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A new flexible Weibull distribution

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Abstract

Many of studies have suggested the modifications on Weibull distribution to model the non-monotone hazards. In this paper, we combine two cumulative hazard functions and propose a new modified Weibull distribution function. The newly suggested distribution will be named as a new flexible Weibull distribution. Corresponding hazard function of the proposed distribution shows flexible (monotone or non-monotone) shapes. We study the characteristics of the proposed distribution that includes ageing behavior, moment, and order statistic. We also discuss an estimation method for its parameters. The performance of the proposed distribution is compared with existing modified Weibull distributions using various types of hazard functions. We also use real data example to illustrate the efficiency of the proposed distribution.

Keywords: Weibull distribution, modified Weibull distribution, bathtub shape, hazard function, maximum likelihood estimate, reliability

1. Introduction

Weibull distribution has been widely used in various fields such as reliability engineering due to its flexibility in fitting failure times, where its survival function is given as

$$\bar{F}(t) = \exp\left(-(\theta t)^{\lambda}\right), \quad t > 0,$$

with parameters $\lambda, \theta > 0$.
The corresponding hazard function (failure rate function) can then be written as

$$h(t) = \lambda\theta^{\lambda}t^{\lambda-1}.$$

However, Weibull distribution is inappropriate to model the non-monotone hazard rate such as bathtub-shaped hazard rate because Weibull distribution can produce only monotonic hazard rates. Hence, many of modifications of Weibull distribution have been suggested, which can fit the non-monotone hazard rate.

It may be a natural approach to combine two different survival functions (with increasing and decreasing hazards) and generate a distribution function as

$$\bar{F}(t) = \alpha\bar{F}_1(t) + (1-\alpha)\bar{F}_2(t),$$

where $0 < \alpha < 1$, which is well-known as a mixture of distributions, or

$$\bar{F}(t) = \bar{F}_1^{\alpha}(t) \times \bar{F}_2^{\beta}(t), \tag{1.1}$$

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with parameters $\alpha, \beta > 0$.

The cumulative distribution function can be written in terms of the cumulative hazard function $H(x)$ as

$$F(t) = 1 - e^{-H(t)},$$

where the cumulative hazard function satisfies the following properties as

1. $H(t)$ is nonnegative and increasing,
2. $\lim_{t\to 0} H(t) = 0$ and $\lim_{t\to\infty} H(t) = \infty$.

Hence, we can also combine two cumulative hazard functions to generate a distribution function as

$$H(t) = \alpha H_1(t) + \beta H_2(t), \tag{1.2}$$

which is eventually equivalent to (1.1), or

$$H(t) = H_1^{\alpha}(t) \times H_2^{\beta}(t), \tag{1.3}$$

with parameters $\alpha, \beta > 0$. We can introduce further an intercept parameter in (1.3) to add more flexibility.

The modified Weibull distributions suggested by Xie and Lai (1995), Almalki and Yuan (2013), and Lemonte *et al.* (2014) belong to the class (1.2), and the modified Weibull distributions suggested by Xie *et al.* (2002), Lai *et al.* (2003), Nadarajah and Kotz (2005), Bebbington *et al.* (2007) and Aryal and Elbatal (2015) belong to the class (1.3), and Park and Park (2016) discussed its generalization. Only the difference in past works is the choice of cumulative hazard functions. If we combine different types of hazard functions, where one is increasing and the other is decreasing, the hazard functions of the above distribution functions are expected to be flexible. However, the popular choices in past works include $\exp(t)$ and $\log(t)$, which are not appropriate as cumulative hazard functions.

In this paper, we choose $\exp(t) - 1$ and $\log(t + 1)$ as two cumulative hazard functions and produce a new flexible Weibull distribution (NFW) by following the aforementioned approach (1.3) as

$$\log H_{NFW}(t) = \mu + \alpha \log(\exp(t) - 1) + \beta \log(\log(t + 1)).$$

We provide the properties including characterization of hazard function and ageing behavior. The proposed distribution is also compared with some current modified Weibull distributions for various types of distribution functions and real data example.

