Korean J. Math. **24** (2016), No. 3, pp. 397–407
http://dx.doi.org/10.11568/kjm.2016.24.3.397

$k-$ DENTING POINTS AND $k-$ SMOOTHNESS OF BANACH SPACES

Suyalatu Wulede*, Shaoqiang Shang, and Wurina Bao

Abstract. In this paper, the concepts of $k-$smoothness, $k-$very smoothness and $k-$strongly smoothness of Banach spaces are dealt with together briefly by introducing three types $k-$denting point regarding different topology of conjugate spaces of Banach spaces. In addition, the characterization of first type $w^* - k$ denting point is described by using the slice of closed unit ball of conjugate spaces.

1. Introduction

Throughout this paper, $(X, \|\cdot\|)$ will denote a real Banach space and X^* will denote its conjugate space. Set

$U(X) = \{x : x \in X, \| x \| \leq 1\},\ U(x_0, \delta) = \{x : \ x \in X, \| x - x_0 \| \leq \delta\},$

$S(X) = \{x : x \in X, \| x \| = 1\},\ S_x = \{f : f \in S(X^*), f(x) = 1 = \| x \|\}.$

For $f \in X^*$ and $\delta > 0$, set $F(f, \delta)$ will denote the slice $\{x \in U(X) : f(x) > 1 - \delta\}$. The symbol $x_n \xrightarrow{w^*} x$ (resp. $x_n \xrightarrow{w} x$, $x_n \longrightarrow x$) will denote the sequence $\{x_n\}$ of X which w^* (resp. w, strong) convergence to x in X. $\sigma(X, w)$ will denote the weak topology of X and the open (resp. compact, closed) set regarding weak topology $\sigma(X, w)$ is said

Received May 12, 2016. Revised July 19, 2016. Accepted August 21, 2016.
2010 Mathematics Subject Classification: 46B09; 46B20.
Key words and phrases: k denting points, k smooth spaces, k very smooth spaces, k strongly smooth spaces, Banach spaces.
* Corresponding author.
This work was supported by the National Natural Science Foundation of China (Grant no.11561053) and Foundation of Inner Mongolia Normal University RCPY-2-2012-K-034.

to be w open (resp. w compact, w closed) set. The symbol $\sigma(X^*, w^*)$ will denote the weak* topology of X^* and the open (resp. compact, closed) set regarding weak* topology $\sigma(X^*, w^*)$ is said to be w^* open (resp. w^* compact, w^* closed) set. The neighborhood regarding weak (weak*)topology is said to be w (w^*) neighborhood. The accumulation point regarding weak* topology is said to be w^* accumulation point. The symbol co$M$ will denote the convex hull of set M and the symbol $\overline{H}^w$ (resp. $\overline{H}^{w^*}$) will denote the w (resp.w^*) closure of set H, where $H \subset X^*$.

Definition 1.1. A point $x^* \in S(X^*)$ is said to be first (resp. second) type weak$^* - k$ (in short $w^* - k$) denting point of $U(X^*)$ if there is a $x \in S(X)$ with $x^*(x) = 1$, dim$S_x \leq k$ such that for every norm (resp. w^*) open set V_{S_x} which includes set S_x, we have $S_x \bigcap \overline{co}^{w*}(U(X^*) \setminus V_{S_x}) = \emptyset$.

Definition 1.2. A point $x^* \in S(X^*)$ is said to be weak$-k$ (in short $w - k$) denting point of $U(X^*)$ if there is a $x \in S(X)$ with $x^*(x) = 1$, dim$S_x \leq k$
such that for every w open set V_{S_x} which includes set S_x, we have $S_x \bigcap \overline{co}^w(U(X^*) \setminus V_{S_x}) = \emptyset$.

Definition 1.3. [4] Let X be a Banach space. A point $x \in S(X)$ is said to be $k-$smooth point of X if the inequality dim$S_x \leq k$ holds for $x \in S(X)$, where dimS_x denote the linear dimension of S_x. X is said to be $k-$smooth space if every point of $S(X)$ is $k-$smooth point of X.

