Korean J. Math. **24** (2016), No. 3, pp. 447–467
http://dx.doi.org/10.11568/kjm.2016.24.3.447

THE EXISTENCE OF RANDOM ATTRACTORS FOR PLATE EQUATIONS WITH MEMORY AND ADDITIVE WHITE NOISE

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Abstract. We prove the existence of random attractors for the continuous random dynamical systems generated by stochastic damped plate equations with linear memory and additive white noise when the nonlinearity has a critically growing exponent.

1. Introduction

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, where

$$\Omega = \{\omega = (\omega_1, \omega_2, \cdots, \omega_m) \in C(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\},$$

is endowed with compact open topology, $\mathscr{F}$ is the $\mathbb{P}$-completion of Borel σ-algebra on Ω, and $\mathbb{P}$ is the corresponding Wiener measure. Define the time shift via

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t),\ t \in \mathbb{R},\ \omega \in \Omega.$$

Thus, $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ is an ergodic metric dynamical system.

Received March 14, 2016. Revised June 21, 2016. Accepted August 29, 2016.
2010 Mathematics Subject Classification: 35Q35, 35B40, 35B41.
Key words and phrases: Plate equation; Random attractors; Fading memory; Additive noise.
This work was supported by the NSFC (11561064,11361053),and NWNU-LKQN-14-6.
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In this paper, we are devoted to consider the existence of random attractors for the following plate equations with linear memory and additive white noise:

$$\begin{cases} u_{tt}+\alpha u_t+\Delta^2 u+\int_0^\infty \mu(s)\Delta^2(u(t)-u(t-s))ds+g(u) \\ \qquad =f(x)+\sum_{j=1}^m h_j\dot{W}_j, & x\in U, t\geq\tau, \\ u(x,\tau)=u_0(x), u_t(x,\tau)=u_1(x), & x\in U, t\leq\tau, \\ u|_{\partial U}=\frac{\partial u}{\partial \mathbf{n}}|_{\partial U}=0, & t\geq\tau,\ \tau\in\mathbb{R}. \end{cases} \tag{1.1}$$

Where U is a bounded open set of $\mathbb{R}^5$ with a smooth boundary ∂U, $u=u(x,t)$ is a real-valued function on $U\times[\tau,+\infty)$, $f(x)\in H_0^1(U)\cap H^2(U)$ is a given external force. $h_j(x)\in H_0^2(U)\cap H^4(U)$, $(j=1,2,3,\cdots,m)$, $\{W_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on $(\Omega,\mathscr{F},\mathbb{P})$. Then we identify $\omega(t)$ with $(W_1(t),W_2(t),\cdots,W_m(t))$, i.e.,

$$\omega(t)=(W_1(t),W_2(t),\cdots,W_m(t)), t\in\mathbb{R}.$$

The memory kernel function $\mu(s)$ and the nonlinear term $g(u)$ satisfy the following conditions:

$(H_1):$ $\mu(s)\in C^1(\mathbb{R}^+)\cap L^1(\mathbb{R}^+)$, $\mu(s)\geq 0$, $\mu'(s)+\delta\mu(s)\leq 0$, $\forall s\in\mathbb{R}^+$ and some $\delta>0$.

$(H_2):$ *Let* $G(s)=\int_0^s g(\tau)d\tau$, and there exists constants $C_0,\ C_1,\ C_2>0$, such that

$$|g'(s)|\leq C_0(1+|s|^4),\ g(0)=0,\ \forall s\in\mathbb{R}, \tag{1.2}$$

$$G(s)\geq C_1(|s|^6-1),\ \forall s\in\mathbb{R}, \tag{1.3}$$

$$\liminf_{|s|\to\infty}\frac{sg(s)-C_2G(s)}{s^2}\geq 0,\ \forall s\in\mathbb{R}. \tag{1.4}$$

Following Dafermos [1], we introduce a Hilbert "history" space $\mathfrak{R}_{\mu,2}=L^2_\mu(\mathbb{R}^+,H_0^2(U))$ with the inner product

$$(\eta_1,\eta_2)_{\mu,2}=\int_0^\infty \mu(s)\left(\Delta\eta_1(s),\Delta\eta_2(s)\right)ds,\ \forall\eta_1,\eta_2\in\mathfrak{R}_{\mu,2},$$

and new variables

$$\eta(t,x,s)=u(t,x)-u(t-s,x).$$

Set $E=H_0^2(U)\times L^2(U)\times\mathfrak{R}_{\mu,2}$, $Z=(u,u_t,\eta)^T$, then the system (1.1) is equivalent to the following initial value problem in the Hilbert space $E:$

$$\begin{cases} Z_t=L(Z)+N(Z,t,W(t)),\ x\in U,\ t\geq\tau,\ s\in\mathbb{R}^+, \\ Z(\tau)=Z_\tau=(u_0(x),u_1(x),\eta_0(x,s)),\ (x,s)\in U\times\mathbb{R}^+, \end{cases} \tag{1.5}$$

where

$$\begin{cases} u(t,x) = \eta(t,x,s) = \eta(t,x,0) = 0, \ x \in \partial U, \ t \geq \tau, \ s \in \mathbb{R}^+, \\ u(\tau,x) = u_0(x), u_t(\tau,x) = u_1(x), \ x \in U, \\ \eta(\tau,x,s) = \eta_0(x,s) = u(\tau,x) - u(\tau - s,x), \ (x,s) \in U \times \mathbb{R}^+, \end{cases} \tag{1.6}$$

$$L(Z) = \begin{pmatrix} u_t \\ -\Delta^2 u - \alpha u_t - \int_0^\infty \mu(s)\Delta^2\eta(s)ds \\ u_t - \eta_s \end{pmatrix}, \tag{1.7}$$

$$N(Z,t,W(t)) = \begin{pmatrix} 0 \\ -g(u) + f(x) + \sum_{j=1}^m h_j \dot{W}_j \\ 0 \end{pmatrix}, \tag{1.8}$$

$$D(L) = \left\{ Z \in E \mid \begin{array}{c} u + \int_0^\infty \mu(s)\eta(s)ds \in H^4(U) \cap H_0^2(U), \\ u_t \in H_0^2(U), \eta(s) \in H_\mu^1(\mathbb{R}^+, H_0^2(U)), \eta(\tau) = 0 \end{array} \right\}, \tag{1.9}$$

here $H_\mu^1(\mathbb{R}^+, H_0^2(U)) = \{\eta : \eta(s), \partial_s\eta(s) \in L_\mu^2(\mathbb{R}^+, H_0^2(U))\}$.

Problem (1.1) models transversal vibrations of thin extensible elastic plate in a history space, which is established based on the framework of elastic vibration by Woinowsky-Krieger([2]) and Berger([3]). It can also be regarded as an elastoplastic flow equation with some kind of memory effect([1]).

When $h_j = 0$ $(1 \leq j \leq m)$ and $\mu = 0$, (1.1) reduces to a determined autonomous damped plate equation. There were a lot of publications concerning the existence of their global attractors, uniform attractors, pullback attractors and exponential attractors, for instance [4,5] for the linear damping and see [6-10] for the nonlinear damping.

When $h_j \neq 0$ $(1 \leq j \leq m)$ and $\mu = 0$, that is the case of without memory kernel, then (1.1) reduces to a stochastic damped plate equation with additive white noise. The existence of random attractors for such system in a bounded domain have been studied in [11,12].