2. New flexible Weibull distribution and its properties

We propose a new modified Weibull distribution as

$$\log H_{NFW}(t; \mu, \alpha, \beta) = \mu + \alpha \log(\exp(t) - 1) + \beta \log(\log(t + 1)),$$

which we call new flexible Weibull distribution.

Since both $\exp(t) - 1$ and $\log(t + 1)$ satisfy the properties of the cumulative hazard function, the resulting new flexible Weibull distribution becomes theoretically rigorous and more flexible compared to other modified Weibull distributions. The hazard function corresponding to $\alpha \log(\exp(t) - 1)$ shows the bathtub-shape ($\alpha < 1$) or the increasing shape ($\alpha \geq 1$), whereas the hazard function corresponding to $\beta \log(\log(t+1))$ may be decreasing ($\beta \leq 1$) or upside-down bathtub-shaped ($\beta > 1$).

The cumulative hazard function of the new flexible Weibull distribution is given by

$$H_{NFW}(t;\mu,\alpha,\beta) = \exp(\mu)\{\exp(t)-1\}^{\alpha}\{\log(t+1)\}^{\beta},$$

and the corresponding hazard function has the following form

$$\begin{aligned} h_{NFW}(t;\mu,\alpha,\beta) = {} & \exp(\mu)\{\exp(t)-1\}^{\alpha-1}\{\log(t+1)\}^{\beta-1} \\ & \times \left[\alpha \exp(t)\log(t+1) + \{\exp(t)-1\}\frac{\beta}{t+1}\right]. \end{aligned}$$

The cumulative distribution function of the new flexible Weibull distribution can be written as

$$F_{NFW}(t;\mu,\alpha,\beta) = 1 - \exp\left[-\exp(\mu)\{\exp(t)-1\}^{\alpha}\{\log(t+1)\}^{\beta}\right],$$

and its probability density function can be obtained as

$$\begin{aligned} f_{NFW}(t;\mu,\alpha,\beta) = {} & \exp(\mu)\{\exp(t)-1\}^{\alpha-1}\{\log(t+1)\}^{\beta-1} \\ & \times \left[\alpha \exp(t)\log(t+1) + \{\exp(t)-1\}\frac{\beta}{t+1}\right] \\ & \times \exp\left[-\exp(\mu)\{\exp(t)-1\}^{\alpha}\{\log(t+1)\}^{\beta}\right]. \end{aligned}$$

Figures 1 and 2 shapes of hazard function $h_{NFW}(t)$ relating to the change of α and β, with μ fixed to 0. As shown in both figures, the hazard functions of the new flexible Weibull distribution can cover increasing, decreasing, bathtub-shaped, modified bathtub-shaped and upside-down bathtub-shaped failure rates.

We also study the limiting behavior of the hazard function of the hazard function. Note that $\lim_{t\to\infty} h_{NFW}(t;\mu,\alpha,\beta) = 0$ when α is close to 0 and $h_{NFW}(t)$ goes to ∞ otherwise. It is straightforward that $\lim_{t\to 0} h_{NFW}(t;\mu,\alpha,\beta) = 0$ when at least one of the parameters α or β is greater or equal to 1. However when neither α nor β is greater or equal to 1, the limiting behavior varies as follows, depending on the parameters. Figure 2 illustrates the different behaviors of hazard function as t goes to 0, by an example of changing β when α is fixed.

1. If $\alpha + \beta$ is less than 1,
$$\lim_{t\to 0} h_{NFW}(t;\mu,\alpha,\beta) = \infty.$$
2. If $\alpha + \beta$ is equal to 1,
$$\lim_{t\to 0} h_{NFW}(t;\mu,\alpha,\beta) = \exp(\mu).$$
3. If $\alpha + \beta$ is greater than 1,
$$\lim_{t\to 0} h_{NFW}(t;\mu,\alpha,\beta) = 0.$$

Limiting behavior plays an important role on the shape of the hazard function. The resulting hazard function is only decreasing if a limiting behavior is fixed, for example, to be $h(0) = \infty$ and