Definition 1.4. [4, 9] Let X be a Banach space. X is said to be k-strongly (resp.k-very) smooth space if and only if X is $k-$smooth space and for any sequence $\{f_n\} \subset S(X^*)$, $x \in S(X)$ and $f_n(x) \to 1$ imply that $\{f_n\}$ is relatively compact (resp. relatively w compact).

Let us recall the concepts of denting point and property (G).

Let M be a subset of X. A point $x \in M$ is said to be denting point of M if $x \notin \overline{co}(M \setminus N(0, \epsilon))$ holds for any $\epsilon > 0$. M is said to be dentable set if for any $\epsilon > 0$ there is a $x_\epsilon \in M$ such that $x_\epsilon \notin \overline{co}(M \setminus N(x_\epsilon, \epsilon)$, where $N(x_\epsilon, \epsilon) = \{x \in X : \ \|x - x_\epsilon\| < \epsilon\}$. The concept of dentabe set was first introduced by Rieffel in 1966 and the following important result has been given in [5]. That is, X has the Radon-Nikodym property whenever every bounded subset of X is dentable. This important result, later improved by Maynard [3] in 1973, is very simply. That is, X has the Radon-Nikodym property if and only if X is dentable.

The property (G) is given by Fan and Glicksberg [1] in 1955. Banach space X has the property (G) if and only if for all $x \in S(X)$ and $\epsilon > 0$, we have $x \notin \overline{co}(H(x,\epsilon))$, where $H(x,\epsilon) = \{y : \ y \in X, \| y - x \| \geq \epsilon\}$. In 1993, the concept of strongly convex Banach spaces were introduced by Wu and Li, and the another important result connected to property (G) has been given in [7]. That is, X is strongly convex space if and only X has the property (G), where X is reflexive Banach space. Noticing that the connection with dentable set and property (G), the above important result can be motivated by the following restatement of property (G). That is, X is strongly convex space if and only if every point of $S(X)$ is denting point of $U(X)$, where X is reflexive Banach space. Up to now, this result is only a result has being known about describing the straight relations between dentability and convexity.

The concept of w^* denting point of $U(X^*)$ was given in [1]. A point $x^* \in S(X^*)$ is said to be denting point of $U(X^*)$ if $x^* \notin \overline{co}^{w^*}(U(X^*) \setminus N(x^*,\epsilon))$ holds for each $\epsilon > 0$, where $N(x^*,\epsilon) = \{y^* : \ y^* \in X^*, \| y^* - x^* \| < \epsilon\})$. About the strongly smooth space which is the dual concept of strongly convex space, Shang, Cui and Fu [6] are greatly inspired to obtain the following important result : X is strongly smooth spaces if and only if the point of $S(X^*)$ which attains its norm is the w^* denting point of $U(X^*)$. Up to now, this important result is only a result has being known about describing the straight relations between dentability and smoothness also.

In this paper, the concepts of $k-$smoothness, $k-$very smoothness and $k-$strongly smoothness of Banach spaces are dealt with together by introducing three types $k-$denting point regarding different topology of conjugate spaces of Banach spaces. In fact, by using the skill of Banach spaces theory, we show that X is $k-$smooth (resp. $k-$strongly smooth) spaces if and only if each point of $S(X^*)$ which attains its norm is the second (resp. first) type $w^* - k$ denting point of $U(X^*)$; X is $k-$very smooth spaces if and only if each point of $S(X^*)$ which attains its norm is the $w - k$ denting point of $U(X^*)$. Specially, as a simple consequence of these results, we obtain the main result of ref [6]. In fact, the first type weak$^* - 1$ denting point coincide with weak* denting point. Also, the characterization of first type $w^* - k$ denting point is described by using the slice of closed unit ball of conjugate spaces.

2. Main results

THEOREM 2.1. *X is $k-$very smooth spaces if and only if each point of $S(X^*)$ which attains its norm is the $w-k$ denting point of $U(X^*)$.*

Proof. Proof of necessity. Firstly, we will prove that if for all $x^* \in S(X^*)$, there exists $x \in S(X)$ such that $x^*(x) = 1$, $\dim S_x \leq k$, and $\{x_n^*\}_{n=1}^{\infty} \subset S(X^*)$ satisfying $x_n^*(x) \to 1 (n \to \infty)$, then

$\overline{\{x_n^*\}_{n=1}^{\infty}}^{w} \cap S_x \neq \emptyset$.

