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DERIVATIONS OF UP-ALGEBRAS

Kaewta Sawika, Rossukon Intasan, Arocha Kaewwasri, and Aiyared Iampan

Abstract. The concept of derivations of BCI-algebras was first introduced by Jun and Xin. In this paper, we introduce the notions of (l, r)-derivations, (r, l)-derivations and derivations of UP-algebras and investigate some related properties. In addition, we define two subsets $\mathrm{Ker}_d(A)$ and $\mathrm{Fix}_d(A)$ for some derivation d of a UP-algebra A, and we consider some properties of these as well.

1. Introduction and Preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [14], BCI-algebras [15], BCH-algebras [11], KU-algebras [29], SU-algebras [18] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [15] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [14, 15] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

In the theory of rings and near rings, the properties of derivations is an important topic to study [20, 28]. In 2004, Jun and Xin [17] applied the notions of rings and near rings theory to BCI-algebras and

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obtained some properties. Several researches were conducted on the generalizations of the notion of derivations and application to many logical algebras such as: In 2005, Zhan and Liu [33] introduced the notion of left-right (right-left) f-derivations of BCI-algebras. In 2006, Abujabal and Al-Shehri [1] investigated some fundamental properties and proved some results on derivations of BCI-algebras. In 2007, Abujabal and Al-Shehri [2] introduced the notion of left derivations of BCI-algebras. In 2009, Javed and Aslam [16] investigated some fundamental properties and established some results of f-derivations of BCI-algebras. Nisar [27] introduced the notions of right F-derivations and left F-derivations of BCI-algebras. Nisar [26] characterized f-derivations of BCI-algebras. Prabpayak and Leerawat [29] studied the notions of left-right (right-left) derivations of BCC-algebras. In 2010, Al-Shehri [4] introduced the notion of derivations of MV-algebras. Al-Shehrie [6] introduced the notion of left-right (right-left) derivations of B-algebras. In 2011, Ilbira, Firat and Jun [13] introduced the notion of left-right (right-left) symmetric bi-derivations of BCI-algebras. Thomys [31] described f-derivations of weak BCC-algebras in which the condition $(xy)z = (xz)y$ is satisfied in the case when elements x, y belong to the same branch. In 2012, Al-Shehri and Bawazeer [5] introduced the notion of left-right (right-left) t-derivations of BCC-algebras. Lee and Kim [21] considered the properties of f-derivations of BCC-algebras. Muhiuddin and Al-Roqi [23] introduced the notion of t-derivations of BCI-algebras. Muhiuddin and Al-Roqi [22] introduced the notion of (regular) (α, β)-derivations of BCI-algebras. So and Ahn [30] introduced the notions of complicatednesses and derivations of BCC-algebras. In 2013, Ardekani and Davvaz [7] introduced the notion of f-derivations and (f, g)-derivations of MV-algebras. Bawazeer, Al-Shehri and Babusal [9] introduced the notion of generalized derivations of BCC-algebras. Ganeshkumar and Chandramouleeswaran [10] introduced the notion of generalized derivations of TM-algebras. Lee [19] introduced a new kind of derivations of BCI-algebras. Torkzadeh and Abbasian [32] introduced the notion of $(\odot, \vee)$-derivations of BL-algebras. In 2014, Al-Roqi [3] introduced the notion of generalized (regular) (α, β)-derivations of BCI-algebras. Ardekani and Davvaz [8] introduced the notion of f-derivations and (f, g)-derivations of B-algebras. Muhiuddin and Al-Roqi [24] introduced the notion of generalized left derivations of BCI-algebras. Muhiuddin and Al-Roqi [25] introduced the notion of (regular) left (θ, ϕ)-derivations of BCI-algebras.

Iampan [12] now introduced a new algebraic structure, called a UP-algebra and a concept of UP-ideals and UP-subalgebras of UP-algebras. The notion of derivations play an important role in studying the many logical algebras. In this paper, we introduce the notions of (l, r)-derivations, (r, l)-derivations and derivations of UP-algebras, and their properties are investigated.