For our problem, to the best of our knowledge, there were no results on the random attractors for the stochastic system(1.1), moreover, we know that there need some different techniques. To prove the existence of random attractors for a RDS(random dynamical system), the key step is to establish the compactness of the system. For our system (1.5), there are two essential difficulties in proving the compactness. One difficulty was caused by the critical growth condition (1.2) of g, which can be overcome by using the decomposition of solutions and the more accurate calculation. Another difficulty was caused by the memory kernel itself, because there is no compact embedding in the history space; moreover, we can't use the finite rank method, that is, we can't use the $(I-P_m)\eta$ to deal with our problem. For our purpose, we introduce a new variable and define a extend Hilbert space, as well as combine with the compactness transform theorem.

This paper is organized as follows. In Section 2, we recall some basic concepts and properties for general random dynamical systems. In Section 3, we first show that the existence and uniqueness of solutions for the random differential equation, which generates a RDS. In Section 4, we consider the dissipativeness of solutions of the random differential equation and obtain the existence of the uniformally absorbing set. In Section 5, we decompose the solution of the random differential equation into two parts: one part decays exponentially and another part is bounded in a higher regular space by using technique in [13] and careful computation. In Section 6, we construct a compact measurable attracting set and prove the existence of random attractor in E.

2. Random dynamical systems

In this section, we recall some basic concepts related to RDS and a random attractors for RDS in [14], which are important for getting our main results.

Let $(X, \|\cdot\|_X)$ be a separable Hilbert space with Borel σ-algebra $\mathscr{B}(X)$ and $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system.

Definition 2.1. Let $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ be a metric dynamical system. Suppose that the mapping $\phi : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathscr{B}(\mathbb{R}^+) \times \mathscr{F} \times \mathscr{B}(X), \mathscr{B}(X))$-measurable and satisfies the following properties:
(i) $\phi(0,\omega)x = x$;
(ii) $\phi(s, \theta_t\omega) \circ \phi(t,\omega)x = \phi(s+t,\omega)x$;
for all $s, t \in \mathbb{R}^+$, $x \in X$ and $\omega \in \Omega$. Then ϕ is called a RDS. Moreover, ϕ is called a continuous RDS if ϕ is continuous with respect to x for $t \geqslant 0$ and $\omega \in \Omega$.

To study the asymptotic behavior of the RDS determined by Eq.(1.1), we first need to recall some concepts and properties.

A set-valued mapping $B : \Omega \to 2^X$ is called a random closed set if $B(\omega)$ is closed, nonempty, and $\omega \mapsto d(x, B(\omega))$ is measurable for all $x \in$

$X,\ \omega \in \Omega$. A random set $\mathscr{B} := \{B(\omega)\}_{\omega\in\Omega}$ is said to tempered if

$$\lim_{t\to\infty} e^{-\eta t} diam(B(\theta_{-t}\omega)) = 0,$$

for a.e. $\omega \in \Omega$ and all $\eta > 0$, where $diam(B) := \sup_{x,y\in B} d(x,y)$.

Let $\mathscr{D}$ be the collection of all tempered random sets in X. We will only deal with the system $\mathscr{D}$ of tempered random sets in this paper.

DEFINITION 2.2. A random set $\mathscr{A} := \{A(\omega)\}_{\omega\in\Omega} \in X$ is called a random attractor for the RDS ϕ if $P - a.s.$
(i) $\mathscr{A}$ is a random compact set, i.e. $A(\omega)$ is nonempty and compact for a.e. $\omega \in \Omega$ and $\omega \mapsto\ d(x, A(\omega))$
is measurable for every $x \in X$;
(ii) $\mathscr{A}$ is ϕ-invariant, i.e. $\phi(t,\omega,A(\omega)) = A(\theta_t\omega)$, for all $t \geqslant 0$ and a.e. $\omega \in \Omega$;
(iii) $\mathscr{A}$ attracts every set in X, $i.e.$ for all bounded(and non-random) $B \subset X$,

$$\lim_{t\to\infty} dist(\phi(t,\theta_{-t}\omega, B(\theta_{-t}\omega)), A(\omega)) = 0,\ a.e.\ \omega \in \Omega.$$

THEOREM 2.3 *Let ϕ be a continuous random dynamical system on E over $(\Omega,\mathscr{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$. Suppose that there exists a random compact set $\{K(\omega)\}_{\omega\in\Omega}$ which absorbs every bounded non-random set $B \in \mathscr{D}$. Then, the set*

$$\mathscr{A} = \{A(\omega)\}_{\omega\in\Omega} = \overline{\cup_{B\subset X}\Lambda_B(\omega)},$$

is a global attractors for ϕ, where the union is taken over all bounded $B \subset X$, and $\Lambda_B(\omega)$ is the ω-limits set of B given by

$$\Lambda_B(\omega) = \bigcap_{\tau\geqslant 0}\overline{\bigcup_{t\geqslant\tau}\phi(t,\theta_{-t}\omega, B(\theta_{-t}\omega))},\ \omega \in \Omega.$$

3. Existence and uniqueness of solutions

From now on, assume that conditions $(H_1) - (H_2)$ hold, the space E, $(\Omega,\mathscr{F},\mathbb{P},(\theta_t)_{t\in\mathbb{R}})$ are defined as in Section 1. With the usual notation, we denote $A = \triangle^2$, and $D(A) = H^4(U) \cap H_0^2(U)$. We can define the powers A^ν of A for $\nu \in \mathbb{R}$. The space $V_{2\nu} = D(A^{\frac{\nu}{2}})$ is a Hilbert space with the following inner product and norm

$$(u,v)_{2\nu} = (A^{\frac{\nu}{2}}u, A^{\frac{\nu}{2}}v),\ \|u\|_{2\nu}^2 = (A^{\frac{\nu}{2}}u, A^{\frac{\nu}{2}}u).$$

The injection $V_{\nu_1} \hookrightarrow V_{\nu_2}$ is compact if $\nu_1 > \nu_2$. In particular, $V_0 = L^2(U), V_1 = H_0^1(U), V_2 = H_0^2(U)$, respectively, the inner product and norm in $L^2(U)$ is denoted by $(\cdot,\cdot)$, $\|\cdot\|$, and in $H_0^2(U)$ is denoted by $((\cdot,\cdot))$, $\|\cdot\|_2$. By (H_1), the space $\mathfrak{R}_{\mu,2\nu} = L^2_\mu(\mathbb{R}^+, V_{2\nu})$ is a Hilbert space of $V_{2\nu}$-valued functions on $\mathbb{R}^+$ with the inner product and norm, respectively

$$(\eta_1,\eta_2)_{\mu,2\nu} = \int_0^\infty \mu(s)\left(A^{\frac{\nu}{2}}\eta_1(s), A^{\frac{\nu}{2}}\eta_2(s)\right) ds,\ \forall \eta_1,\eta_2 \in V_{2\nu},$$

$$\|\eta(s)\|^2_{\mu,2\nu} = (\eta,\eta)_{\mu,2\nu} = \int_0^\infty \mu(s)\|A^{\frac{\nu}{2}}\eta(s)\|^2 ds,$$

and on $\mathfrak{R}_{\mu,2\nu}$, the linear operator $-\partial_s$ has domain

$$D(-\partial_s) = \{\eta \in H^1_\mu(\mathbb{R}^+, V_{2\nu}) : \eta(0) = 0\},$$
$$\text{where } H^1_\mu(\mathbb{R}^+, V_{2\nu}) = \{\eta : \eta(s), \partial_s\eta(s) \in L^2_\mu(\mathbb{R}^+, V_{2\nu})\},$$

which generates a right-translation semigroup. The symbol C and $C_i (i = 1, 2, \cdots)$ are used for a general positive number which may change from line to line.