In fact, by the $k-$very smoothness of X, we know that $\dim S_x \leq k$ and there exists a subsequence $\{x_{n_k}^*\}_{k=1}^{\infty}$ of $\{x_n^*\}_{n=1}^{\infty}$ such that $x_{n_k}^* \xrightarrow{w} y^* (k \to \infty)$. It follows that $x_{n_k}^*(x) \to y^*(x) = 1$, hence $\|y^*\| \geq 1$.

On the other hand, noticing that $U(X^*)$ is w^* closed set, we know that $\|y^*\| \leq 1$. Moreover, we have $y^* \in S_x$. This shows that

$\overline{\{x_n^*\}_{n=1}^{\infty}}^{w} \cap S_x \neq \emptyset$.

Secondly, we will prove that for all $x^* \in S(X^*)$, there exists $x \in S(X)$ such that $x^*(x) = 1$, and for each w open set V_{S_x} which includes S_x there exists a scalar $m > 0$ such that

$x^*(x) \geq z^*(x) + m$, if $z^* \in U(X^*) \setminus V_{S_x}$.

If it is not true, then there exists $z_n^* \in U(X^*) \setminus V_{S_x}$ such that $z_n^*(x) \to x^*(x) = 1 (n \to \infty)$, so we have

$\overline{\{z_n^*\}_{n=1}^{\infty}}^{w} \cap S_x \neq \emptyset$, $\{z_n^*\}_{n=1}^{\infty} \cap V_{S_x} = \emptyset$,

which is a contradiction.

Moreover, we have

$$\begin{aligned} x^*(x) - m &\geq \sup\{z^*(x) : z^* \in U(X^*) \setminus V_{S_x}\} \\ &= \sup\{z^*(x) : z^* \in co(U(X^*) \setminus V_{S_x})\} \\ &= \sup\{z^*(x) : z^* \in \overline{co}^{w}(U(X^*) \setminus V_{S_x})\}. \end{aligned}$$

This shows that $x^* \notin \overline{co}^{w}(U(X^*) \setminus V_{S_x})$, hence $S_x \cap \overline{co}^{w}(U(X^*) \setminus V_{S_x}) = \emptyset$. By Definition 2.1 we know that each point of $S(X^*)$ which attains its norm is the $w-k$ denting point of $U(X^*)$.

Proof of sufficiency.

Firstly, we will prove that X is $k-$smooth spaces.

For all $x \in S(X)$, by Hahn-Banach theorem, there exists $x^* \in S(X^*)$ such that $x^*(x) = 1$, hence x^* is a point of $S(X^*)$ which attains its norm. By the assumption of Theorem 2.1, we know that x^* is $w-k$ denting point of $U(X^*)$. It follows that $\dim S_x \leq k$, this shows that X is $k-$smooth spaces.

Secondly, we will prove that if

$x \in S(X),\ \{x_n^*\}_{n=1}^\infty \subset S(X^*),\ x_n^*(x) \to 1(n \to \infty),$

then $\{x_n^*\}_{n=1}^\infty$ is relatively w compact set and there exist

$x^* \in S_x$, net $\{x_\alpha^*\}_{\alpha\in\Delta} \subset \{x_n^*\}_{n=1}^\infty$

such that $x_\alpha^* \xrightarrow{w^*} x^*$ (here, we may assume that $x_n^* \neq x_m^*$ for all $m \neq n$).