Before we begin our study, we will introduce to the definition of a UP-algebra.

DEFINITION 1.1. [12] An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* if it satisfies the following axioms: for any $x, y, z \in A$,
(UP-1): $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,
(UP-2): $0 \cdot x = x$,
(UP-3): $x \cdot 0 = 0$, and
(UP-4): $x \cdot y = y \cdot x = 0$ implies $x = y$.

EXAMPLE 1.2. [12] Let X be a set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = B \cap A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); \cdot, \emptyset)$ is a UP-algebra.

In what follows, let A denote a UP-algebra unless otherwise specified. The following proposition is very important for the study of UP-algebras.

PROPOSITION 1.3. [12] *In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \cdot x = 0$,
(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,
(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,
(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,
(5) $x \cdot (y \cdot x) = 0$,
(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*
(7) $x \cdot (y \cdot y) = 0$.

On a UP-algebra $A = (A; \cdot, 0)$, we define a binary relation $\leq$ on A as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0. \tag{1}$$

Proposition 1.4 obviously follows from Proposition 1.3.

PROPOSITION 1.4. [12] *In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \leq x$,
(2) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,
(3) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$,
(4) $x \leq y$ *implies* $z \cdot x \leq z \cdot y$,
(5) $x \leq y$ *implies* $y \cdot z \leq x \cdot z$,
(6) $x \leq y \cdot x$, *and*
(7) $x \leq y \cdot y$.

From Proposition 1.4 and UP-3, we have Proposition 1.5.

PROPOSITION 1.5. [12] *Let* A *be a UP-algebra with a binary relation* $\leq$ *defined by* (1). *Then* $(A, \leq)$ *is a partially ordered set with* 0 *as the greatest element.*

We often call the partial ordering $\leq$ defined by (1) the *UP-ordering* on A. From now on, the symbol $\leq$ will be used to denote the UP-ordering, unless specified otherwise.

DEFINITION 1.6. [12] A nonempty subset B of A is called a *UP-ideal* of A if it satisfies the following properties:
(1) the constant 0 of A is in B, and
(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.
Clearly, A and $\{0\}$ are UP-ideals of A.

THEOREM 1.7. [12] *Let* A *be a UP-algebra and* B *a UP-ideal of* A. *Then the following statements hold: for any* $x, a, b \in A$,
(1) if $b \cdot x \in B$ *and* $b \in B$, *then* $x \in B$. *Moreover, if* $b \cdot X \subseteq B$ *and* $b \in B$, *then* $X \subseteq B$,
(2) if $b \in B$, *then* $x \cdot b \in B$. *Moreover, if* $b \in B$, *then* $X \cdot b \subseteq B$, *and*
(3) if $a, b \in B$, *then* $(b \cdot (a \cdot x)) \cdot x \in B$.

THEOREM 1.8. [12] *Let* A *be a UP-algebra and* $\{B_i\}_{i \in I}$ *a family of UP-ideals of* A. *Then* $\bigcap_{i \in I} B_i$ *is a UP-ideal of* A.

DEFINITION 1.9. [12] A subset S of A is called a *UP-subalgebra* of A if it constant 0 of A is in S, and $(S; \cdot, 0)$ itself forms a UP-algebra. Clearly, A and $\{0\}$ are UP-subalgebras of A.

Applying Proposition 1.3 (1), we can then easily prove the following Proposition.

PROPOSITION 1.10. [12] *A nonempty subset* S *of a UP-algebra* $A = (A; \cdot, 0)$ *is a UP-subalgebra of* A *if and only if* S *is closed under the* $\cdot$ *multiplication on* A.

