2) =&-2q\cdot\widehat{\Theta}_{40}^{-1}\big(\mathrm{tr}_{L(\bigwedge_{(18)};\,41)}q^{L_0}-\mathrm{tr}_{L(\bigwedge_{(2)};\,41)}q^{L_0}\big)+2q\widehat{\eta}_{40,18}(\tau)\\&-2q\widehat{\eta}_{40,2}(\tau)+\frac{(j(-q^2,q^{20})j(q^{12},q^{40})+2q(q^{40};q^{40})_\infty^3)}{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty j(q^8,q^{40})}\\&\times\left\{\frac{1}{1-}\,\frac{q^2}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^6}{1+}\,\frac{q^4(1-q^4)}{1-}\,\frac{q^{10}}{1+}\,\frac{q^6(1-q^6)}{1-}\cdots\right\};\end{aligned}\tag{2.13}$$

$$\begin{aligned}\chi(-q^2) =&-2q^3\cdot\widehat{\Theta}_{40}^{-1}\big(\mathrm{tr}_{L(\bigwedge_{(14)};\,41)}q^{L_0}+q^2\cdot\mathrm{tr}_{L(\bigwedge_{(6)};\,41)}q^{L_0}\big)+2q^3\widehat{\eta}_{40,14}(\tau)\\&+2q^5\widehat{\eta}_{40,6}(\tau)+q^2\frac{(2q(q^{40};q^{40})_\infty^3-j(-q^6,q^{20})^2j(q^4,q^{40}))}{(q^{20};q^{20})_\infty(q^{40};q^{40})_\infty j(q^{16},q^{40})}\\&\times\left\{\frac{1}{1-}\,\frac{q^2}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^6}{1+}\,\frac{q^4(1-q^4)}{1-}\,\frac{q^{10}}{1+}\,\frac{q^6(1-q^6)}{1-}\cdots\right\}.\end{aligned}\tag{2.14}$$

Here the universal mock theta functions (see, *e.g.*, [10, pp. 440-441]) $g_2(w;q)$ and $g_3(w;q)$ are given by Gordon and Mclntosh [11] who generalize the original mock theta functions of Ramanujan:

$$g_2(w;q):=\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{\frac{1}{2}(n^2+n)}}{(w;q)_{n+1}(qw^{-1};q)_{n+1}}\tag{2.15}$$

and

$$g_3(w;q):=\sum_{n=0}^{\infty}\frac{q^{(n^2+n)}}{(w;q)_{n+1}(qw^{-1};q)_{n+1}}.\tag{2.16}$$

The general form of Kac-Wakimoto character formula (see [10, p. 442]) is given as follows:

$$\mathrm{tr}_{L(\bigwedge_{(s)};\,r+1)}\cdot q^{L_0}:=2q^{\frac{r-1}{24}-\frac{s}{2}}\cdot\frac{\eta^2(2\tau)}{\eta^{r+3}(\tau)}\cdot L_{r,s}(\tau)\quad(r\in\mathbb{N};\ s\in\mathbb{Z}),\tag{2.17}$$

where L_0 is the energy operator or Hamiltonian,

$$L_{r,s}(\tau) := \sum_{k=(k_1,k_2,\ldots,k_r)\in\mathbb{Z}^r} \frac{q^{\frac{1}{2}\sum_i k_i(k_i+1)}}{1+q^{-s+\sum_i k^i}} \quad (r \in \mathbb{N};\ s \in \mathbb{Z}), \tag{2.18}$$

and the function $\eta(\tau)$ is the Dedekind η-function, a classical weight 1/2 modular form, defined by

$$\eta(\tau) := q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1-q^n). \tag{2.19}$$

It is noted that $L(\bigwedge_{(s)};\ r+1)$ is the irreducible $s\ell(r+1,1)^{\wedge}$ module with highest weight $\bigwedge_{(s)}$.