In this section, we show the existence, uniqueness and continuous dependence of (mild) solutions of initial problem (1.5) in E which generates a continuous RDS on E over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$. For our purpose, it is convenient to convert the problem (1.5) into a deterministic system with a random parameter, and then show that it generates a RDS. Consider Ornstein-Uhlenbeck equations

$$dz_j + z_j dt = dW_j(t),\ j = \{1, 2, \cdots, m\}, \tag{3.1}$$

and Ornstein-Uhlenbeck processes

$$z_j(\theta_t\omega_j) = -\int_{-\infty}^0 e^s(\theta_t\omega_j)(s)ds,\ t \in \mathbb{R}.$$

From [12,13], it is known that the random variable $|z_j(\omega_j)|$ is tempered, and for every $\omega \in \Omega$, there is a θ_t-invariant set $\widetilde{\Omega} \subset \Omega$ of full $\mathbb{P}$ measure such that $t \mapsto z_j(\theta_t\omega_j),\ j = 1, 2, \cdots, m$, is continuous in t. Put

$$z(\theta_t\omega) = z(x, \theta_t\omega) = \sum_{j=1}^m h_j z_j(\theta_t\omega_j), \tag{3.2}$$

which is a solution to

$$dz + zdt = \sum_{j=1}^{m} h_j dW_j.$$

Let

$$v(t,\omega,x) = u_t(t,\omega,x) + \varepsilon u(t,\omega,x),\ t \geq \tau,\ \psi(t,\omega,x) = (u,v,\eta)^T,$$

here

$$\varepsilon = \frac{2\alpha}{3+\kappa\alpha+\alpha^2/\lambda_1+\sqrt{(3+\kappa\alpha+\alpha^2/\lambda_1)^2-12\kappa\alpha}} > 0,\ \kappa = \frac{2\|\mu\|_{L^1(\mathbb{R}^+)}}{\delta} > 0,$$

where $\lambda_1(>0)$ is the smallest eigenvalue of operator A with the Neumann boundary condition on U. The initial problem (1.5) can be written as the following equivalent system in E :

$$\dot{\psi} + H(\psi) = N(Z,t,W(t)),\ \psi_\tau(\omega) = (u_0, u_1+\varepsilon u_0, \eta_0)^T,\ t \geq \tau,\ \tau \in \mathbb{R}, \tag{3.3}$$

where

$$H(\psi) = \begin{pmatrix} \varepsilon u - v \\ Au + (\alpha-\varepsilon)v - \varepsilon(\alpha-\varepsilon)u + \int_0^\infty \mu(s)A\eta(s)ds \\ \varepsilon u - v + \eta_s \end{pmatrix} = -T_\varepsilon L T_\varepsilon(\psi),$$

$$T_\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ \varepsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.4}$$

Let

$$w(t,\omega,x) = u_t(t,\omega,x) + \varepsilon u(t,\omega,x) - z(\theta_t\omega),\ \varphi = (u,w,\eta)^T,$$

then the problem (3.3) is equivalent to the following determined system with random parameters in E :

$$\dot{\varphi} + H(\varphi) = F(\varphi,\theta_t\omega,t),\ \varphi_\tau(\omega) = (u_0, u_1+\varepsilon u_0 - z(\omega), \eta_0)^T,\ t \geq \tau,\ \tau \in \mathbb{R}, \tag{3.5}$$

where

$$F(\varphi,\theta_t\omega,t) = \begin{pmatrix} z(\theta_t\omega) \\ -g(u) + f(x) - (\alpha-\varepsilon-1)z(\theta_t\omega) \\ z(\theta_t\omega) \end{pmatrix}. \tag{3.6}$$

We know from [17] that the operator L in (1.5) is the infinitesimal generator of a C_0 semigroup $\{e^{Lt}\}$ of contractions on the space E. Since $-H =$

$T_\varepsilon L T_{-\varepsilon}$, and T_ε is an isomorphism of E, the operator $-H$ also generates a C_0 -semigroup $\{e^{-Ht}\}$ of contractions on E .

By (H_2) and the embedding relation $H_0^2 \hookrightarrow L^{10}$, the function $F(\varphi, \theta_t\omega, t)$ is locally Lipschitz with respect to φ from E into E for t in bounded interval and $\omega \in \Omega$, and $F(\varphi, \theta_t\omega, t)$ is continuous in (φ, t) and measurable in ω w.r.t. $\mathscr{F}$. By the standard theory of operators semigroup concerning the existence and uniqueness of solutions of evolution equations [17], we have the following theorem.

Theorem 3.1. *If* $(H_1)-(H_2)$ *and* (1.2)−(1.4) *hold. Then for each* $\omega \in \Omega$ *and* $\varphi_\tau \in E$, *there exists* $T > 0$, *such that* (3.5) *has a unique mild solution* $\varphi(\cdot, \omega, \psi_\tau) \in C([\tau, \tau + T); E)$ *with* $\varphi(\tau, \omega, \varphi_\tau) = \varphi_\tau$, *and*

$$\varphi(t, \omega, \varphi_\tau) = e^{-H(t-\tau)}\varphi_\tau(\omega) + \int_\tau^t e^{-H(t-s)} F(s, \theta_s\omega, \varphi(s, \omega, \varphi_\tau))ds. \quad (3.7)$$

Furthermore, $\varphi(t, \omega, \varphi_\tau)$ *is jointly continuous in* φ_τ, *and measurable in* ω.

From Theorem 3.1 *and Lemma* 4.1 *below, the solution* $\varphi(\cdot, \omega, \varphi_\tau)$ *exists globally for* $t \in [\tau, \infty)$. *Then the solution* $\varphi(\cdot, \omega, \varphi_\tau) \in C([\tau, +\infty); E)$, *which can define a continuous random dynamical system over* $\mathbb{R}$ *and* $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$:

$$\Phi(t, \omega) : \varphi_\tau \mapsto \varphi(t, \omega, \varphi_\tau). \quad (3.8)$$

It is easy to see that

$$\Upsilon(t, \omega, Z_\tau) = R^{-1}_{\varepsilon,\theta_t\omega} \Phi(t, \omega) R_{\varepsilon,\theta_t\omega} : Z_\tau \to Z(t, \omega, Z_\tau) \quad (3.9)$$

and

$$\Psi(t, \omega, \psi_\tau) = T_\varepsilon \Upsilon T_{-\varepsilon} : \psi_\tau \to \psi(t, \omega, \psi_\tau) \quad (3.10)$$

are continuous random dynamical systems over $\mathbb{R}$ and $(\Omega, \mathscr{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ associated with systems (1.5) and (3.3), respectively, where

$$\psi(t, \omega, \psi_\tau) = \varphi(t, \omega, \psi_\tau) + (0, z(\theta_t\omega), 0)^T, \quad (3.11)$$

and $R_{\varepsilon,\theta_t\omega} : (a, b, c)^T \mapsto (a, b + \varepsilon a - z(\theta_t\omega), c)^T$ is an isomorphism of E . Therefore, Φ, Υ, and Ψ are equivalent to each other in dynamics. In this article, we will study the existence of a random attractor for RDS Φ based on Theorem 2.3.