THEOREM 1.11. [12] *Let A be a UP-algebra and $\{B_i\}_{i\in I}$ a family of UP-subalgebras of A. Then $\bigcap_{i\in I} B_i$ is a UP-subalgebra of A.*

THEOREM 1.12. [12] *Let A be a UP-algebra and B a UP-ideal of A. Then $A \cdot B \subseteq B$. In particular, B is a UP-subalgebra of A.*

We can easily show the following example.

EXAMPLE 1.13. [12] Let $A = \{0, a, b, c, d\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

(2)

$\cdot$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	d
b	0	0	0	c	d
c	0	0	b	0	d
d	0	0	0	0	0

Using the following program in the software "MATLAB", we know that $(A; \cdot, 0)$ is a UP-algebra, where we use numbers $1, 2, 3, 4$ and 5 instead of $0, a, b, c$ and d, respectively.

Program for test UP-1

```
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
if n == 4
    b = [   1   2   3   4;
            1   1   1   1;
            1   2   1   4;
            1   2   3   1 ];
else
    b = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
end
tc = 0;
cp = 0;
np = 0;
```

```
for i = 1:n
    for j = 1:n
        for k = 1:n
            tc = tc + 1;
            rc = b(b(j,k),b(b(i,j),b(i,k)));
            if rc == 1
                cp = cp + 1;
            else
                np = np + 1;
             end
        end
    end
end
```

We can check condition (2) in Definition 1.6 that the set $\{0, a, c\}$ is a UP-ideal of A by using the following program.

Program for test Definition 1.6 (2)

```
clc,clear
display(['Input n = 4 or n = 5']);
n = input('n = ');
b = zeros(n,n);
if n == 4
    b = [   1   2   3   4;
            1   1   1   1;
            1   2   1   4;
            1   2   3   1   ];
else
    b = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1  ];
end
tc  = 0;
cp  = 0;
scp = 0;
```

```
ncp = 0;
np  = 0;
for i = 1:n
    for j = 1:4
        for k = 1:n
            rc = b(i,b(j,k));
            if (rc <= 2) | (rc == 4)
                tc = tc + 1;
                   if j ~= 3
                    cp = cp + 1;
                    src = b(i,k);
                    if (src <= 2) | (src == 4)
                        scp = scp + 1;
                    else
                        ncp = ncp + 1;
                    end
                end
            end
            if ((rc == 3) | (rc ==5)) & (j == 3)
                np = np + 1;
            end
        end
    end
end
```

By Proposition 1.10, we can check that the set $\{0, a, b, c\}$ is a UP-subalgebra of A.

Definition 1.14. For any $x, y \in A$, we define a binary operation $\wedge$ on A by $x \wedge y = (y \cdot x) \cdot x$.

Definition 1.15. A UP-algebra A is called *meet-commutative* if $x \wedge y = y \wedge x$ for all $x, y \in A$, that is, $(y \cdot x) \cdot x = (x \cdot y) \cdot y$ for all $x, y \in A$.

We can easily show the following example.

EXAMPLE 1.16. [12] Let $A = \{0, a, b\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

(3)

$\cdot$	0	a	b
0	0	a	b
a	0	0	a
b	0	0	0

Using the following program in the software "MATLAB", we know that $(A; \cdot, 0)$ is a UP-algebra, where we use numbers $1, 2$ and 3 instead of $0, a$ and b, respectively.

Program for test UP-1

```
clc,clear
display(['Input n = 3 or n = 5']);
n = input('n = ');
b = zeros(n,n);
if n == 3
    b = [   1   2   3;
            1   1   3;
            1   2   1;   ];
else
    b = [   1   2   3   4   5;
            1   1   3   4   5;
            1   1   1   4   5;
            1   1   3   1   5;
            1   1   1   1   1   ];
end
tc = 0;
cp = 0;
np = 0;
for i = 1:n
    for j = 1:n
        for k = 1:n
            tc = tc + 1;
            rc = b(b(j,k),b(b(i,k),b(j,k)));
            if rc == 1
                cp = cp + 1;
            else
```

```
                    np = np + 1;
                end
            end
        end
end
```

We can check Definition 1.15 that A is meet-commutative by using the following program.