The function $j(x;q)$ (see [10, p. 454, Table 8]) is defined by

$$j(x;q) := (x;q)_\infty (x^{-1}q;q)_\infty (q;q)_\infty. \tag{2.20}$$

For the details of the other notations whose definitions are not given here, one may refer to the work [10].

3. Main Results

Here we state and prove certain interesting interrelations among character formulas, combinatorial partition identities and continued partition identities asserted by the following theorem.

Theorem 3.1. *Each of the following relationships holds true:*

$$\begin{aligned} f(-q) = &- 4q \cdot \widehat{\Theta}_{12}^{-1}\big(q^{\frac{1}{2}} \cdot \mathrm{tr}_{L(\bigwedge_{(1)};\ 13)} q^{L_0} + q^{-\frac{1}{2}} \cdot \mathrm{tr}_{L(\bigwedge_{(5)};\ 13)} q^{L_0}\big) + 4q\widehat{\beta}_{12,1}(\tau) \\ &+ \frac{(q^2;q^2)_\infty (q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty \{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^2}{(q^4,q^{16};q^{20})_\infty \{(q^{16};q^{16})_\infty\}^2} \\ &\times [R(8,8,1,1,1,2)]^2\,[R(1,1,1,1,1,2)]^3 \\ &\times \left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+} \cdots\right\}; \end{aligned} \tag{3.1}$$

$$\begin{aligned} \phi(q) = &- 2q \cdot \widehat{\Theta}_{12}^{-1}\big(q^{\frac{1}{2}} \cdot \mathrm{tr}_{L(\bigwedge_{(1)};\ 13)} q^{L_0} + q^{-\frac{1}{2}} \cdot \mathrm{tr}_{L(\bigwedge_{(5)};\ 13)} q^{L_0}\big) + 2q\widehat{\beta}_{12,1}(\tau) \\ &+ \frac{(q^2;q^2)_\infty (q^2,q^6,q^8,q^{10},q^{12},q^{14},q^{18};q^{20})_\infty \{(q^2,q^6,q^8,q^{10},q^{14};q^{16})_\infty\}^2}{(q^4,q^{16};q^{20})_\infty \{(q^{16};q^{16})_\infty\}^2} \\ &\times [R(8,8,1,1,1,2)]^2\,[R(1,1,1,1,1,2)]^3 \\ &\times \left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+} \cdots\right\}; \end{aligned} \tag{3.2}$$

$$\begin{aligned}\chi(-q) = &-q\cdot\widehat{\Theta}_{12}^{-1}\big(q^{\frac{1}{2}}\cdot \mathrm{tr}_{L(\bigwedge_{(1)};\ 13)}q^{L_0} + q^{-\frac{1}{2}}\cdot \mathrm{tr}_{L(\bigwedge_{(5)};\ 13)}q^{L_0}\big)\\ &+ q\widehat{\beta}_{12,1}(\tau) + \frac{(q^4;q^4)_\infty (q^6;q^6)_\infty}{\{(q^{12};q^{12})_\infty\}^2\{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})_\infty\}^2}\\ &\times [R(3,3,1,1,1,2)]\,[R(10,10,1,1,1,2)]^2\\ &\times \left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\}^2\\ &\times \left\{1+\frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\}^2;\end{aligned} \tag{3.3}$$

$$\begin{aligned}\upsilon(q) = &-\ q\cdot\widehat{\Theta}_{12}^{-1}\big(\mathrm{tr}_{L(\bigwedge_{(-2)};\ 13)}q^{L_0} + \mathrm{tr}_{L(\bigwedge_{(2)};\ 13)}q^{L_0}\big)\\ &+\ q\widehat{\beta}_{12,-2}(\tau) + \frac{(q^4;q^4)_\infty [R(10,10,1,1,1,2)]^2}{\{(q^2,q^6,q^{14},q^{18},q^{20};q^{20})_\infty\}^2}\\ &\times \left\{\frac{1}{1+}\ \frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\}^2\\ &\times \left\{1+\frac{q^4}{1+}\ \frac{q^8}{1+}\ \frac{q^{12}}{1+}\ \frac{q^{16}}{1+}\ \frac{q^{20}}{1+}\ \frac{q^{24}}{1+}\cdots\right\}^2;\end{aligned} \tag{3.4}$$