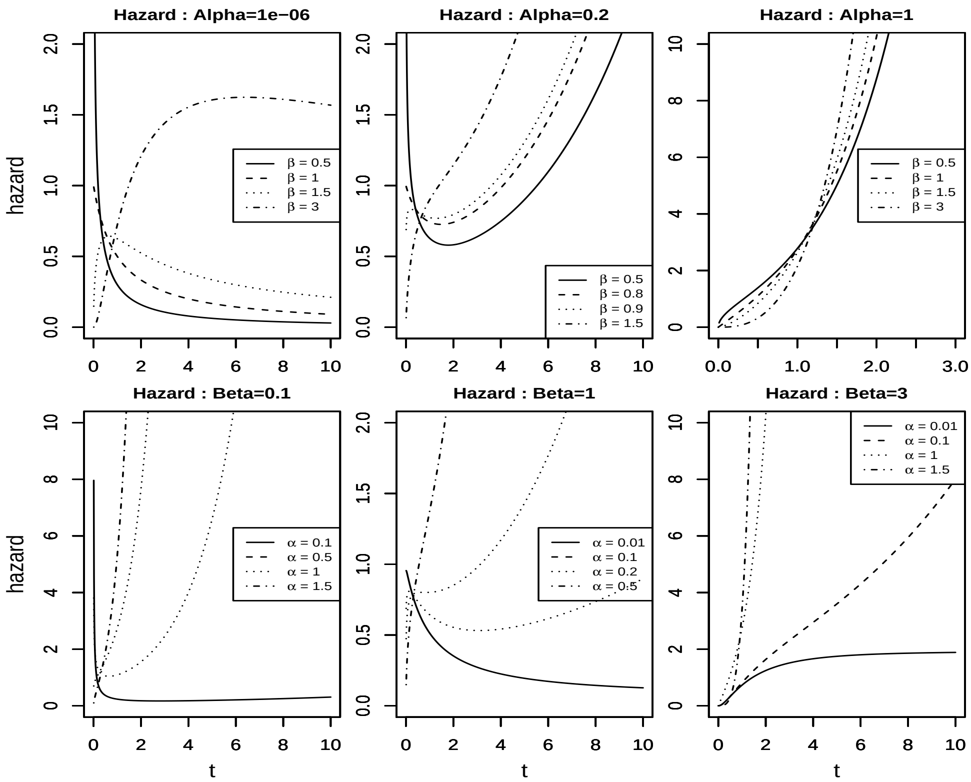

Figure 1: *Hazard function of new flexible Weibull distribution with* $\mu = 0$.

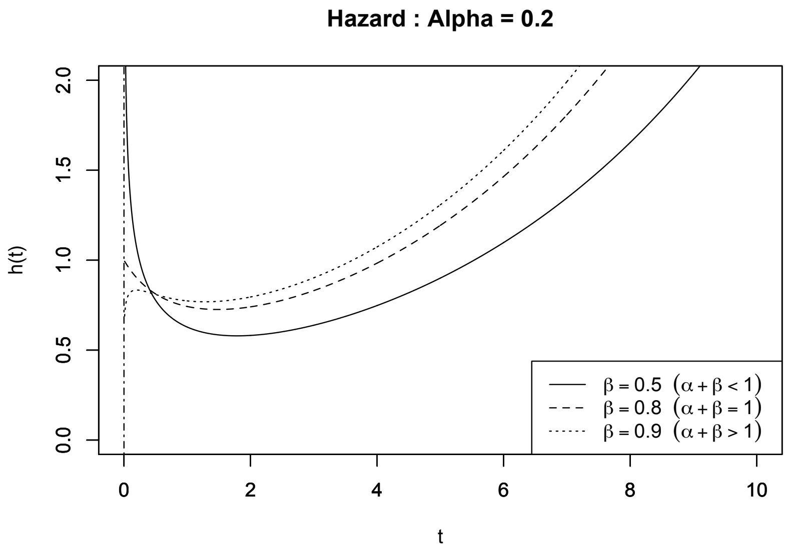

Figure 2: *Change of limiting behavior of hazard function when* $\alpha = 0.2$.