Program for test Definition 1.15

```
clc,clear
display(['Input n = 3 or n = 4']);
n = input('n = ');
b = zeros(n,n);
if n == 3
    b = [   1   2   3;
            1   1   2;
            1   1   1   ];
else
    b = [   1   2   3   4;
            2   1   4   3;
            3   4   3   4;
            4   4   4   3   ];
end
tc = 0;
ac = 0;
nc = 0;
for i = 1:n
    for j = 1:n
        for k = 1:n
            tc = tc + 1;
            v1 = b(b(j,i),i);
            v2 = b(b(i,j),j);
            ass = v1-v2;
            if ass == 0
                ac = ac + 1;
            else
```

```
                nc = nc + 1;
            end
        end
    end
end
```

2. Main Results

In this section, we first introduce the notions of an (l, r)-derivation, an (r, l)-derivation and a derivation of a UP-algebra and study some of their basic properties. Finally, we define two subsets $\mathrm{Ker}_d(A)$ and $\mathrm{Fix}_d(A)$ for some derivation d of a UP-algebra A, and we consider some properties of these as well.

DEFINITION 2.1. A self-map $d\colon A \to A$ is called an (l, r)-*derivation* of A if it satisfies the identity $d(x \cdot y) = (d(x) \cdot y) \wedge (x \cdot d(y))$ for all $x, y \in A$. Similarly, a self-map $d\colon A \to A$ is called an (r, l)-*derivation* of A if it satisfies the identity $d(x \cdot y) = (x \cdot d(y)) \wedge (d(x) \cdot y)$ for all $x, y \in A$. Moreover, if d is both an (l, r)-derivation and an (r, l)-derivation of A, it is called a *derivation* of A.

EXAMPLE 2.2. [12] Let $A = \{0, a, b, c\}$ be a UP-algebra in which the operation $\cdot$ is defined as follows:

(4)

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Define a self-map $d\colon A \to A$ by, for any $x \in A$,

$$d(x) = \begin{cases} 0 & \text{if } x \neq b, \\ b & \text{if } x = b. \end{cases}$$

Then it is easily checked that d is both an (l, r)-derivation and an (r, l)-derivation of A.

Define two self-maps $1_A\colon A \to A$ and $0_A\colon A \to A$ by, for any $x \in A$,

$$1_A(x) = x \text{ and } 0_A(x) = 0.$$

Then, for any $x, y \in A$,

$$1_A(x \cdot y) = x \cdot y$$

(By Proposition 2.3 (3)) $$= (x \cdot y) \wedge (x \cdot y),$$

so $1_A(x \cdot y) = (1_A(x) \cdot y) \wedge (x \cdot 1_A(y)) = (x \cdot 1_A(y)) \wedge (1_A(x) \cdot y)$, and

$$0_A(x \cdot y) = 0$$

(By Proposition 2.3 (2)) $$= y \wedge 0$$

(By Proposition 2.3 (1)) $$= 0 \wedge y,$$

so $0_A(x \cdot y) = (0_A(x) \cdot y) \wedge (x \cdot 0_A(y)) = (x \cdot 0_A(y)) \wedge (0_A(x) \cdot y)$. Hence, 1_A and 0_A are both an (l, r)-derivation and an (r, l)-derivation of A.

Proposition 2.3. *In a UP-algebra A, the following properties hold: for any $x \in A$,*

(1) $0 \wedge x = 0$,
(2) $x \wedge 0 = 0$, *and*
(3) $x \wedge x = x$.