Because $U(X^*)$ is w^* compact set, so there exists $x^* \in U(X^*)$ such that x^* become w^* accumulation point of $\{x_n^*\}_{n=1}^\infty$.
Let

$\Delta = \{R_{x^*} : R_{x^*}$ is w^* neighborhood of point $x^*\}$

and define a order by inclusive relation, i.e., $R_{x^*} \subset Q_{x^*}$ if and only if $R_{x^*} \succ Q_{x^*}$. Then

$\{R_{x^*} \cap \{x_n^*\}_{n=1}^\infty : R_{x^*}$ is w^* neighborhood of point $x^*\}$

is a semi-ordered set. By Zermelo principle, there is a mapping f such that

$f(R_{x^*} \cap \{x_n^*\}_{n=1}^\infty) \in R_{x^*} \cap \{x_n^*\}_{n=1}^\infty.$

Put $x_\alpha^* = f(R_{x^*} \cap \{x_n^*\}_{n=1}^\infty)$, then $\{x_\alpha^*\}_{\alpha\in\Delta} \subset \{x_n^*\}_{n=1}^\infty$ is a net. From $x_n^*(x) \to 1(n \to \infty)$ and the structure of this net, we know that $x_\alpha^* \xrightarrow{w^*} x^*$ and $x^* \in S_x$.

It remains to prove that $\{x_n^*\}_{n=1}^\infty$ is relatively w compact set.

Case 1° : If $\{x_n^*\}_{n=1}^\infty \cap S_x = \emptyset$, then $\{x_n^*\}_{n=1}^\infty$ must be a relatively w compact set. If it is not true, then any point of S_x is not w accumulation point of $\{x_n^*\}_{n=1}^\infty$, i.e., for all $x^* \in S_x$ there exists a w neighborhood V_{x^*} of point 0 such that $x^* + V_{x^*}$ does not contain any point of $\{x_n^*\}_{n=1}^\infty$. We construct a w open set

$V_{S_x} = \cup_{x^*\in S_x}\{y^* : y^* \in x^* + V_{x^*}\}.$

Obviously, V_{S_x} includes S_x and $\{x_n^*\}_{n=1}^\infty \cap V_{S_x} = \emptyset$. Because $U(X^*)$ is w^* compact set, so $\overline{co}^{w^*}(U(X^*)\backslash V_{S_x})$ is w^* compact set also. Noticing that S_x is w^* closed set, by separating theorem, we know that there exists $y \in X$ such that

$y(S_x) > \sup y(\overline{co}^{w^*}(U(X^*)\backslash V_{S_x}).$

Moreover, we choose a scalar $r > 0$ such that

$y(S_x) - y(\overline{co}^{w^*}(U(X^*) \setminus V_{S_x}) > r.$

Obviously,

$\{x_n^*\}_{n=1}^\infty \subset \overline{co}^{w^*}(U(X^*) \setminus V_{S_x}).$

On the other hand, by we have proved above, we know that there exists net $\{x^*_\alpha\}_{\alpha\in\Delta} \subset \{x^*_n\}_{n=1}^\infty$, such that $x^*_\alpha \xrightarrow{w^*} x^*$ and $x^* \in S_x$. This contradicts that

$$y(S_x) - y(\overline{co}^{w^*}(U(X^*) \setminus V_{S_x}) > r.$$

Hence, we obtain the desired result that $\{x^*_n\}_{n=1}$ is a relatively w compact set.

Case 2° : If $\{x^*_n\}_{n=1}^\infty \cap S_x \neq \emptyset$, then by case 1° we know that $\{x^*_n\}_{n=1}^\infty \backslash S_x$ is a relatively w compact set. Because S_x is a bounded closed set of finite dimensional spaces, so $\{x^*_n\}_{n=1}^\infty \cap S_x$ is a relatively w compact set. Noticing that

$$\{x^*_n\}_{n=1}^\infty = (\{x^*_n\}_{n=1}^\infty \cap S_x) \cup (\{x^*_n\}_{n=1}^\infty \backslash S_x),$$

we have

$$\overline{\{x^*_n\}_{n=1}^\infty}^w = \overline{\{x^*_n\}_{n=1}^\infty \cap S_x}^w \cup \overline{(\{x^*_n\}_{n=1}^\infty \backslash S_x)}^w.$$

Thus $\{x^*_n\}_{n=1}^\infty$ is a relatively w compact set. □

THEOREM 2.2. *X is $k-$strongly smooth spaces if and only if each point of $S(X^*)$ which attains its norm is the first type $w^* - k$ denting point of $U(X^*)$.*

Proof. Proof of necessity. Firstly, we will prove that if for all $x^* \in S(X^*)$, there exists $x \in S(X)$ such that $x^*(x) = 1$, $\dim S_x \leq k$, and each norm open set V_{S_x} which includes S_x there exists a scalar $r > 0$ such that the inequality $\mathrm{dist}(z^*, S_x) \geq r$ holds for $z^* \notin V_{S_x}$.