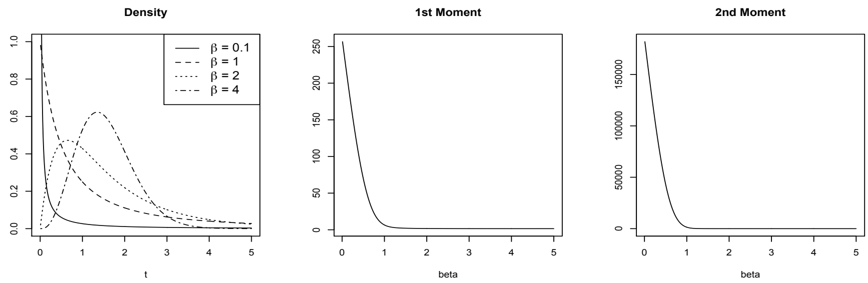

Figure 3: *Change of density and moment when* $\alpha = 1e - 06$.

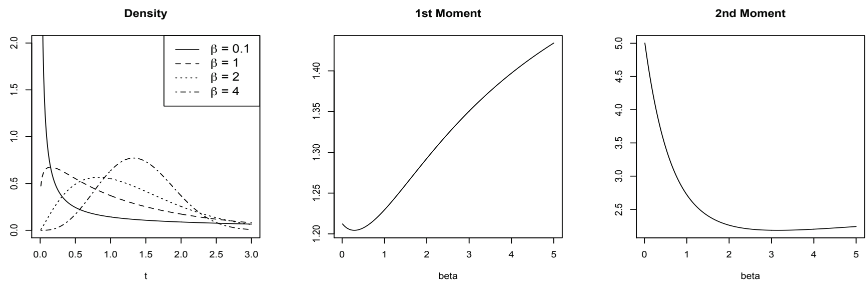

Figure 4: *Change of density and moment when* $\alpha = 0.2$.

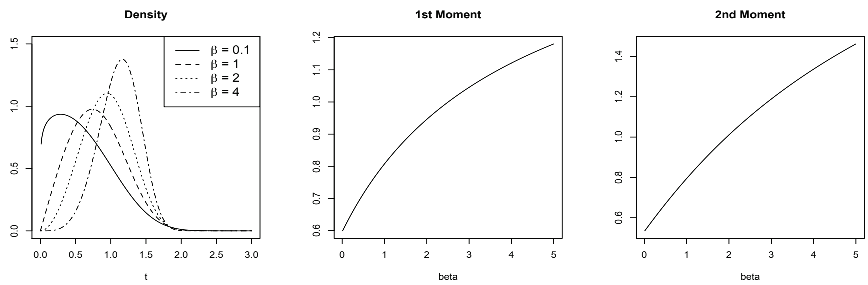

Figure 5: *Change of density and moment when* $\alpha = 1$.

$h(\infty) = 0$. Most previous studies have fixed limiting behaviors, which result in limited types of capable hazard functions. The new flexible Weibull distribution, however, is able to cover various types of hazard functions by an appropriate choice of parameters.

We can write the r^{th} moment of the new flexible Weibull distribution as

$$\begin{aligned} E[T^r] &= \int_0^\infty t^r f(t)dt = \int_0^\infty rt^{r-1} S(t)dt \\ &= \int_0^\infty rt^{r-1} \exp\left[-\exp(\mu)\{\exp(t) - 1\}^\alpha \{\log(t+1)\}^\beta\right] dt. \end{aligned}$$

We use Gauss-Kronrod quadrature for numerical integration since the above integral cannot be computed in closed-form. Figures 3–5 shows the changing density and moments as β varies when α is fixed.