Proof. (1) By UP-3, we have

$$0 \wedge x = (x \cdot 0) \cdot 0 = 0 \text{ for all } x \in A.$$

(2) By UP-2 and using Proposition 1.3 (1), we have

$$x \wedge 0 = (0 \cdot x) \cdot x = x \cdot x = 0 \text{ for all } x \in A.$$

(3) By UP-2 and using Proposition 1.3 (1), we have

$$x \wedge x = (x \cdot x) \cdot x = 0 \cdot x = x \text{ for all } x \in A.$$

□

Definition 2.4. An (l, r)-derivation (resp. (r, l)-derivation, derivation) d of A is called *regular* if $d(0) = 0$.

Theorem 2.5. *In a UP-algebra A, the following statements hold:*

(1) every (l, r)-derivation of A is regular, and
(2) every (r, l)-derivation of A is regular.

Proof. (1) Assume that d is an (l, r)-derivation of A. Then

$$
\begin{aligned}
&\text{(By UP-3)} & d(0) &= d(0 \cdot 0) \\
& & &= (d(0) \cdot 0) \wedge (0 \cdot d(0)) \\
&\text{(By UP-2 and UP-3)} & &= 0 \wedge d(0) \\
&\text{(By Proposition 2.3 (1))} & &= 0.
\end{aligned}
$$

Hence, d is regular.
(2) Assume that d is an (r, l)-derivation of A. Then

$$
\begin{aligned}
&\text{(By UP-3)} & d(0) &= d(0 \cdot 0) \\
& & &= (0 \cdot d(0)) \wedge (d(0) \cdot 0) \\
&\text{(By UP-2 and UP-3)} & &= d(0) \wedge 0 \\
&\text{(By Proposition 2.3 (2))} & &= 0.
\end{aligned}
$$

Hence, d is regular. □

Corollary 2.6. *Every derivation of A is regular.*

Theorem 2.7. *In a UP-algebra A, the following statements hold:*

(1) if d is an (l, r)-derivation of A, then $d(x) = x \wedge d(x)$ for all $x \in A$, and
(2) if d is an (r, l)-derivation of A, then $d(x) = d(x) \wedge x$ for all $x \in A$.

Proof. (1) Assume that d is an (l, r)-derivation of A. Then, for all $x \in A$,

$$
\begin{aligned}
&\text{(By UP-2)} & d(x) &= d(0 \cdot x) \\
& & &= (d(0) \cdot x) \wedge (0 \cdot d(x)) \\
&\text{(By UP-2 and Theorem 2.5 (1))} & &= (0 \cdot x) \wedge d(x) \\
&\text{(By UP-2)} & &= x \wedge d(x).
\end{aligned}
$$

(2) Assume that d is an (r, l)-derivation of A. Then, for all $x \in A$,

$$
\begin{aligned}
&\text{(By UP-2)} & d(x) &= d(0 \cdot x) \\
& & &= (0 \cdot d(x)) \wedge (d(0) \cdot x) \\
&\text{(By UP-2 and Theorem 2.5 (2))} & &= d(x) \wedge (0 \cdot x) \\
&\text{(By UP-2)} & &= d(x) \wedge x.
\end{aligned}
$$

□

COROLLARY 2.8. *If d is a derivation of A, then $d(x) \wedge x = x \wedge d(x)$ for all $x \in A$.*

DEFINITION 2.9. Let d be an (l, r)-derivation (resp. (r, l)-derivation, derivation) of A. We define a subset $\mathrm{Ker}_d(A)$ of A by

$$\mathrm{Ker}_d(A) = \{x \in A \mid d(x) = 0\}.$$

PROPOSITION 2.10. *Let d be an (l, r)-derivation of A. Then the following properties hold: for any $x, y \in A$,*

(1) $x \leq d(x)$,
(2) $d(x) \cdot y \leq d(x \cdot y)$,
(3) $d(x \cdot d(x)) = 0$,
(4) $d(d(x) \cdot x) = 0$, *and*
(5) $x \leq d(d(x))$.