4. Random absorbing set

THEOREM 4.1. *Assume that* $(H_1)-(H_2)$ *and* $(1.2)-(1.4)$ *hold. There exists a random variable* $r_1(\omega) > 0$ *and a bounded ball* $B_0 \subset E$, *centered at 0 with random radius* $r_0(\omega) > 0$ *such that for any bounded non-random set* $B \subset E$, *there exists a deterministic* $T(B) \leq -1$, *such that the solution* $\varphi(t,\omega;\varphi_\tau(\omega))$ *of* (3.5)
with initial value $(u_0, u_1+\varepsilon u_0, \eta_0)^T \in B$ *satisfies, for* P-*a.s.*$\omega \in \Omega$,

$$\|\varphi(-1,\omega;\varphi(\tau,\omega))\|_E \leq r_0(\omega),\ \ \tau \leq T(B),$$

and for all $\tau \leq t \leq 0$,

$$\begin{aligned}\|\varphi(t,\omega;\varphi(\tau,\omega))\|_E^2 \leq& 2e^{-\beta_1(t-\tau)}\bigg(\|u_0\|_2^2 + \|u_1+\varepsilon u_0\|^2 + \|\eta_0\|_{\mu,2} + \|z(\omega)\| \\ &+ \int_U G(u_0)dx\bigg) + r_1^2(\omega) \\ \doteq& r_2(\omega).\end{aligned} \tag{4.1}$$

Proof. Taking the inner product $(\cdot,\cdot)_E$ of (3.5) with $\varphi(r) = (u(r), w(r), \eta(r))^T$, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\varphi\|_E^2 + (H(\varphi),\varphi)_E =& ((z(\theta_t\omega),u)) - (g(u),w) \\ &+ (f(x) - (\alpha-\varepsilon-1)z(\theta_t\omega),w) + (z(\theta_t\omega),\eta)_{\mu,2}.\end{aligned} \tag{4.2}$$

Similar to the estimates of Lemma 2 in [16],

$$(H(\varphi),\varphi)_E \geq \frac{\varepsilon}{2}(\|u\|_2^2 + \|w\|^2) + \frac{\delta}{4}\|\eta\|_{\mu,2}^2 + \frac{\alpha}{2}\|w\|^2. \tag{4.3}$$

We estimate each term of the right-hand side of (4.2) as follows,

$$((z(\theta_t\omega),u)) \leq \frac{\varepsilon}{8}\|u\|_2^2 + \frac{2}{\varepsilon}\|z(\theta_t\omega)\|_2^2, \tag{4.4}$$

$$|(f(x)-(\alpha-\varepsilon-1)z(\theta_t\omega),w)| \leq \frac{\alpha}{2}\|w\|^2 + \alpha\|z(\theta_t\omega)\|^2 + \frac{1}{\alpha}\|f(x)\|^2, \tag{4.5}$$

$$(z(\theta_t\omega),\eta)_{\mu,2} \leq \frac{1}{\delta}\|z(\theta_t\omega)\|_2^2 + \frac{\delta}{8}\|\eta\|_{\mu,2}^2. \tag{4.6}$$

Using (1.2) and (1.3), we have

$$
\begin{aligned}
(g(u), z(\theta_t\omega)) \leq & C_0 \int_U (1+|u|^5)|z(\theta_t\omega)|dx \\
\leq & C_0 \int_U |z(\theta_t\omega)|dx + C_0 \left(\int_U |u|^6 dx\right)^{\frac{5}{6}} \|z(\theta_t\omega)\|_6 \\
\leq & C_0 \int_U |z(\theta_t\omega)|dx + C_0 C_1^{-\frac{5}{6}} \left(\int_U (G(u)+C_1)dx\right)^{\frac{5}{6}} \|z(\theta_t\omega)\|_6 \\
\leq & C_0|z(\theta_t\omega)| + \frac{\varepsilon C_2}{2}\int_U G(u)dx + \frac{\varepsilon C_1 C_2 |U|}{2} + C_0\|z(\theta_t\omega)\|_6^6,
\end{aligned}
\tag{4.7}
$$

and by condition (1.4), there exists a constant $M_1 > 0$, such that

$$
(g(u), u) - C_2 \int_U G(u)dx + \frac{\lambda_1}{8}\|u\|_2^2 \geq -M_1, \tag{4.8}
$$

then by (4.7) – (4.8) and Poincaré inequality, it derives

$$
\begin{aligned}
-(g(u), w) = & -(g(u), u_t + \varepsilon u - z(\theta_t\omega)) \\
\leq & -\frac{d}{dt}\int_U G(u)dx - \varepsilon C_2 \int_U G(u)dx + \frac{\varepsilon\lambda_1}{8}\|u\|^2 + \varepsilon M_1 \\
& + C_0|z(\theta_t\omega)| + \frac{\varepsilon C_2}{2}\int_U G(u)dx + \frac{\varepsilon C_1 C_2|U|}{2} + C_0\|z(\theta_t\omega)\|_6^6 \\
\leq & -\frac{d}{dt}\int_U G(u)dx - \frac{\varepsilon C_2}{2}\int_U G(u)dx + \frac{\varepsilon}{8}\|u\|_2^2 + \varepsilon M_1 \\
& + C_0|z(\theta_t\omega)| + \frac{\varepsilon C_1 C_2|U|}{2} + C_0\|z(\theta_t\omega)\|_6^6.
\end{aligned}
\tag{4.9}
$$

Collecting all above (4.3) – (4.9), it yields

$$
\begin{aligned}
& \frac{d}{dt}\left(\|\varphi\|_E^2 + 2\int_U G(u)dx\right) \\
\leq & -\frac{\varepsilon}{2}\|u\|_2^2 - \varepsilon\|w\|^2 - \frac{\delta}{4}\|\eta\|_{\mu,2}^2 - \varepsilon C_2\int_U G(u)dx + \left(\frac{4}{\varepsilon}+\frac{2}{\delta}\right)\|z(\theta_t\omega)\|_2^2 \\
& + 2\varepsilon M_1 + 2\alpha\|z(\theta_t\omega)\|^2 + 2C_0|z(\theta_t\omega)| + \varepsilon C_1 C_2|U| + \frac{2}{\alpha}\|f(x)\|^2 \\
& + 2C_0\|z(\theta_t\omega)\|_6^6.
\end{aligned}
\tag{4.10}
$$

Let $R_0(\theta_t\omega) = 1+\|z(\theta_t\omega)\|_2^2+\|z(\theta_t\omega)\|^2+|z(\theta_t\omega)|+\|z(\theta_t\omega)\|_6^6$, it follows that

$$\frac{d}{dt}\left(\|\varphi\|_E^2+2\int_U G(u)dx\right)+\beta_1\left(\|\varphi\|_E^2+2\int_U G(u)dx\right)\le 2CR_0(\theta_t\omega),$$

where $C=\max\{\frac{2}{\varepsilon}+\frac{1}{\delta},\alpha,\varepsilon M_1+\frac{\varepsilon C_1C_2|U|}{2}+\frac{1}{\alpha}\|f(x)\|^2,C_0\}$, $\beta_1=\min\{\frac{\varepsilon}{2},\frac{\delta}{4},\frac{\varepsilon C_2}{2}\}$. Applying Gronwall's inequality

$$\begin{aligned}
&\|\varphi(t,\omega,\varphi(\tau,\omega))\|_E^2\\
\le& e^{-\beta_1 t}\left(e^{\beta_1\tau}\left(\|\varphi(\tau,\omega)\|_E^2+2\int_U G(u_0)dx\right)\right)+2C\int_\tau^t R_0(\theta_s\omega)e^{-\beta_1(t-s)}ds\\
\le& 2Ce^{-\beta_1(t-\tau)}\left(\|u_0\|_2^2+\|u_1+\varepsilon u_0\|^2+\|\eta_0\|_{\mu,2}^2+\|z(\omega)\|^2+\int_U G(u_0)dx\right)\\
&+2C\int_\tau^t R_0(\theta_s\omega)e^{-\beta_1(t-s)}ds,
\end{aligned}\tag{4.11}$$

by Lemma 3.1 of [12],

$$\int_\tau^t R_0(\theta_s\omega)e^{-\beta_1(t-s)}ds\le\int_\tau^t R_1(s,\omega)e^{-\beta_1(t-s)}ds\le\int_{-\infty}^0 R_1(s,\omega)e^{-\beta_1(t-s)}ds$$
$$<+\infty,$$

where

$$\begin{aligned}
R_1(t,\omega)=&1+(e^{-\sigma t}r(\omega))^2+\frac{(e^{-\sigma t}r(\omega))^2}{\lambda_1}+\frac{e^{-\sigma t}r(\omega)}{\sqrt{\lambda_1}}+(e^{-\sigma t}r(\omega))^6\\
=&1+\frac{\lambda_1+1}{\lambda_1}(e^{-\sigma t}r(\omega))^2+\frac{e^{-\sigma t}r(\omega)}{\sqrt{\lambda_1}}+(e^{-\sigma t}r(\omega))^6.
\end{aligned}$$

