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WHEN THE NAGATA RING $D(X)$ IS A SHARP DOMAIN

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Abstract. Let D be an integral domain, X be an indeterminate over D, $D[X]$ be the polynomial ring over D, and $D(X)$ be the Nagata ring of D. Let $[d]$ be the star operation on $D[X]$, which is an extension of the d-operation on D as in [5, Theorem 2.3]. In this paper, we show that D is a sharp domain if and only if $D[X]$ is a $[d]$-sharp domain, if and only if $D(X)$ is a sharp domain.

1. Introduction

Let D be an integral domain and $*$ be a star operation on D. (The definitions related to star operations will be reviewed in Section 2.) As in [2], we say that D is a $*$-*sharp domain* if whenever $I \supseteq AB$ with I, A, B nonzero ideals of D, there exist nonzero ideals H and J of D such that $I^* = (HJ)^*$, $H^* \supseteq A$, and $J^* \supseteq B$. Following [1], we say that a d-sharp domain is a *sharp domain*, i.e., D is a sharp domain if whenever $I \supseteq AB$ with I, A, B nonzero ideals of D, there exist ideals $A_0 \supseteq A$ and $B_0 \supseteq B$ of D such that $I = A_0B_0$. Assume that D is a $*$-sharp domain. It is known that D is a t-sharp domain (and hence D is a PvMD whose prime t-ideals are maximal t-ideal); if $* = *_w$, then D is a P$*$MD whose maximal $*$-ideals have height-one; and I_v is $*$-invertible for all nonzero fractional ideals I of D [2, Propositions 2.2, 3.1, 2.3 and

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2.4]. Also, D is a v-sharp domain if and only if D is completely integrally closed [2, Corollary 2.6].

In [2, Theorem 3.7(a)], the authors showed that D is a t-sharp domain if and only if $D[X]$, the polynomial ring over D, is a t-sharp domain. They then remarked that "we do not have a "d-analogue" of [2, Theorem 3.7(a)] because a sharp domain has Krull dimension ≤ 1". In fact, if D is a sharp domain, then D is a Prüfer domain with $\dim(D) \leq 1$ [1, Theorem 11]. Hence, $D[X]$ is a sharp domain if and only if D is a field. However, in this paper, we use the star operation $[d]$ on $D[X]$ (see Lemma 2) to prove the d-operation analogue of [2, Theorem 3.7(a)]. Precisely, we prove that if $*$ is a star operation on D such that $*_w = *$, then D is a $*$-sharp domain if and only if $D[X]$ is a $[*]$-sharp domain, if and only if $D[X]_{N_*}$ is a sharp domain, where $N_* = \{f \in D[X] \mid c(f)^* = D\}$. Let $D(X)$ be the Nagata ring of D, i.e., $D(X) = \{\frac{f}{g} \mid f, g \in D[X]$ and $c(g) = D\}$. As a corollary, we have that D is a sharp domain if and only if $D[X]$ is a $[d]$-sharp domain, if and only if $D(X)$ is a sharp domain. Finally, we study when $D[X]$ is a $\star$-sharp domain if $\star$ is a star operation on $D[X]$ such that $\star_w = \star$.