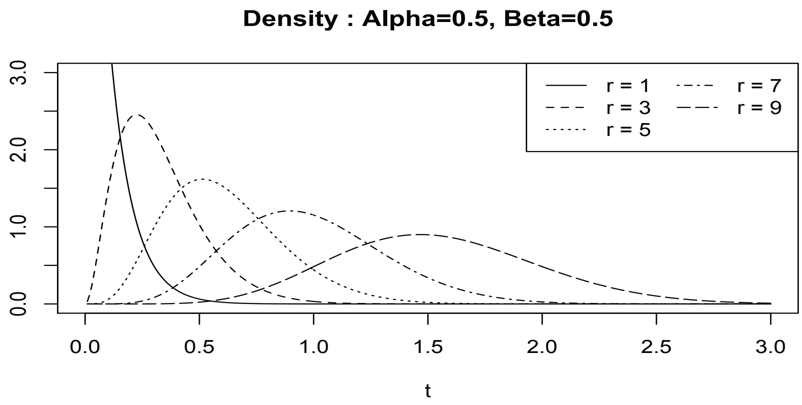

Figure 6: *Probability density function of r^{th} order statistic.*

We also study the order statistic of the proposed distribution. We denote the probability density function of r^{th} order statistic $T_{(r)}$ as $f_{r:n}(t)$. The probability density function $f_{r:n}(t)$ can be written as

$$\begin{aligned} f_{r:n}(t) &= \frac{1}{B(r, n-r+1)} F^{r-1}(t)[1-F(t)]^{n-r} f(t) \\ &= \frac{1}{B(r, n-r+1)} \sum_{k=0}^{r-1} \binom{r-1}{k} (-1)^k \exp\{-H(t)(n+k+1-r)\} h(t). \end{aligned}$$

Since the probability density function $f(t)$ can be written in terms of hazard function $h(t)$ and cumulative hazard function $H(t)$ as

$$f(t) = h(x) e^{-H(t)},$$

we can derive $f_{r:n}(t)$ as

$$f_{r:n}(t) = \frac{1}{B(r, n-r+1)} \sum_{k=0}^{r-1} \binom{r-1}{k} \frac{(-1)^k}{(n+k+1-r)} f(t; \mu^*, \alpha, \beta),$$

where $\mu^* = \mu + \log(n+k+1-r)$. Figure 6 shows the probability density function of r^{th} order statistic when α and β are both fixed to 0.5 and n fixed to 10.

3. Parameter estimation

In order to estimate the unknown parameters of the distribution, we can consider the Weibull-type probability plot employed in Bebbington *et al.* (2007), Lai *et al.* (2003) and Park and Park (2016) by letting the theoretical cumulative hazard function be as close to the empirical cumulative hazard function as

$$\log \hat{H}(t) = \mu + \alpha \log(\exp(t) - 1) + \beta \log(\log(t+1)),$$

where $\hat{H}(t)$ is the nonparametric estimate of the cumulative hazard function at t.

Hence, we can obtain the (weighted) least square estimation by estimating $H(t)$ with the Nelson-Aalen estimator or Kaplan-Meier estimator. For simplicity, one may consider the ordinary least square estimation; however, we consider the maximum likelihood estimation as follows.

The likelihood function of the new flexible Weibull distribution given $t_1, \ldots, t_n$ has the following form as

$$
\begin{aligned}
L_{NFW}(\mu, \alpha, \beta) &= \prod_{i=1}^{n} f_{NFW}(t_i; \mu, \alpha, \beta) \\
&= \prod_{i=1}^{n} \exp(\mu)\{\exp(t_i) - 1\}^{\alpha-1}\{\log(t_i + 1)\}^{\beta-1} \\
&\quad \times \left[\alpha \exp(t_i)\log(t_i + 1) + \{\exp(t_i) - 1\}\frac{\beta}{t_i + 1}\right] \\
&\quad \times \exp\left[-\exp(\mu)\{\exp(t_i) - 1\}^{\alpha}\{\log(t_i + 1)\}^{\beta}\right].
\end{aligned}
$$

Then the score functions for parameters μ, α and β can be obtained as follows.