Put

$$r_0^2(\omega)=2Ce^{\beta_1}\left(1+\sup_{\tau\le -1}e^{\beta_1\tau}\|z(\omega)\|^2+\int_{-\infty}^{-1}R_1(s,\omega)e^{\beta_1(s)}ds\right),$$

$$r_1^2(\omega)=\int_{-\infty}^0 e^{\beta_1 s}R_1(s,\omega)ds.$$

Obviously, the quantities $r_0^2(\omega)$ and $r_1^2(\omega)$ are finite P-a.s. as $s\to\infty$, for any bounded set $B\subset E$, choose $T(B)\le -1$, such that for all $(u_0,u_1+\varepsilon u_0,\eta_0)^T\in B$, one concludes

$$e^{\beta_1(-1-\tau)}\left(\|u_0\|_2^2+\|u_1+\varepsilon u_0\|^2+\|\eta_0\|_{\mu,2}^2+\int_U G(u_0)dx\right)\le 1,\ \tau\le T(B),$$

and

$$-\tau e^{\beta_1\tau}\left(\|u_0\|_2^2+\|u_1+\varepsilon u_0\|^2+\|\eta_0\|_{\mu,2}^2+\int_U G(u_0)dx\right)\leq 1,\ \tau\leq T(B),$$

the proof is completed. □

5. Decomposition of solutions

In order to obtain the regularity estimates later, as in [13], we decompose the nonlinear term $g(u)$ as

$$g(u)=g_1(u)+g_2(u),$$

where $g_1(u),\ g_2(u)$ satisfy

$$g_1^{'}(0)=0,\ g_1(u)u\geq 0,\ |g_1^{''}(u)|\leq C_3(1+|u|^3), \tag{5.1}$$

$$g_2(0)=0,\ |g_2^{'}(u)|\leq C_4(1+|u|^\gamma),\ 0\leq\gamma<4. \tag{5.2}$$

$$C_5|u|^6-C_6\leq G_i(u)\leq C_7ug_i(u)+C_8,\ G_i(u)=\int_0^u g_i(r)dr\geq 0,\ i=1,2. \tag{5.3}$$

And we decompose $\varphi=(u,w,\eta)^T$ of the system (3.5) into $\varphi=\varphi_L+\varphi_N$, where $\varphi_L=(u_L,v_L,\eta_L)^T$ and $\varphi_N=(u_N,w_N,\eta_N)^T$ solve respectively the following equatins,

$$\dot{\varphi}_L+H(\varphi_L)+F_1(\varphi_L)=0,\ \varphi_L(\tau,\omega)=(u_0,u_1+\varepsilon u_0,\eta_0)^T,\ t\geq\tau,\ \tau\in\mathbb{R}, \tag{5.4}$$

and

$$\dot{\varphi_N}+H(\varphi_N)+F_2(\varphi,\varphi_L)=\tilde{F}_2(\omega),\ \varphi_N(\tau,\omega)=(0,-z(\omega),0)^T,\ t\geq\tau,\ \tau\in\mathbb{R}, \tag{5.5}$$

where

$$\begin{aligned} F_1(\varphi_L)&=\begin{pmatrix}0\\ g_1(u_L)\\ 0\end{pmatrix}, F_2(\varphi,\varphi_L)=\begin{pmatrix}0\\ g(u)-g_1(u_L)\\ 0\end{pmatrix},\\ \tilde{F}_2(\omega)&=\begin{pmatrix}z(\theta_t\omega)\\ f(x)-(\alpha-\varepsilon-1)z(\theta_t(\omega))\\ z(\theta_t\omega)\end{pmatrix}.\end{aligned} \tag{5.6}$$

For the solutions of equations (5.4) and (5.5), we have the following estimates and regularity results, respectively.

THEOREM 5.1. *Let B be a bounded non-random subset of E, for any $\varphi_L(\tau,\omega) = (u_0, u_1+\varepsilon u_0, \eta_0)^T \in B$, there holds*

$$\|\varphi_L(0,\omega;\varphi_L(\tau,\omega))\|_E^2 \le r_3^2(\omega), \tag{5.7}$$

where $\varphi_L = (u_L, v_L, \eta_L)^T$ satisfies (5.4).

Proof. Taking the inner $(\cdot,\cdot)_E$ of (5.4) with $\varphi_L = (u_L, v_L, \eta_L)^T$, in which $v_L = u_{Lt} + \varepsilon u_L$, whose initial value is $(u_0, u_1+\varepsilon u_0, \eta_0)^T$,

$$\frac{1}{2}\frac{d}{dt}\left(\|\varphi_L\|_E^2 + 2\int_U G_1(u_L)dx\right) + (H(\varphi_L),\varphi_L)_E + \varepsilon\,(g_1(u_L),u_L) = 0, \tag{5.8}$$

by simple computation there holds

$$(H(\varphi_L),\varphi_L)_E \ge \frac{\varepsilon}{2}\left(\|u_L\|_2^2 + \|v_L\|^2\right) + \frac{\delta}{4}\|\eta_L\|_{\mu,2}^2 + \frac{\alpha}{2}\|v_L\|^2. \tag{5.9}$$

By (5.1) and (5.3), we have

$$G_1(u_L) \ge 0,\ (g_1(u_L),u_L) \ge \frac{1}{C_7}\int_U (G_1(u_L) - C_8)dx. \tag{5.10}$$

Thus, by $(5,9)-(5.10)$, we have

$$\begin{aligned}\frac{d}{dt}\left(\|\varphi_L\|_E^2 + 2\int_U G_1(u_L)dx\right) + \varepsilon(\|u_L\|_2^2 + \|v_L\|^2) + \frac{\delta}{2}\|\eta_L\|_{\mu,2}^2 \\ + \alpha\|v_L\|^2 + \frac{2\varepsilon}{C_7}\int_U G_1(u_L)dx \le C_9,\end{aligned} \tag{5.11}$$

that is,

$$\frac{d}{dt}\|y_L\|_E^2 + \beta_2\|y_L\|_E^2 \le C_9, \tag{5.12}$$

where $\beta_2 = \min\{\varepsilon, \frac{\delta}{2}, \frac{\varepsilon}{C_7}\}$, and

$$y_L = \|\varphi_L\|_E^2 + 2\int_U G_1(u_L)dx \ge \|\varphi_L\|_E^2 \ge 0.$$

Since $\varphi(0,\omega;\varphi(\tau,\omega)) = \varphi_L(0,\omega;\varphi_L(\tau,\omega)) + (0,z(\omega),0) \in B_0(\omega)$, by definition of $B_0(\omega)$, it follows that

$$\|\varphi_L(0,\omega,\varphi_L(\tau,\omega))\|_E \le r_2(\omega) + |z(\omega)| = M_1(\omega).$$