In fact, by the $k-$strongly smoothness of X, we know that $\dim S_x \leq k$. Because V_{S_x} is a norm open set which includes S_x, so there exists $\delta' > 0$ such that $U(x^*, \delta') \subset V_{S_x}$ holds for $x^* \in S_x$ and such δ' exists a minimum value δ. Obviously, $\bigcup_{x^*\in S_x} U(x^*, \delta) \subset V_{S_x}$. Let $r = \frac{\delta}{2}$, then we have $\mathrm{dist}(z^*, S_x) \geq r$. Otherwise, there exists $x^* \in S_x$ such that $\|z^* - x^*\| < r < \delta$, hence $z^* \in \bigcup_{x^*\in S_x} U(x^*, \delta) \subset V_{S_x}$. This contradicts that $z^* \notin V_{S_x}$.

Secondly, we will prove that for all $x^* \in S(X^*)$, there exists $x \in S(X)$ such that $x^*(x) = 1$, and for each norm open set V_{S_x} which includes S_x there exists a scalar $m > 0$ such that

$$x^*(x) \geq z^*(x) + m, \text{ if } z^* \in U(X^*) \setminus V_{S_x}.$$

If it is not true, then there exists $z^*_n \in U(X^*) \setminus V_{S_x}$ such that $z^*_n(x) \to x^*(x) = 1 (n \to \infty)$. By the $k-$strongly smoothness of X, we can deduce that $\mathrm{dist}(z^*_n, S_x) \to 0 (n \to \infty)$. Otherwise, we may find a $\epsilon_0 > 0$ such that for every $n_0 > 0$, there exists $n_k > n_0$, $k = 1, 2, \cdots$, satisfying

$\text{dist}(z^*_{n_k}, S_x) > \epsilon_0$. On the other hand, $z^*_n(x) \to 1$ implies that $z^*_{n_k}(x) \to 1$. Hence, by the $k-$strongly smoothness of X we know that $\{z^*_{n_k}\}$ is a relatively compact set. It follows that there exists subsequence $\{z^*_{n_{k_l}}\} \subset \{z^*_{n_k}\}$ such that $z^*_{n_{k_l}} \to z^*_0$. Hence $z^*_{n_{k_l}}(x) \to z^*_0(x) = 1$ and $z^*_0 \in S_x$. Which leads to that $\text{dist}(z^*_{n_{k_l}}, S_x) \to 0$. This contradicts that $\text{dist}(z^*_{n_k}, S_x) > \epsilon_0$.

Moreover, we have

$$\begin{aligned} x^*(x) - m &\geq \sup\{z^*(x) : z^* \in U(X^*) \setminus V_{S_x}\} \\ &= \sup\{z^*(x) : z^* \in co(U(X^*)\backslash V_{S_x})\} \\ &= \sup\{z^*(x) : z^* \in \overline{co}^{w^*}(U(X^*) \setminus V_{S_x})\}. \end{aligned}$$

This shows that $x^* \notin \overline{co}^{w^*}(U(X^*)\backslash V_{S_x})$, it follows that $S_x \cap \overline{co}^{w^*}(U(X^*)\backslash V_{S_x}) = \emptyset$. Hence, we obtain the desired result that each point of $S(X^*)$ which attains its norm is the first type $w^* - k$ denting point of $U(X^*)$.

Proof of sufficiency. Suppose that $x \in S(X)$, $\{x^*_n\}^\infty_{n=1} \subset S(X^*)$, $x^*_n(x) \to 1 (n \to \infty)$. Greatly similarly to the proof of Theorem 2.1, by using the given conditions in Theorem 2.2, we can prove that there exists a net $x^* \in S_x \{x^*_n\}^\infty_{n=1} \subset \{x^*_\alpha\}_{\alpha \in \Delta}$ such that $x^*_\alpha \xrightarrow{w^*} x^*$ and X is $k-$smooth spaces. Now we prove that $\{x^*_n\}^\infty_{n=1}$ is a relatively compact set.