Proof. (1) For all $x \in A$,

$$\begin{aligned} &\text{(By Theorem 2.7 (1))} & x \cdot d(x) &= x \cdot (x \wedge d(x)) \\ & & &= x \cdot ((d(x) \cdot x) \cdot x) \\ &\text{(By Proposition 1.3 (5))} & &= 0. \end{aligned}$$

Hence, $x \leq d(x)$ for all $x \in A$.
(2) For all $x, y \in A$,

$$\begin{aligned} (d(x) \cdot y) \cdot d(x \cdot y) &= (d(x) \cdot y) \cdot ((d(x) \cdot y) \wedge (x \cdot d(y))) \\ &= (d(x) \cdot y) \cdot (((x \cdot d(y)) \cdot (d(x) \cdot y)) \cdot (d(x) \cdot y)) \end{aligned}$$

(By Proposition 1.3 (5))

$$= 0.$$

Hence, $d(x) \cdot y \leq d(x \cdot y)$ for all $x, y \in A$.
(3) For all $x \in A$,

$$\begin{aligned} & & d(x \cdot d(x)) &= (d(x) \cdot d(x)) \wedge (x \cdot d(d(x))) \\ &\text{(By Proposition 1.3 (1))} & &= 0 \wedge (x \cdot d(d(x))) \\ &\text{(By Proposition 2.3 (1))} & &= 0. \end{aligned}$$

(4) For all $x \in A$,

$$\begin{aligned} & & d(d(x) \cdot x) &= (d(d(x)) \cdot x) \wedge (d(x) \cdot d(x)) \\ &\text{(By Proposition 1.3 (1))} & &= (d(d(x)) \cdot x) \wedge 0 \\ &\text{(By Proposition 2.3 (2))} & &= 0. \end{aligned}$$

(5) For all $x \in A$,

$$
\begin{aligned}
&\text{(By Theorem 2.7 (1))} \quad d(d(x)) &&= d(x \wedge d(x)) \\
& &&= d((d(x) \cdot x) \cdot x) \\
& &&= (d(d(x) \cdot x) \cdot x) \wedge ((d(x) \cdot x) \cdot d(x)) \\
&\text{(By (4))} &&= (0 \cdot x) \wedge ((d(x) \cdot x) \cdot d(x)) \\
&\text{(By UP-2)} &&= x \wedge ((d(x) \cdot x) \cdot d(x)) \\
& &&= (((d(x) \cdot x) \cdot d(x)) \cdot x) \cdot x.
\end{aligned}
$$

Thus

$$
\begin{aligned}
x \cdot d(d(x)) &= x \cdot ((((d(x) \cdot x) \cdot d(x)) \cdot x) \cdot x) \\
\text{(By Proposition 1.3 (5))} \quad &= 0.
\end{aligned}
$$

Hence, $x \leq d(d(x))$ for all $x \in A$. □

PROPOSITION 2.11. *Let d be an (r, l)-derivation of A. Then the following properties hold: for any $x, y \in A$,*

(1) $x \cdot d(y) \leq d(x \cdot y)$,
(2) $d(x \cdot d(x)) = 0$, *and*
(3) $d(d(x) \cdot x) = 0$.

Proof. (1) For all $x, y \in A$,

$$
\begin{aligned}
(x \cdot d(y)) \cdot d(x \cdot y) &= (x \cdot d(y)) \cdot ((x \cdot d(y)) \wedge (d(x) \cdot y)) \\
&= (x \cdot d(y)) \cdot (((d(x) \cdot y) \cdot (x \cdot d(y))) \cdot (x \cdot d(y)))
\end{aligned}
$$

(By Proposition 1.3 (5))

$$= 0.$$

Hence, $x \cdot d(y) \leq d(x \cdot y)$ for all $x, y \in A$.
(2) For all $x \in A$,

$$
\begin{aligned}
d(x \cdot d(x)) &= (x \cdot d(d(x))) \wedge (d(x) \cdot d(x)) \\
\text{(By Proposition 1.3 (1))} \quad &= (x \cdot d(d(x))) \wedge 0 \\
\text{(By Proposition 2.3 (2))} \quad &= 0.
\end{aligned}
$$