$$
\begin{aligned}
\frac{\partial \log L_{NFW}}{\partial \mu} &= n - \sum_{i=1}^{n} \exp(\mu)\{\exp(t_i) - 1\}^{\alpha}\{\log(t_i + 1)\}^{\beta}, \\
\frac{\partial \log L_{NFW}}{\partial \alpha} &= \sum_{i=1}^{n} \log\{\exp(t_i) - 1\} + \sum_{i=1}^{n} \frac{\exp(t_i)\log(t_i + 1)(t_i + 1)}{\alpha \exp(t_i)\log(t_i + 1)(t_i + 1) + \beta\{\exp(t_i) - 1\}} \\
&\quad - \exp(\mu)\sum_{i=1}^{n}\{\exp(t_i) - 1\}^{\alpha}\{\log(t_i + 1)\}^{\beta}\log\{\exp(t_i) - 1\}, \\
\frac{\partial \log L_{NFW}}{\partial \beta} &= \sum_{i=1}^{n} \log\{\log(t_i + 1)\} + \sum_{i=1}^{n} \frac{\{\exp(t_i) - 1\}}{\alpha \exp(t_i)\log(t_i + 1)(t_i + 1) + \beta\{\exp(t_i) - 1\}} \\
&\quad - \exp(\mu)\sum_{i=1}^{n}\{\exp(t_i) - 1\}^{\alpha}\{\log(t_i + 1)\}^{\beta}\log\{\log(t_i + 1)\}.
\end{aligned}
$$

In order to obtain maximum likelihood estimates of μ, α and β, we solve the above three equations using the quasi-Newton method with initial values set to the ordinary least square estimates.

4. Simulation studies

In order to evaluate the performance of the new flexible Weibull distribution, we consider the following five different types of distributions. The hazard functions are given in Figure 7.

Example 1. Constant hazard function: Exponential distribution with a rate parameter $\lambda = 1/3$.

Example 2. Decreasing hazard function: Additive Weibull distribution suggested by Lemonte *et al.* (2014) with parameters $a = 0.1, b = 1, c = 0.2, d = 0.8$.

Example 3. Increasing hazard function: Additive Weibull distribution suggested by Lemonte *et al.* (2014) with parameters $a = 0.1, b = 1.5, c = 0.2, d = 1$.

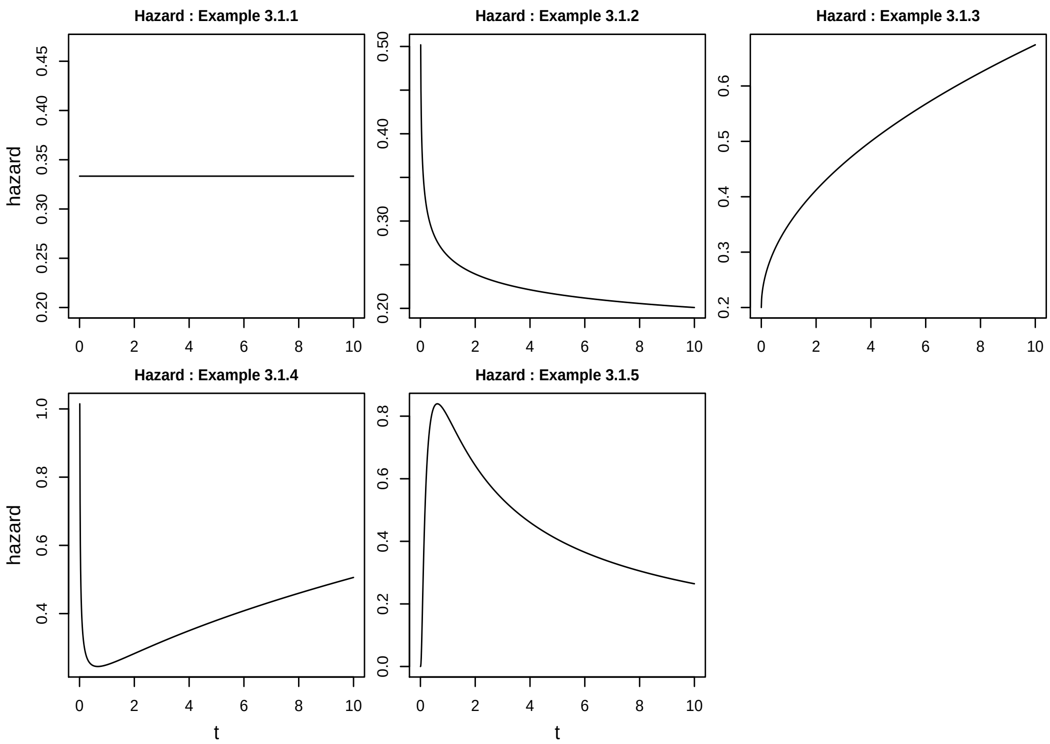

Figure 7: *Hazard functions for simulated examples.*

Example 4. Bathtub-shaped hazard function: Additive Weibull distribution suggested by Lemonte *et al.* (2014) with parameters $a = 0.1, b = 1.5, c = 0.2, d = 0.5$.