By Gronwall's inequality to (5.12), we have

$$\begin{aligned}\|\varphi_L(0,\omega,\varphi_L(\tau,\omega))\|_E^2 &\le y_L(0,\omega,\varphi_L(\tau,\omega))\\ &\le y_L(\tau,\omega,\varphi_L(\tau,\omega))e^{\beta_2\tau}+\frac{C_9}{\beta_2}\\ &\le \left(\|\varphi_L(\tau,\omega)\|_E^2+C_{10}(|U|+\|u_L\|^5)\right)e^{\beta_2\tau}+\frac{C_9}{\beta_2}\\ &\le \left(M_1^2(\omega)+C_{10}\left(|U|+M_1^5(\omega)\right)\right)e^{\beta_2\tau}+\frac{C_9}{\beta_2}=r_3^2(\omega),\end{aligned} \tag{5.13}$$

the proof is completed. □

Theorem 5.2. *Let B be a bounded non-random subset of E, for any $\varphi_L(\tau,\omega)=(u_0,u_1+\varepsilon u_0,\eta_0)^T\in B$, we have*

$$\|\varphi_L(0,\omega,\varphi_L(\tau,\omega))\|_E^2\le r_4^2(\omega)e^{2\sigma_1(\omega)\tau},\ \tau\le 0, \tag{5.14}$$

where $\varphi_L=(u_L,v_L,\eta_L)^T$ satisfies (5.4).

Proof. Consider (5.8). By (5.1), $(g_1(u_L),u_L)\ge 0, g_1(0)=0$ and $|g_1(u_L)|\le C_{11}(|u_L|^5+|u_L|)$. By Sobolev embedding $H_0^2\subset L^6\subset L^2$ and (5.7), there exists $M_2(\omega)$ such that

$$0\le\int_U G_1(u_L)\le C_{11}(|u_L|_{L^6}^6+|u_L|^2)\le M_2(\omega)\|u_L\|^2, \tag{5.15}$$

i.e.,

$$\|u_L\|^2\ge\frac{1}{M_2(\omega)}\int_U G_1(u_L)dx. \tag{5.16}$$

By (5.8), (5.9) and (5.16), we have

$$\begin{aligned}\frac{d}{dt}\left(\|\varphi_L\|_E^2+2\int_U G_1(u_L)dx\right)+\frac{\varepsilon}{2}(\|u_L\|^2+|v_L|^2)\\ +\frac{\delta}{2}\|\eta_L\|_{\mu,2}^2+\frac{\varepsilon}{2M_2(\omega)}\int_U G_1(u_L)dx\le 0.\end{aligned} \tag{5.17}$$

Thus,

$$\frac{d}{dt}\left(\|\varphi_L\|_E^2+2\int_U G_1(u_L)dx\right)+2\sigma_1(\omega)\left(\|\varphi_L\|_E^2+2\int_U G_1(u_L)dx\right)\le 0, \tag{5.18}$$

where

$$\sigma_1(\omega)=\min\{\frac{\varepsilon}{4},\frac{\delta}{4},\frac{\varepsilon}{8M_2(\omega)}\}>0. \tag{5.19}$$

By (5.18), it yields

$$\|\varphi_L(0,\omega;\varphi_L(\tau,\omega))\|_E^2 \le \left(\|\varphi_L(\tau,\omega)\|_E^2 + 2\int_U G_1\left(u_L(\tau)\right)dx\right)e^{2\sigma_1(\omega)\tau}$$
$$\le \left(M_1^2(\omega) + C(|U| + M_1^2(\omega))\right)e^{2\sigma_1(\omega)\tau}$$
$$\le r_4^2(\omega)e^{2\sigma_1(\omega)\tau}. \tag{5.20}$$

□

THEOREM 5.3. *Assume that* (5.1) − (5.3) *hold, there exists a random radius* $r_5(\omega)$*, such that for* P*-a.e.*$\omega \in \Omega$,

$$\|A^{\frac{\nu}{2}}u_N\|_2^2 + \|A^{\frac{\nu}{2}}u_{Nt}\|^2 + \|A^{\frac{\nu}{2}}\eta_N\|_{\mu,2}^2 \le r_5(\omega). \tag{5.21}$$

Proof. By (5.7), (4.1) and $\varphi_N = \varphi - \varphi_L$, there exists a random variables $r(\omega) > 0$, such that

$$\max\{\|\varphi(0,\omega,\varphi(\tau,\omega))\|_E, \|\varphi_N(0,\omega,\varphi_N(\tau,\omega))\|_E\} \le r(\omega).$$

Taking the inner of $(\cdot,\cdot)_E$ of (5.5) with $A^\nu\varphi_N = (A^\nu u_N, A^\nu w_N, A^\nu\eta_N)^T$, $\nu = \min\{\frac{1}{4}, \frac{4-\gamma}{4}\}$, we find

$$\frac{1}{2}\frac{d}{dt}\left(\|A^{\frac{\nu}{2}}u_N\|_2^2 + \|A^{\frac{\nu}{2}}w_N\|^2 + \|A^{\frac{\nu}{2}}\eta_N\|_{\mu,2}^2 + 2\int_U \left(g(u) - g_1(u_L)\right)A^\nu u_N dx\right)$$
$$+ (H(\varphi_N), A^\nu\varphi_N)_E + \varepsilon\int_U \left(g(u) - g_1(u_L)\right)A^\nu u_N dx$$
$$- \int_U \left(\left(g_1'(u) - g_1'(u_L)\right)u_t + g_2'(u)u_t + g_1'(u_L)u_{Nt}\right)A^\nu u_N dx$$
$$= ((z(\theta_t\omega), A^\nu u_N)) + (f(x) - (\alpha-\varepsilon)z(\theta_t\omega), A^\nu w_N) + (z(\theta_t\omega), A^\nu\eta_N)_{\mu,2}$$
$$+ (g(u) - g_1(u_L), A^\nu z(\theta_t\omega)). \tag{5.22}$$

Now, we deal with the right terms in (5.22) one by one as follows

$$(H(\varphi_N), A^\nu\varphi_N)_E \ge \frac{\varepsilon}{2}\left(\|A^{\frac{\nu}{2}}u_N\|_2^2 + \|A^{\frac{\nu}{2}}w_N\|^2\right) + \frac{\delta}{4}\|A^{\frac{\nu}{2}}\eta_N\|_{2\mu,2}^2 + \frac{\alpha}{2}\|A^{\frac{\nu}{2}}w_N\|^2, \tag{5.23}$$

$$((z(\theta_t\omega), A^\nu u_N)) \le \frac{\varepsilon}{4}\|A^{\frac{\nu}{2}}u_N\|_2^2 + \frac{1}{\varepsilon}\|A^{\frac{\nu}{2}}z(\theta_t\omega)\|_2^2, \tag{5.24}$$

$$|\left(f(x) - (\alpha-\varepsilon-1)z(\theta_t\omega)\,, A^\nu w_N\right)| \le \frac{\alpha}{2}\|A^{\frac{\nu}{2}}w_N\|^2 + \frac{1}{\alpha}\|A^{\frac{\nu}{2}}z(\theta_t\omega)\|^2 + \alpha\|f(x)\|^2, \tag{5.25}$$

$$(z(\theta_t\omega), A^{\nu}\eta_N)_{\mu,2} \leq \frac{1}{\delta}\|A^{\frac{\nu}{2}}z(\theta_t\omega)\|_2^2 + \frac{\delta}{4}\|A^{\frac{\nu}{2}}\eta_N\|_{\mu,2}^2, \tag{5.26}$$