(3) For all $x \in A$,

$$
\begin{aligned}
d(d(x) \cdot x) &= (d(x) \cdot d(x)) \wedge (d(d(x)) \cdot x) \\
\text{(By Proposition 1.3 (1))} \quad &= 0 \wedge (d(d(x)) \cdot x) \\
\text{(By Proposition 2.3 (1))} \quad &= 0.
\end{aligned}
$$

□

THEOREM 2.12. *Let $d_1, d_2, \ldots, d_n$ be (l,r)-derivations of A for all $n \in \mathbb{N}$. Then*

$$x \leq d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots)) \text{ for all } x \in A. \tag{5}$$

In particular, if d is an (l,r)-derivation of A, then $x \leq d^n(x)$ for all $n \in \mathbb{N}$ and $x \in A$.

Proof. For $n = 1$, it follows from Proposition 2.10 (1) that $x \leq d_1(x)$ for all $x \in A$. Let $n \in \mathbb{N}$ and assume that $x \leq d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots))$ for all $x \in A$. Let

$$D_n := d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots)).$$

Then

$$\begin{aligned}
&\text{(By UP-2)} & d_{n+1}(D_n) &= d_{n+1}(0 \cdot D_n) \\
& & &= (d_{n+1}(0) \cdot D_n) \wedge (0 \cdot d_{n+1}(D_n)) \\
&\text{(By Theorem 2.5 (1))} & &= (0 \cdot D_n) \wedge (0 \cdot d_{n+1}(D_n)) \\
&\text{(By UP-2)} & &= D_n \wedge d_{n+1}(D_n) \\
& & &= (d_{n+1}(D_n) \cdot D_n) \cdot D_n.
\end{aligned}$$

Thus

$$\begin{aligned}
& & D_n \cdot d_{n+1}(D_n) &= D_n \cdot ((d_{n+1}(D_n) \cdot D_n) \cdot D_n) \\
&\text{(By Proposition 1.3 (5))} & &= 0.
\end{aligned}$$

Therefore, $D_n \leq d_{n+1}(D_n)$. By assumption, we get

$$x \leq D_n \leq d_{n+1}(D_n) = d_{n+1}(d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots))) \text{ for all } x \in A.$$

Hence,

$$x \leq d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots)) \text{ for all } n \in \mathbb{N} \text{ and } x \in A.$$

In particular, put $d = d_n$ for all $n \in \mathbb{N}$. Hence, $x \leq d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots)) = d^n(x)$ for all $n \in \mathbb{N}$ and $x \in A$. □

DEFINITION 2.13. *An ideal B of A is called invariant (with respect to an (l,r)-derivation (resp. (r,l)-derivation, derivation) d of A) if $d(B) \subseteq B$.*

THEOREM 2.14. *Every ideal of A is invariant with respect to any (l,r)-derivation of A.*

Proof. Assume that B is an ideal of A and d is an (l, r)-derivation of A. Let $y \in d(B)$. Then $y = d(x)$ for some $x \in B$. By Proposition 2.10 (1), we obtain $x \leq d(x)$; that is, $x \cdot d(x) = 0$. Thus $x \cdot y = x \cdot d(x) = 0 \in B$. Since $x \in B$, it follows from Theorem 1.7 (1) that $y \in B$. Hence, $d(B) \subseteq B$, which implies that B is invariant. □

Corollary 2.15. *Every ideal of A is invariant with respect to any derivation of A.*

Theorem 2.16. *In a UP-algebra A, the following statements hold:*

(1) if d is an (l, r)-derivation of A, then $y \wedge x \in \mathrm{Ker}_d(A)$ for all $y \in \mathrm{Ker}_d(A)$ and $x \in A$, and

(2) if d is an (r, l)-derivation of A, then $y \wedge x \in \mathrm{Ker}_d(A)$ for all $y \in \mathrm{Ker}_d(A)$ and $x \in A$.