Case 1° : If $\{x^*_n\}^\infty_{n=1} \cap S_x = \emptyset$, then $\{x^*_n\}^\infty_{n=1}$ must be a relatively compact set. If it is not true, then any point of S_x is not accumulation point of $\{x^*_n\}^\infty_{n=1}$. Hence, for all $x^* \in S_x$ there is a $\epsilon > 0$ such that the set $\{y^* : \|y^* - x^*\| < \epsilon\}$ does not contain any point of $\{x^*_n\}^\infty_{n=1}$. We construct a norm open set

$$V_{S_x} = \cup_{x^* \in S_x} \{y^* : \|y^* - x^*\| < \epsilon\}.$$

Obviously, V_{S_x} includes S_x and $\cup_{x^* \in S_x} \{y^* : \|y^* - x^*\| < \epsilon\} \cap \{x^*_n\}^\infty_{n=1} = \emptyset$. Greatly similarly to the proof of Theorem 2.1, we can deduce that $\{x^*_n\}^\infty_{n=1}$ is a relatively compact set.

Case 2° : If $\{x^*_n\}^\infty_{n=1} \cap S_x \neq \emptyset$, then by case 1° we know that $\{x^*_n\}^\infty_{n=1} \backslash S_x$ is a relatively compact set. Because S_x is a bounded closed set of finite dimensional spaces, so $\{x^*_n\}^\infty_{n=1} \cap S_x$ is a relatively compact set. Noticing that

$$\{x^*_n\}^\infty_{n=1} = (\{x^*_n\}^\infty_{n=1} \cap S_x) \cup (\{x^*_n\}^\infty_{n=1} \backslash S_x),$$

we have

$$\overline{\{x^*_n\}^\infty_{n=1}} = \overline{\{x^*_n\}^\infty_{n=1} \cap S_x} \cup \overline{(\{x^*_n\}^\infty_{n=1} \backslash S_x)},$$

Thus $\{x^*_n\}^\infty_{n=1}$ is a relatively compact set. □

When $k = 1$, the first type $w^* - 1$ denting point coincide with w^* denting point. It is well known that $1-$strongly smooth space coincide with usual strongly smooth spaces [8]. Hence we obtained the following corollary.

Corollary 2.1. [6] *X is strongly smooth spaces if and only if each point of $S(X^*)$ which attains its norm is the w^* denting point of $U(X^*)$.*

In what follows, using the slice of closed unit ball of conjugate spaces X^*, we will describe the characterization of first type $w^* - k$ denting point.

Theorem 2.3. *$x^* \in S(X^*)$ is first $w^* -$ k denting point of $U(X^*)$ if and only if there exists $x \in S(X)$ such that $x^* \in S_x$, $dimS_x \le k$ and for $\forall \epsilon > 0$, there exists slice*

$$F(x, \delta) = \{z^* : \; z^* \in U(X^*), \; z^*(x) > 1 - \delta\}$$

satisfying the inclusive relation

$$F(x, \delta) \subset \{y^* : \; y^* \in U(X^*), \; d(y^*, S_x) < \epsilon\}.$$

Proof. Proof of necessity. Suppose that $x^* \in S(X^*)$ is first $w^* - k$ denting point of $U(X^*)$, then there exists $x \in S(X)$ such that $x^* \in S_x$, $\dim S_x \le k$. Let

$$H_{S_x} = \{y^* : \; y^* \in U(X^*), d(y^*, S_x) < \epsilon\},$$

then H_{S_x} is norm open set which includes S_x, hence $S_x \cap \overline{co}^{w^*}(U(X^*) \backslash H_{S_x}) = \emptyset$. Moreover, we can deduce that