2. Definitions related to star operations

Let D be an integral domain with quotient field K, $F(D)$ be the set of nonzero fractional ideals of D, and $f(D)$ be the set of nonzero finitely generated fractional ideals of D; so $f(D) \subseteq F(D)$, and equality holds if and only if D is Notherian. We say that a mapping $* : F(D) \to F(D)$, $I \mapsto I^*$, is a *star operation* on D if the following three conditions are satisfied for all $0 \neq a \in K$ and $I, J \in F(D)$: (i) $(aD)^* = aD$ and $(aI)^* = aI^*$, (ii) $I \subseteq I^*$ and if $I \subseteq J$, then $I^* \subseteq J^*$, and (iii) $(I^*)^* = I^*$. Given a star operation $*$ on D, two new star operations $*_f$ and $*_w$ on D can be constructed as follows for all $I \in F(D)$; $I^{*_f} = \bigcup\{J^* \mid J \subseteq I$ and $J \in f(D)\}$ and $I^{*_w} = \{x \in K \mid xJ \subseteq I$ for some $J \in f(D)$ with $J^* = D\}$. Clearly, $(*_f)_f = *_f$ and $(*_f)_w = (*_w)_f = *_w$. An $I \in F(D)$ is called a $*$-*ideal* if $I^* = I$, and a $*$-ideal is a *maximal* $*$-*ideal* if it is maximal among proper integral $*$-ideals. Let $*$-Max(D) be the set of maximal $*$-ideals of D. It is known that $*_f$-Max$(D) \neq \emptyset$ when D is not a field; $*_f$-Max$(D) = *_w$-Max(D) [3, Theorem 2.16]; and $I^{*_w} = \bigcap_{P \in *_f\text{-Max}(D)} ID_P$ for all $I \in F(D)$ [3, Corollary 2.10]. For $I \in F(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$; then $I^{-1} \in F(D)$. We say that $I \in F(D)$ is $*$-*invertible* if $(II^{-1})^* = D$, and D is a *Prüfer $*$-multiplication domain* (P$*$MD) if every nonzero finitely generated ideal of D is $*_f$-invertible. Examples of star operations include the d-, v-, t-, and w-operations. The d-operation is the identity function of $F(D)$, i.e., $I^d = I$ for all $I \in F(D)$, the v-operation is defined by $I^v = (I^{-1})^{-1}$, the t-operation is defined by $t = v_f$, and the w-operation is given by $w = v_w$. For more on basic properties of star operations, see [8, Sections 32 and 34].

Let X be an indeterminate over D, $D[X]$ be the polynomial ring over D, and $c(f)$ be the ideal of D generated by the coefficients of $f \in D[X]$. The next lemma is nice characterizations of P$*$MDs, which appear in [7, Theorem 3.1 and Proposition 3.15] in a more general setting of semistar operations.

Lemma 1. *Let $*$ be a star operation on D with $*_f = *$. Then the following statements are equivalent.*

1. *D is a P$*$MD.*
2. *D is a PvMD and $* = t$.*
3. *D is a PvMD and $*_w = t$.*
4. *$D[X]_{N_*}$ is a Prüfer domain, where $N_* = \{f \in D[X] \mid c(f)^* = D\}$.*

In this case, $*_f = *_w = t = w$.

Let $*$ be a star operation on D. Then there is a star operation $[*]$ on $D[X]$, which is an extension of $*_w$ to $D[X]$ in the sense that $(I[X])^{[*]} \cap K = I^{*_w}$ for each $I \in F(D)$. We recall this result for easy reference of the reader.

Lemma 2. [5, Theorem 2.3] *Let X, Y be two indeterminates over D, $*$ be a star operation on D, and let*

$$\mathbf{\Delta} = \{Q \in Spec(D[X]) \mid Q \cap D = (0) \text{ with } htQ = 1 \\ \text{ or } Q = (Q \cap D)[X] \text{ and } (Q \cap D)^{*_f} \subsetneq D\}.$$

Set $\mathcal{S} = D[X][Y] \setminus (\bigcup\{Q[Y] \mid Q \in \mathbf{\Delta}\})$ and define

$$A^{[*]} = A[Y]_{\mathcal{S}} \cap K(X) \quad \text{for all } A \in F(D[X]).$$

1. *The mapping $[*] : F(D[X]) \to F(D[X])$, given by $A \mapsto A^{[*]}$, is a star operation on $D[X]$ such that $[*] = [*]_f = [*]_w$.*
2. *$[*] = [*_f] = [*_w]$.*
3. *$(ID[X])^{[*]} \cap K = I^{*_w}$ for all $I \in F(D)$.*
4. *$(ID[X])^{[*]} = I^{*_w} D[X]$ for all $I \in F(D)$.*

5. $[*]$-*Max*$(D[X]) = \{Q \mid Q \in Spec(D[X])$ *such that* $Q \cap D = (0)$, $htQ = 1$, *and* $(\sum_{g\in Q} c(g))^{*_f} = D\} \cup \{P[X] \mid P \in *_f\text{-}Max(D)\}$.
6. $[v]$ *is the* w*-operation on* $D[X]$.