$$\begin{aligned}(g(u)-g_1(u_L), A^{\nu}z(\theta_t\omega)) \leq& \frac{\varepsilon}{4}\|g(u)-g_1(u_L)\|^2 + \frac{1}{\varepsilon}\|A^{\nu}z(\theta_t\omega)\|^2 \\ \leq& C_3(1+\|u\|_2^5 + 1 + \|u_L\|_2^5) + \frac{1}{\varepsilon}\|A^{\nu}z(\theta_t\omega)\|^2 \\ \leq& C_{12}(\omega) + \frac{1}{\varepsilon}\|A^{\frac{\nu}{2}}z(\theta_t\omega)\|_2^2\end{aligned} \tag{5.27}$$

$$\begin{aligned}\int_U g_1^{'}(u_L)u_{Nt}A^{\nu}u_N dx \leq& C_3\left(\int_U (1+|u_L|^4)^{\frac{5}{2}}\right)^{\frac{2}{5}} \cdot \left(\int_U |u_{Nt}|^{\frac{10}{5-4\nu}}\right)^{\frac{5-4\nu}{10}} \\ &\cdot \left(\int_U |A^{\nu}u_N|^{\frac{10}{1+4\nu}}\right)^{\frac{1+4\nu}{10}} \\ \leq& C_3(1+\|u_L\|_2^4)\cdot\|A^{\frac{\nu}{2}}u_{Nt}\|\cdot\|A^{\frac{\nu}{2}}u_N\|_2 \\ \leq& C_{13}(\omega)\|A^{\frac{\nu}{2}}u_N\|_2(\|A^{\frac{\nu}{2}}w_N\|+\varepsilon),\end{aligned} \tag{5.28}$$

$$\begin{aligned}\int_U g_2^{'}(u)u_t A^{\nu}u_N dx \leq& C_4\int_U |u_t|(1+|u|^{\gamma})|A^{\nu}u_N|dx \\ \leq& C_4\left(\int_U |u_t|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int_U (1+|u|^{\gamma})^{\frac{10}{4-4\nu}}\right)^{\frac{4-4\nu}{10}} \\ &\cdot \left(\int_U |A^{\nu}u_N|^{\frac{10}{1+4\nu}}\right)^{\frac{1+4\nu}{10}} \\ \leq& C_4|u_t|\cdot(1+\|u\|_2^{\gamma})\cdot\|A^{\frac{\nu}{2}}u_N\|_2 \leq C_{14}(\omega)\|A^{\frac{\nu}{2}}u_N\|_2,\end{aligned} \tag{5.29}$$

$$\begin{aligned}\int_U [g_1^{'}(u)-g_1^{'}(u_L)]u_t A^{\nu}u_N dx \leq& C_3\int_U |u_t|(1+|u_L|^3+|u_N|^3|)|u_N|A^{\nu}u_N|dx \\ \leq& C_3\left(\int_U |u_t|^2 dx\right)^{\frac{1}{2}} \cdot \left(\int_U (1+|u_L|^3+|u_N|^3)^{\frac{10}{3}}\right)^{\frac{3}{10}} \\ &\times\left(\int_U |u_N|^{\frac{10}{1-4\nu}}\right)^{\frac{1-4\nu}{10}} \cdot \left(\int_U |A^{\nu}u_N|^{\frac{10}{1+4\nu}}\right)^{\frac{1+4\nu}{10}} \\ \leq& C_{15}(\omega)\|A^{\frac{\nu}{2}}u_N\|_2.\end{aligned} \tag{5.30}$$

Let

$$y = \|A^{\frac{\nu}{2}} u_N\|_2^2 + \|A^{\frac{\nu}{2}} w_N\|^2 + \|A^{\frac{\nu}{2}} \eta_N\|_{\mu,2}^2 + 2\int_U [g(u) - g_1(u_L)] A^\nu u_N dx, \tag{5.31}$$

and by putting (5.23)-(5.31) into (5.22), yields

$$\begin{aligned} \frac{d}{dt} y + C_{16}(\omega) y \le C_{17}(\omega) + \left(\frac{2}{\varepsilon} + \frac{1}{\alpha} + \frac{1}{\delta}\right) \|A^{\frac{\nu}{2}} z(\theta_t \omega)\|_2^2 \\ \le C_{17}(\omega) + C_{18} \|A^{\frac{\nu}{2}} z(\theta_t \omega)\|_2^2, \end{aligned} \tag{5.32}$$

by Gronwall's lemma and Lemma 3.1 in [12], it follows that

$$\begin{aligned} y(0, \omega, \varphi(\tau, \omega)) \le & e^{C_{16}(\omega)\tau} y(\tau, \omega, \varphi(\tau, \omega)) \\ & + \int_\tau^0 \left(C_{17}(\omega) e^{C_{16}(\omega)s} + C_{18} e^{\frac{C_{16}(\omega)}{2} s} r^{(3)}(\omega)\right) ds \\ \le & e^{C_{16}(\omega)\tau} y(\tau, \omega, \varphi(\tau, \omega)) + C_{19}(\omega) \\ & + C_{18} \int_\tau^0 e^{\frac{C_{16}(\omega)}{2} s} r^{(3)}(\omega) ds. \end{aligned} \tag{5.33}$$

Note that

$$\int_U (g(u) - g_1(u_L)) A^\nu u_N dx = \int_U [g_2(u) + g_1(u) - g_1(u_L)] A^\nu u_N dx, \tag{5.34}$$

where

$$\begin{aligned} \left| \int_U g_2(u) A^\nu u_N dx \right| \le & C_4 \int_U (1 + |u|^{\gamma+1}) |A^\nu u_N| dx \\ & \le C_4 \left(\int_U (1 + |u|^{\gamma+1})^{\frac{10}{9-4\nu}} dx \right)^{\frac{9-4\nu}{10}} \cdot \left(\int_U |A^\nu u_N|^{\frac{10}{1+4\nu}} dx \right)^{\frac{1+4\nu}{10}} \\ & \le C_{20}(\omega) \|A^{\frac{\nu}{2}} u_N\|_2 \quad \le C_{20}^2(\omega) + \frac{1}{4} \|A^{\frac{\nu}{2}} u_N\|_2^2, \end{aligned} \tag{5.35}$$

and

$$\begin{aligned}\int_U (g_1(u)-g_1(u_L))\, A^\nu u_N dx \leq& C_3\int_U (1+|u_L|^4+|u_N|^4|)|u_N||A^\nu u_N|dx\\ &\leq C_3\left(\int_U (1+|u_L|^4+|u_N|^4)^{\frac{5}{2}}dx\right)^{\frac{2}{5}}\\ &\quad\times\left(\int_U |u_N|^{\frac{10}{5-4\nu}}dx\right)^{\frac{5-4\nu}{10}}\cdot\left(\int_U |A^\nu u_N|^{\frac{10}{1+4\nu}}dx\right)^{\frac{1+4\nu}{10}}\\ &\leq C_{21}(\omega)\|A^{\frac{\nu}{2}}u_N\|_2^2 \quad \leq C_{21}^2(\omega)+\frac{1}{4}\|A^{\frac{\nu}{2}}u_N\|^2.\end{aligned} \tag{5.36}$$