$$\sup x(\overline{co}^{w^*}(U(X^*) \setminus H_{S_x}) < 1.$$

Otherwise, there exists sequence $y_n^* \in \overline{co}^{w^*}(U(X^*)) \setminus H_{S_x})$ such that $y_n^*(x) \to 1$ $(n \to \infty)$. Let $x_n^* = \frac{y_n^*}{\|y_n^*\|}$, then $x_n^*(x) \to 1$ $(n \to \infty)$. From the proof of Theorem 2.2, we know that x is $k-$smooth point of X and $\{x_n^*\}_{n=1}^{\infty}$ is relatively compact set. Therefore, sequence $\{x_n^*\}_{n=1}^{\infty}$ has the convergent subsequence, without loss of generality, let the convergent subsequence be $\{x_n^*\}_{n=1}^{\infty}$ itself and suppose that $x_n^* \to x_0^*$ $(n \to \infty)$. Clearly,

$$x_n^*(x) \to 1 = x_0^*(x) \; (n \to \infty), \; x_0^* \in S_x.$$

On the other hand,

$$\|y_n^* - x_0^*\| \le \|\frac{y_n^*}{\|y_n^*\|} - y_n^*\| + \|\frac{y_n^*}{\|y_n^*\|} - x_0^*\| \to 0 (n \to \infty),$$

it follows that x_0^* belong to the norm closure of set $\overline{co}^{w^*}(U(X^*) \setminus H_{S_x})$. Noticing that this set is closed set regarding norm topology, we know that $x_0^* \in \overline{co}^{w^*}(U(X^*) \setminus H_{S_x})$, hence $x_0^* \notin H_{S_x}$. It is impossible.

Let $1 - \delta = \sup x(\overline{co}^{w^*}(U(X^*) \setminus H_{S_x}))$. It is easy to see that if

$$z^* \in F(x, \delta) = \{z^* : \; z^* \in U(X^*), z^*(x) > 1 - \delta\},$$

then $z^* \notin \overline{co}^{w^*}(U(X^* \backslash H_{S_x}))$. Hence $z^* \in H_{S_x}$, this shows that $F(x, \delta) \subset H_{S_x}$.

Proof of sufficiency. Suppose that there exists $x \in S(X)$ such that $x^* \in S_x$, $\dim S_x \leq k$ and for $\forall \epsilon > 0$, there exists slice

$F(x,\delta) = \{z^* : \ z^* \in U(X^*),\ z^*(x) > 1-\delta\}$

satisfying the inclusive relation

$F(x,\delta) \subset \{y^* : \ y^* \in U(X^*),\ d(y^*, S_x) < \epsilon\}$.

For the convenient, we denote $\{y^* : \ y^* \in U(X^*),\ d(y^*, S_x) < \epsilon\}$ by H_{S_x}, then

$1-\delta \geq \sup\{z^*(x) : z^* \in co(U(X^*)\backslash H_{S_x})\} = \sup\{z^*(x) : z^* \in \overline{co}^{w^*}(U(X^*)\backslash H_{S_x})\}$.

Moreover, we can deduce that $S_x \cap \overline{co}^{w^*}(U(X^*)\backslash H_{S_x}) = \emptyset$ from the structure of S_x. Hence $x^* \in S(X^*)$ is first $w^* - k$ denting point of $U(X^*)$. □

Theorem 2.4. *X is $k-$ smooth spaces if and only if each point of $S(X^*)$ which attains its norm is the second type $w^* - k$ denting point of $U(X^*)$.*

Proof. The sufficiency is immediate from the definition of $k-$ smooth spaces. It remains to prove the necessity.

Firstly, we will prove that for all $x^* \in S(X^*)$, there exists $x \in S(X)$ such that $x^*(x) = 1$, and $\{x_n^*\}_{n=1}^\infty \subset S(X^*)$ satisfying $x_n^*(x) \to 1 (n \to \infty)$, then $\overline{\{x_n\}_{n=1}^\infty}^{w^*} \cap S_x \neq \emptyset$.

If it is not true, then there exists w^* neighborhood V_{S_x} which includes S_x such that $\overline{\{x_n^*\}_{n=1}^\infty}^{w^*} \cap S_x = \emptyset$. From the proof of sufficient of Theorem 2.2, we know that there exists net $\{x_\alpha^*\}_{\alpha\in\Delta} \subset \{x_n^*\}_{n=1}^\infty$ satisfying $x_\alpha^* \xrightarrow{w^*} x^*,\ x^* \in S_x$. Hence $\overline{\{x_n^*\}_{n=1}^\infty}^{w^*} \cap S_x \neq \emptyset$. This contradicts that $\overline{\{x_n^*\}_{n=1}^\infty}^{w^*} \cap S_x = \emptyset$.