Let $*$ be a star operation on D. It is known that D is a P$*$MD if and only if $D[X]$ is a P$[*]$MD [5, Corollary 2.5]; hence D is a PvMD if and only if $D[X]$ is a PvMD. Also, since a PdMD is just the Prüfer domain, D is a Prüfer domain if and only if $D[X]$ is a P$[d]$MD.

3. Main Results

Let D be an integral domain with quotient field K, and we assume that $D \neq K$ in order to avoid the trivial case. Let X be an indeterminate over D, $D[X]$ be the polynomial ring over D, and $N_v = \{f \in D[X] \mid c(f)^v = D\}$.

Lemma 3. *Let* $N_v = \{f \in D[X] \mid c(f)^v = D\}$, $I \in F(D)$, *and* $A \in F(D[X])$.

1. $ID[X]_{N_v} \cap K = I^w$, *and hence* $I^w D[X]_{N_v} = ID[X]_{N_v}$.
2. $A_{N_v} = (A^w)_{N_v}$.

Proof. (1) [4, Lemma 2.1].

(2) It suffices to show that $A^w \subseteq A_{N_v}$. For this, let $0 \neq f \in A^w$. Then there is a nonzero finitely generated ideal J of $D[X]$ such that $J^v = D[X]$ and $fJ \subseteq A$. Since $J^v = D[X]$, $J \nsubseteq P[X]$ for all $P \in t$-Max(D), and hence $(\sum_{h\in J} c(h))^t = D$. Hence, there is a $0 \neq g \in J$ with $c(g)^v = D$; so $g \in N_v$ and $fg \in A$. Thus, $f \in A_{N_v}$. □

It is known that D is a t-sharp domain if and only if $D[X]$ is a t-sharp domain, if and only if $D[X]_{N_v}$ is a sharp domain [2, Theorem 3.7]. We next generalize this result to an arbitrary star operation $*$ on D with $*_w = *$.

Theorem 4. *Let* $*$ *be a star operation on D such that* $*_w = *$. *Then the following statements are equivalent.*

1. D *is a* $*$*-sharp domain.*
2. $D[X]$ *is a* $[*]$*-sharp domain.*
3. $D[X]_{N_*}$ *is a sharp domain, where* $N_* = \{f \in D[X] \mid c(f)^* = D\}$.

In this case, $* = t = w$ *on* D *and* $[*] = t = w$ *on* $D[X]$.

Proof. (1) $\Rightarrow$ (2) If D is a $*$-sharp domain, then D is a P$*$MD [2, Proposition 2.3], and hence D is a PvMD with $* = t = w$ on D by Lemma 1. Hence, D is a t-sharp doman, and thus $D[X]$ is a t-sharp domain. However, note that $D[X]$ is a PvMD and $[t] = w = t$ on $D[X]$ by Lemmas 1 and 2; so $[*] = t$. Thus, $D[X]$ is a $[*]$-sharp domain.

(2) $\Rightarrow$ (3) Assume that $D[X]$ is a $[*]$-sharp domain. Then $D[X]$ is a PvMD and $[*] = t = w$ on $D[X]$; hence $I^* = (ID[X])^{[*]} \cap K = (ID[X])^t \cap K = I^t D[X] \cap K = I^t$ for all $I \in F(D)$ by Lemma 2 and [9, Proposition 4.3]. Thus, D is a PvMD and $* = t = w$ on D.

Let $I_{N_*} \supseteq (A_{N_*})(B_{N_*})$ with I, A, B nonzero ideals of $D[X]$, and let $C = ID[X]_{N_*} \cap D[X]$. Then $C \supseteq AB$, and hence by assumption, there exist nonzero ideals A_0 and B_0 of $D[X]$ such that $C^w = (A_0B_0)^w$, $(A_0)^w \supseteq A$, and $(B_0)^w \supseteq B$. Thus, by Lemma 3, $I_{N_*} = C_{N_*} = (C^w)_{N_*} = ((A_0B_0)^w)_{N_*} = (A_0B_0)_{N_*} = ((A_0)_{N_*})((B_0)_{N_*})$, $(A_0)_{N_*} = ((A_0)^w)_{N_*} \supseteq A_{N_*}$, and $(B_0)_{N_*} = ((B_0)^w)_{N_*} \supseteq B_{N_*}$. Thus, $D[X]_{N_*}$ is a sharp domain.