$$\begin{aligned}\rho(q) = &-\ \frac{1}{2}\cdot\widehat{\Theta}_{6}^{-1}\big(\mathrm{tr}_{L(\bigwedge_{(-1)};\ 7)}q^{L_0} + \mathrm{tr}_{L(\bigwedge_{(1)};\ 7)}q^{L_0}\big)\\ &+ \frac{1}{2}\widehat{\beta}_{6,-1}(\tau) + \frac{3}{2\,(q^2;q^2)_\infty}\big[R(3,3,1,1,1,2)\big]^2;\end{aligned} \tag{3.5}$$

$$\begin{aligned}\sigma(-q) = &q^2\cdot\widehat{\Theta}_{36}^{-1}\big(q^{\frac{3}{2}}\cdot \mathrm{tr}_{L(\bigwedge_{(3)};\ 37)}q^{L_0} + q^{-\frac{3}{2}}\cdot \mathrm{tr}_{L(\bigwedge_{(15)};\ 37)}q^{L_0}\big)\\ &-\ q^2\widehat{\beta}_{36,3}(\tau) + \frac{(q^2,q^{10};q^{12})_\infty^2 (q^6;q^{12})_\infty}{(q;q^2)_\infty}\, R(6,6,1,1,1,2);\end{aligned} \tag{3.6}$$

$$\begin{aligned}A(q^2) = &q\cdot\widehat{\Theta}_{8}^{-1}\cdot \mathrm{tr}_{L(\bigwedge_{(2)};9)}q^{L_0}\\ &- q\cdot\widehat{\eta}_{8,2}(\tau) - q(-q^2;q^2)_\infty(-q^4;q^4)_\infty\, R(4,4,1,1,1,2);\end{aligned} \tag{3.7}$$

$$\mu(q^4) = -\ 2q\cdot\widehat{\Theta}_{4}^{-1}\cdot \mathrm{tr}_{L(\bigwedge_{(0)};\ 5)}q^{L_0} + 2q\,\widehat{\eta}_{4,0}(\tau) + \frac{12\,(q^8;q^8)_\infty R(1,1,1,1,1,2)}{(q;q)_\infty (q^2;q^4)_\infty}; \tag{3.8}$$

$$\begin{aligned}\phi(q^4) = &-\ 2q\cdot\widehat{\Theta}_{12}^{-1}\cdot \mathrm{r}_{L(\bigwedge_{(4)};\ 13)}q^{L_0} + 2q\,\widehat{\eta}_{12,4}(\tau)\\ &+ \frac{(q^2,q^4,q^6;q^8)_\infty (q^{12};q^{24})_\infty (q^3;q^6)_\infty^2}{(q;q)_\infty} R(1,1,1,1,1,2)\, R(6,6,1,1,1,2);\end{aligned} \tag{3.9}$$

$$\begin{aligned}\psi(q^4) = &- q^3 \cdot \widehat{\Theta}_{12}^{-1} \cdot \mathrm{tr}_{L(\bigwedge_{(0)};\ 13)}\, q^{L_0} + q^3\, \widehat{\eta}_{12,0}(\tau) \\ &+ \frac{(q^4, q^{12}, q^{20}, q^{24}; q^{24})_\infty}{(q^8, q^{16}; q^{24})_\infty}\, \frac{q^3}{(q^3; q^3)_\infty}\, R(1,1,1,1,1,2);\end{aligned} \tag{3.10}$$

$$\phi(q) = 2q\cdot\widehat{\Theta}_{10}^{-1}\cdot\mathrm{tr}_{L(\bigwedge_{(1)};\ 11)} q^{L_0} - 2q\,\widehat{\eta}_{10,1}(\tau) + \frac{j(-q^2; q^5)}{j(q^2; q^{10})}\, R(5,5,1,1,1,2); \tag{3.11}$$