Example 5. Upside-down bathtub-shaped hazard function: Lognormal distribution with mean $\mu = 0$ and variance $\sigma = 1$.

We compare the new flexible Weibull distribution with the standard Weibull distribution (two-parameter), modified Weibull distribution (three- parameter), flexible Weibull distribution (three-parameter), very flexible Weibull distribution (three-parameter), and two-parameter lifetime distribution suggested by Chen (2000) as

1. Weibull

$$\log H(t) = \alpha + \beta \log(t).$$

2. Modified Weibull by Lai *et al.* (2003)

$$\log H(t) = \mu + \alpha t + \beta \log(t).$$

3. Flexible Weibull by Bebbington *et al.* (2007)

$$\log H(t) = \alpha t - \frac{\beta}{t}.$$

Table 1: Maximum likelihood fit of Examples 1–5

Model	Example 1			Example 2		
	$\log L$	AIC	K-S	$\log L$	AIC	K-S
Weibull	−40.8832	85.7664	0.1323	−47.2960	98.5919	0.1325
Modified Weibull	−40.7016	87.4032	0.1266	−46.9051	99.8103	0.1258
Flexible Weibull	−43.8693	91.7386	0.2816	−51.3680	106.7360	0.3132
Very flexible Weibull	−40.2433	86.4866	0.1215	−46.6124	99.2249	0.1217
Chen	−41.5697	87.1394	0.1419	−47.9022	99.8043	0.1406
New flexible Weibull	−40.2352	86.4703	0.1217	−46.6118	99.2237	0.1219
Model	Example 3			Example 4		
	$\log L$	AIC	K-S	$\log L$	AIC	K-S
Weibull	−36.9378	77.8755	0.1328	−41.3283	86.6566	0.1476
Modified Weibull	−36.5120	79.0239	0.1247	−39.8640	85.7279	0.1219
Flexible Weibull	−39.5647	83.1295	0.2703	−46.8934	97.7868	0.4069
Very flexible Weibull	−36.2870	78.5740	0.1212	−39.8316	85.6633	0.1228
Chen	−37.3808	78.7615	0.1385	−40.4268	84.8537	0.1300
New flexible Weibull	−36.2783	78.5566	0.1212	−39.8612	85.7224	0.1235
Model	Example 5					
	$\log L$	AIC	K-S			
Weibull	−43.3817	90.7635	0.1319			
Modified Weibull	−43.3713	92.7427	0.1318			
Flexible Weibull	−42.7599	89.5197	0.1691			
Very flexible Weibull	−41.7097	89.4194	0.1100			
Chen	−46.8947	97.7893	0.1681			
New flexible Weibull	−41.7001	89.4003	0.1101			

AIC = Akaike Information Criterion; K-S = Kolmogorov-Smirnov.

4. Very flexible Weibull by Park and Park (2016)

$$\log H(t) = \mu + \alpha t + \beta \log \log(t + 1).$$

5. Two-parameter lifetime distribution by Chen (2000)

$$\log H(t) = \mu + \log\left(\exp\left(t^{\beta}\right) - 1\right).$$

As we can see, the above four modified Weibull distributions contain at least one inappropriate cumulative hazard function.

In evaluating the performance, we generated a random sample of size 30 from each distribution, and calculated the log-likelihood value, Akaike Information Criterion (AIC), and Kolmogorov-Smirnov statistic (K-S). We repeated 100,000 Monte Carlo simulations and calculated the averages which are tabulated in Table 1.