(3) $\Rightarrow$ (1) Suppose that $D[X]_{N_*}$ is a sharp domain. Then $D[X]_{N_*}$ is a Prüfer domain [2, Theorem 11], and hence D is a P$*$MD by Lemma 1 and every ideal of $D[X]_{N_*}$ is extended from D [10, Theorem 3.1]. Let $I \supseteq AB$ with I, A, B nonzero ideals of D. Then $ID[X]_{N_*} \supseteq (AD[X]_{N_*})(BD[X]_{N_*})$, and hence by assumption, there exist nonzero ideals H and J of D such that

$$ID[X]_{N_*} = (HD[X]_{N_*})(JD[X]_{N_*}) = (HJ)D[X]_{N_*},$$

$HD[X]_{N_*} \supseteq AD[X]_{N_*}$, and $JD[X]_{N_*} \supseteq BD[X]_{N_*}$. Thus, by Lemma 3, $I^* = ID[X]_{N_*} \cap K = (HJ)D[X]_{N_*} \cap K = (HJ)^*$, $H^* = HD[X]_{N_*} \cap K \supseteq AD[X]_{N_*} \cap K \supseteq A$, and $J^* = JD[X]_{N_*} \cap K \supseteq BD[X]_{N_*} \cap K \supseteq B$. Thus, D is a $*$-sharp domain. □

The next result is the d-operation analogue of [2, Theorem 3.7(a)] that D is a t-sharp domain if and only if $D[X]$ is a t-sharp domain.

Corollary 5. *The following statements are equivalent for an integral domain D.*

1. *D is a sharp domain.*
2. *$D[X]$ is a $[d]$-sharp domain.*
3. *$D(X)$ is a sharp domain.*

Proof. This follows directly from Theorem 4 because $D(X) = D[X]_{N_d}$. □

Let $\star$ be a star operation on $D[X]$, and let $I^* = (ID[X])^\star \cap K$ for all $I \in F(D)$. Then it is easy to see that $*$ is a star operation on D (cf. [6, Lemma 5]). A nonzero prime ideal Q of $D[X]$ is said to be an *upper to zero* in $D[X]$ if $Q \cap D = (0)$; so each upper to zero in $D[X]$ has height-one. Let $\star = \star_f$, and note that if every upper to zero in $D[X]$ is a maximal $\star$-ideal, then $\star$-Max$(D[X]) = t$-Max$(D[X])$ [11, Theorem 2.9], and hence $\star_w = w$ on $D[X]$.

Corollary 6. *Let $\star$ be a star operation on $D[X]$ with $\star_w = \star$, and let $*$ be the star operation on D defined by $I^* = (ID[X])^\star \cap K$ for all $I \in F(D)$. Suppose that every upper to zero in $D[X]$ is a maximal $\star$-ideal. Then $D[X]$ is a $\star$-sharp domain if and only if D is a $*$-sharp domain.*

Proof. ($\Rightarrow$) Assume that $D[X]$ is a $\star$-sharp domain. Then $D[X]$ is a PvMD and $\star = t = w$ on $D[X]$ as in the proof of Theorem 4, and hence D is a PvMD and $* = t = w$ on D [6, Lemma 5]. Note that $[t] = w$ on $D[X]$; so $[*] = \star$. Thus, D is a $*$-sharp domain by Theorem 4.

($\Leftarrow$) Suppose that D is a $*$-sharp domain, and note that $*_w = *$ [6, Lemma 5] because $\star_w = \star$. Hence, $D[X]$ is a $[*]$-sharp domain and $[*] = t = w$ on $D[X]$ by Theorem 4. Since every upper to zero in $D[X]$ is a maximal $\star$-ideal, $\star = w = [*]$ on $D[X]$. Thus, $D[X]$ is a $\star$-sharp domain. □

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