Therefore, by $(5.31),(5.33)-(5.36)$, we conclude

$$\begin{aligned}&\|A^{\frac{\nu}{2}}u_N\|_2^2+\|A^{\frac{\nu}{2}}u_{Nt}\|^2+\|A^{\frac{\nu}{2}}\eta_N\|_{\mu,2}^2\\ \leq& 2y+4\left(C_{20}^2(\omega)+C_{21}^2(\omega)\right)\\ \leq& 2\left(e^{C_{16}(\omega)\tau}y(\tau,\omega,\varphi(\tau,\omega))+C_{19}(\omega)+C_{18}\int_\tau^0 e^{\frac{C_{16}(\omega)}{2}s}r^{(3)}(\omega)ds\right)\\ &+2\left(C_{20}^2(\omega)+C_{21}^2(\omega)\right) \doteq r_5(\omega),\end{aligned} \tag{5.37}$$

the proof is completed. □

6. Existence of random attractor

Lemma 6.1. [13, 18] *Let X_0, X, X_1 be three Banach spaces such that $X_0 \hookrightarrow X \hookrightarrow X_1$, ,the first injection being compact. Let $Y \subset L^2_\mu(\mathbb{R}^+, X)$ satisfy the following hypotheses:*

(i) Y is bounded in $L^2_\mu(\mathbb{R}^+, X_0)\cap H^1_\mu(\mathbb{R}^+, X_1)$,

(ii) $\sup_{\eta\in Y}\|\eta(s)\|_X^2 \leq K_0, \forall s\in\mathbb{R}^+$, for some $K_0>0$.

Then Y is relatively compact in $L^2_\mu(\mathbb{R}^+, X)$.

Note that for any $\forall\tau\in\mathbb{R},\omega\in\Omega, t\geq 0$

$$\eta_N(t,\omega,\varphi(\tau,\omega),s)=\begin{cases} u_N(t,\omega,\varphi(\tau,\omega))-u_N(t-s,\omega,\varphi(\tau,\omega)), & s\leq t,\\ \eta_N(t,\omega,\varphi(\tau,\omega)), & s\geq t,\end{cases} \tag{6.1}$$

$$\eta_{Ns}(t,\omega,\varphi(\tau,\omega),s)=\begin{cases} u_{Nt}(t-s,\omega,\varphi(\tau,\omega)), & s\leq t,\\ 0, & s\geq t.\end{cases} \tag{6.2}$$

Define a set

$$\tilde{B}(\tau,\omega) = \overline{\cup_{\varphi(\tau,\omega)\in B_0(r_0(\omega))} \cup_{t\geq 0} \eta_N(t,\omega,\varphi(\tau,\omega),s)},$$

where $\varphi = (u, w, \eta)^T$ is the solution of (3.5), then from Lemma 5.3 and (6.1)-(6.2), it follows that

$$\max\{\|\eta_{Ns}(t,\omega,\varphi(\tau,\omega),s)\|^2_{\mu,2\nu}, \|\eta_N(t,\omega,\varphi(\tau,\omega),s)\|^2_{\mu,2\nu+2}\} \leq 2r_5(\omega),\ \forall s\geq 0, \tag{6.3}$$

which imply $\tilde{B}(\tau,\omega)$ is bounded in $L^2_\mu(\mathbb{R}^+, V_{2\nu+2}) \cap H^1_\mu(\mathbb{R}^+, V_{2\nu})$. Again, by Lemma 4.1, Lemma 5.3 and (6.2), there holds

$$\sup_{\eta\in\tilde{B}(\tau,\omega),s\geq 0} \|\eta(s)\|^2 = \sup_{t\geq 0} \sup_{\varphi(\tau,\omega)\in B_0(r_0(\omega))} \|\eta_N(t,\omega,\varphi(\tau,\omega),s)\|^2 \leq 2r_2(\omega). \tag{6.4}$$

Thus, by (H_1), it follows that for any $\eta \in \tilde{B}(\tau,\omega)$

$$\|\eta(s)\|^2_{\mu,2} = \int_0^\infty \mu(s)\|\eta(s)\|^2 ds \leq 2r_2(\omega)\int_0^\infty \mu(s)e^{-\delta s}ds \leq \frac{2r_2(\omega)}{\delta}, \tag{6.5}$$

which shows that $\tilde{B}(\tau,\omega) \subset L^2_\mu(\mathbb{R}^+, H^2_0(U))$ is bounded. By Lemma 6.1,we know that the set $\tilde{B}(\tau,\omega)$ is compact in $L^2_\mu(\mathbb{R}^+, H^2_0(U))$. we prove our result about the existence of a random attractor for the RDS Φ as follows.

THEOREM 6.2. *Suppose* (1.2) − (1.4) *and* $(H1) - (H2)$ *hold, then for any* $\tau \in R, \omega \in \Omega$, *the RDS* Φ *associated with* (3.5) *possesses a uniformally attracting set* $\Lambda(\tau,\omega) \subset E$, *and possesses a random attractor* $\mathcal{A}(\tau,\omega) \subseteq \Lambda(\tau,\omega) \cap B_0(\omega)$.

Proof. For any $\tau \in R,\ \omega \in \Omega$, in view of Lemma 5.3, let $B_\nu(\tau,\omega)$ be the closed ball in $H_{2\nu+2} \times H_{2\nu}$, which radius is $r_5(\omega)$. Set

$$\Lambda(\tau,\omega) = B_\nu(\tau,\omega) \times \tilde{B}(\tau,\omega), \tag{6.6}$$

then $\Lambda(\tau,\omega) \in \mathscr{D}(E)$. Since $H_{2\nu+2} \times H_{2\nu} \hookrightarrow H^2_0(U) \times L^2(U)$, $B_\nu(\tau,\omega) \hookrightarrow H^2_0(U) \times L^2(U)$. Again, $\tilde{B}(\tau,\omega)$ is compact in $\mathfrak{R}_{\mu,2}$, thus $\Lambda(\tau,\omega)$ is compact in E. Now we show the following attraction property of $\Lambda(\tau,\omega)$:

for every $B(\tau,\omega) \in \mathscr{D}(E)$,

$$\lim_{t\to\infty} d_H(\Phi(t,\tau-t,\theta_{-t}\omega, B(\tau-t,\theta_{-t}\omega)), \Lambda(\tau,\omega)) = 0. \tag{6.7}$$

From Lemma 5.2, we have

$$\varphi_N(0,\omega,\varphi(\tau,\omega)) = \varphi(0,\omega,\varphi(\tau,\omega)) - \varphi_L(0,\omega,\varphi_L(\tau,\omega)) \in \Lambda(\tau,\omega). \tag{6.8}$$

Thus, by Lemma 5.2, yields

$$\inf_{\psi\in\Lambda(\tau,\omega)} \|\varphi(0,\omega,\varphi(\tau,\omega))-\psi\|_E^2 \le \|\varphi_L(0,\omega,\varphi_L(\tau,\omega))\|_E^2 \le r_4^2(\omega)e^{2\sigma_1(\omega)\tau}, \quad \tau\le 0. \tag{6.9}$$

Furthermore, for all $t>0$

$$dist\,(\Phi(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega)),\Lambda(\tau,\omega)) \le r_4^2(\omega)e^{-2\sigma_1(\omega)t}. \tag{6.10}$$

Finally, from the relation between Φ and Ψ, one can easily obtain that for any non-random bounded $B\subset E$ P-a.s.

$$dist(\Psi(t,\tau-t,\theta_{-t}\omega,B(\tau-t,\theta_{-t}\omega)),\Lambda(\tau,\omega)) \to 0,\ t\to+\infty. \tag{6.11}$$

Hence, the RDS Φ associated with (3.12) possesses a random attractor $\mathcal{A}(\tau,\omega)\subseteq\Lambda(\tau,\omega)\cap B_0(\omega)$.
The proof is completed. □

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