$$\psi(q) = 2q \cdot \widehat{\Theta}_{10}^{-1} \cdot \mathrm{tr}_{L(\bigwedge_{(3)};\ 11)} q^{L_0} - 2q\,\widehat{\eta}_{10,3}(\tau) - q\frac{j(-q; q^5)}{j(q^4; q^{10})}\, R(5,5,1,1,1,2); \tag{3.12}$$

$$\begin{aligned}X(-q^2) = &-2q \cdot \widehat{\Theta}_{40}^{-1}\big(\mathrm{tr}_{L(\bigwedge_{(18)};\ 41)} q^{L_0} - \mathrm{tr}_{L(\bigwedge_{(2)};\ 41)} q^{L_0}\big) + 2q\,\widehat{\eta}_{40,18}(\tau) \\ &- 2q\,\widehat{\eta}_{40,2}(\tau) + \frac{(j(-q^2, q^{20}) j(q^{12}, q^{40}) + 2q(q^{40}; q^{40})_\infty^3)}{(q^{20}; q^{20})_\infty (q^{40}; q^{40})_\infty j(q^8, q^{40})}\, R(2,2,1,1,1,2);\end{aligned} \tag{3.13}$$

$$\begin{aligned}\chi(-q^2) = &-2q^3 \cdot \widehat{\Theta}_{40}^{-1}\big(\mathrm{tr}_{L(\bigwedge_{(14)};\ 41)} q^{L_0} + q^2 \cdot \mathrm{tr}_{L(\bigwedge_{(6)};\ 41)} q^{L_0}\big) + 2q^3\widehat{\eta}_{40,14}(\tau) \\ &+ 2q^5\widehat{\eta}_{40,6}(\tau) + q^2\frac{(2q(q^{40}; q^{40})_\infty^3 - j(-q^6, q^{20})^2 j(q^4, q^{40}))}{(q^{20}; q^{20})_\infty (q^{40}; q^{40})_\infty j(q^{16}, q^{40})}\, R(2,2,1,1,1,2).\end{aligned} \tag{3.14}$$

Proof. Applying the identity (1.15) with (1.22) ($m = 1$ and $m = 8$) in (2.1) and (2.2), respectively, yields the desired assertions (3.1) and (3.2).

Using the identity (1.15) with (1.22) ($m = 3$ and $m = 10$) in (2.3) yields the desired relation (3.3). Applying the identity (1.15) with (1.22) ($m = 10$) in (2.4) proves the desired result (3.4).

Using the identity (1.22) ($m = 3$), we get

$$R(3,3,1,1,1,2) = \frac{(q^6; q^6)_\infty}{(q^3; q^6)_\infty}. \tag{3.15}$$

Applying (3.15) with (1.15) to the identity (2.5) yields the relation (3.5). Similarly, using the identity (1.22) ($m = 6$, 4, 1) gives, like (3.15), the three corresponding identities, which are applied with (1.15) in the identities (2.6), (2.7), and (2.8), respectively, yields the desired assertions (3.6), (3.7), and (3.8).

Using the identity (1.22) ($m = 1$ and $m = 6$), we get

$$R(1,1,1,1,1,2) = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} \quad \text{and} \quad R(6,6,1,1,1,2) = \frac{(q^{12}; q^{12})_\infty}{(q^6; q^{12})_\infty},$$

which are applied with (1.15) in the identity (2.9), yields the result (3.9).

A similar argument as in the above process can establish the other identities (3.10)-(3.14). We, therefore, choose to skip the details involved.

□

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M. P. Chaudhary

International Scientific Research and Welfare Organization, New Delhi-110018, India

E-mail address: `mpchaudhary_2000@yahoo.com`

J. Choi

Department of Mathematics, Dongguk University, Gyeongju 780-714, Republic of Korea

E-mail address: `junesang@mail.dongguk.ac.kr`