The numerical results in Table 1 indicate that the new flexible Weibull distribution shows the robust performances over the five different distributions. For the first three examples where the Weibull distribution can fit well, the Weibull distribution shows the lowest AIC values but the new flexible Weibull shows the second lowest AIC values. For the fourth and fifth examples where the Weibull distribution can not fit well, the new flexible Weibull distribution shows the second lowest AIC value for the fourth case and lowest AIC value for the fifth case.

5. Application

We use a real data example to illustrate the efficiency of new flexible Weibull distribution. The failure time data studied in Murthy *et al.* (2004) and Aryal and Elbatal (2015) were used for application. The

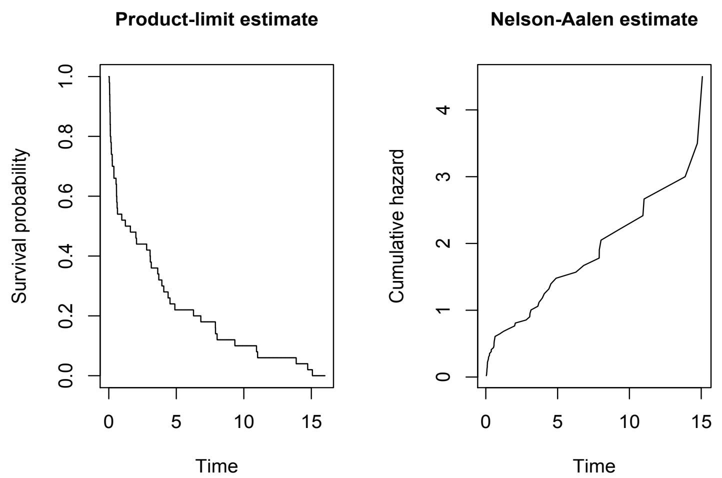

Figure 8: *Survival function and cumulative hazard function for real data example.*

Table 2: Maximum likelihood fit of Example 6

Model	$\log L$	AIC	K-S
Weibull	−102.3643	208.7286	0.1270
Modified Weibull	−101.3636	208.7273	0.1326
Flexible Weibull	−95.9229	195.8457	0.1878
Very flexible Weibull	−99.4470	204.8941	0.1123
Chen *et al.*	−103.4204	210.8408	0.1397
New flexible Weibull	−99.3345	204.6691	0.1103

AIC = Akaike Information Criterion; K-S = Kolmogorov-Smirnov.

data are as follows.

Example 6. 0.036, 0.058, 0.061, 0.074, 0.078, 0.086, 0.102, 0.103, 0.114, 0.116, 0.148, 0.183, 0.192, 0.254, 0.262, 0.379, 0.381, 0.538, 0.570, 0.574, 0.590, 0.618, 0.645, 0.961, 1.228, 1.600, 2.006, 2.054, 2.804, 3.058, 3.076, 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893, 6.274, 6.816, 7.896, 7.904, 8.022, 9.337, 10.940, 11.020, 13.880, 14.730, 15.080.

Figure 8 illustrates the survival function and cumulative hazard function using product-limit estimate and Nelson-Aalen estimate, respectively. We compare the efficiency of new flexible Weibull distribution with aforementioned distributions using log-likelihood value, AIC and K-S values (Table 2).

The results in Table 2 indicate that flexible Weibull distribution shows the largest log-likelihood value and lowest AIC value; however, flexible Weibull distribution has the poorest fit in terms of K-S value. On the other hand, new flexible Weibull distribution shows second best fit in terms of log-likelihood value and AIC value along with lowest K-S value.

6. Conclusions

Some well-known modified Weibull distributions can be represented as a multiplication of two cumulative hazard functions, but some cumulative hazard functions do not satisfy the properties as a cumulative hazard function. We consider $\exp(t) - 1$ and $\log(t + 1)$, and suggest a new modified Weibull distribution called a new flexible Weibull distribution which is theoretically rigorous and shows more flexibility. The hazard function of the new flexible Weibull distribution can cover the monotone shape as well as non-monotone shape that include bathtub-shaped, modified bathtub-shaped or upside-down bathtub-shaped. We presented the parameter estimation methods and compared their performance with some modified Weibull distributions for various types of distributions and real data application.

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