Secondly, we will prove that if for all $x^* \in S(X^*)$, there exists $x \in S(X)$ such that $x^*(x) = 1$, and each w^* open set V_{S_x} which includes S_x there exists a scalar $m > 0$ such that $x^*(x) \geq z^*(x) + m$ holds for $z^* \in U(X^*)\backslash V_{S_x}$.

If it is not true, then there exists $z_n^* \in U(X^*)\backslash V_{S_x}$ such that $z_n^*(x) \to x^*(x) = 1 (n \to \infty)$. Hence we have $\overline{\{z_n^*\}_{n=1}^\infty}^{w^*} \cap S_x \neq \emptyset$. On the other hand, for $z_n^* \in U(X^*)/V_{S_x}$, we have $\{z_n^*\}_{n=1}^\infty \cap V_{S_x} = \emptyset$. This contradicts that $\overline{\{z_n^*\}_{n=1}^\infty}^{w^*} \cap S_x \neq \emptyset$.

Moreover, we have

$x^*(x) - m \geq \sup\{z^*(x) : z^* \in U(X^*)\backslash V_{S_x}\} = \sup\{z^*(x) : z^* \in co(U(X^*)\backslash V_{S_x})\} = \sup\{z^*(x) : z^* \in \overline{co}^{w^*}(U(X^*)\backslash V_{S_x})\}$.

This shows that $x^* \notin \overline{co}^{w^*}(U(X^*)\backslash V_{S_x})$, it follows that $S_x \cap \overline{co}^{w^*}(U(X^*)\backslash V_{S_x}) = \emptyset$. By the definition of $k-$ smooth spaces, we know that $\dim S_x \leq k$. Hence, we obtain the desired result that each point of $S(X^*)$ which attains its norm is the second type $w^* - k$ denting point of $U(X^*)$. $\square$

References

[1] Ky Fan and I. Glicksburg, *Fully convex normed linear spaces*, Proc. Natl. Acad. Sci. USA **41** (1955), 947–953.

[2] P.D.Liu, Martingle and Geometric of Banach Space. Science Press. Shanghai (in Chinese), 2007.

[3] H.B.Maynard, A geometrical characterization of Banach spaces having the Radon-Nikodym property, Trans.Amer.Math.Soc. 185(1973) 493-500.

[4] C.X.Nan and J.H.Wang, k-strict convexity and k-smoothness, Chinese Ann.Math (in Chinese).(Series A). 11(3)(1990) 321-324.

[5] M.A.Rieffel, Dentable subsets of Banach spaces, with applications to a Radon-Nikodym theorem, Funct.Anal.Proc. (Conf. Irvine, Calif., 1966), Acad.Press,London,Thompson, Washington, D.C. 71-77.

[6] S.Q.Shang,Y.A.Cui,Y.Q.Fu, Denting point, smoothness of Banach spaces and approximation compactness[J]. Acta Mathematica Sinica (in Chinese). 53(6)(2010) 1217-1224.

[7] C.X.Wu, Y.J.Li, Strong convexity in Banach spaces,Chin.J.Math. 13(1993) 105-108.

[8] X.T. Yu. Geometric Theory of Banach Space.East China Normal Univ (in Chinese), 1984.

[9] Z.H.Zhang On k-strictly convex and k-very smooth Banach space, Northeast.Math.J. 12(2)(1996) 165-170.

Suyalatu Wulede
College of Mathematics Science
Inner Mongolia Normal University
Hohhot 010022,P.R.China
E-mail: Suyila@imnu.edu.cn

Shaoqiang Shang
School of Science
Northeast Forestry University
Harbin 150040,P.R.China
E-mail: yizhitu@163.com

Wurina Bao
Xilingol Vacation College
Xilinhot 026000,P.R.China
E-mail: baowurina@sohu.com