Proof. (1) Assume that d is an (l, r)-derivation of A. Let $y \in \mathrm{Ker}_d(A)$ and $x \in A$. Then $d(y) = 0$. Thus

$$\begin{aligned}
d(y \wedge x) &= d((x \cdot y) \cdot y) \\
&= (d(x \cdot y) \cdot y) \wedge ((x \cdot y) \cdot d(y)) \\
&= (d(x \cdot y) \cdot y) \wedge ((x \cdot y) \cdot 0) \\
\text{(By UP-3)} \quad &= (d(x \cdot y) \cdot y) \wedge 0 \\
\text{(By Proposition 2.3 (2))} \quad &= 0.
\end{aligned}$$

Hence, $y \wedge x \in \mathrm{Ker}_d(A)$.

(2) Assume that d is an (r, l)-derivation of A. Let $y \in \mathrm{Ker}_d(A)$ and $x \in A$. Then $d(y) = 0$. Thus

$$\begin{aligned}
d(y \wedge x) &= d((x \cdot y) \cdot y) \\
&= ((x \cdot y) \cdot d(y)) \wedge (d(x \cdot y) \cdot y) \\
&= ((x \cdot y) \cdot 0) \wedge (d(x \cdot y) \cdot y) \\
\text{(By UP-3)} \quad &= 0 \wedge (d(x \cdot y) \cdot y) \\
\text{(By Proposition 2.3 (1))} \quad &= 0.
\end{aligned}$$

Hence, $y \wedge x \in \mathrm{Ker}_d(A)$. □

Corollary 2.17. *If d is a derivation of A, then $y \wedge x \in \mathrm{Ker}_d(A)$ for all $y \in \mathrm{Ker}_d(A)$ and $x \in A$.*

Theorem 2.18. *In a meet-commutative UP-algebra A, the following statements hold:*

(1) if d is an (l,r)-derivation of A and for any $x,y \in A$ is such that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$, then $x \in \mathrm{Ker}_d(A)$, and
(2) if d is an (r,l)-derivation of A and for any $x,y \in A$ is such that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$, then $x \in \mathrm{Ker}_d(A)$.

Proof. (1) Assume that d is an (l,r)-derivation of A. Let $x,y \in A$ be such that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$. Then $y \cdot x = 0$ and $d(y) = 0$. Thus

$$\begin{aligned}
\text{(By UP-2)} \qquad d(x) &= d(0 \cdot x) \\
&= d((y \cdot x) \cdot x) \\
&= d((x \cdot y) \cdot y) \\
&= (d(x \cdot y) \cdot y) \wedge ((x \cdot y) \cdot d(y)) \\
&= (d(x \cdot y) \cdot y) \wedge ((x \cdot y) \cdot 0) \\
\text{(By UP-3)} \qquad &= (d(x \cdot y) \cdot y) \wedge 0 \\
\text{(By Proposition 2.3 (2))} \qquad &= 0.
\end{aligned}$$

Hence, $x \in \mathrm{Ker}_d(A)$.
(2) Assume that d is an (r,l)-derivation of A. Let $x,y \in A$ be such that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$. Then $y \cdot x = 0$ and $d(y) = 0$. Thus

$$\begin{aligned}
\text{(By UP-2)} \qquad d(x) &= d(0 \cdot x) \\
&= d((y \cdot x) \cdot x) \\
&= d((x \cdot y) \cdot y) \\
&= ((x \cdot y) \cdot d(y)) \wedge (d(x \cdot y) \cdot y) \\
&= ((x \cdot y) \cdot 0) \wedge (d(x \cdot y) \cdot y) \\
\text{(By UP-3)} \qquad &= 0 \wedge (d(x \cdot y) \cdot y) \\
\text{(By Proposition 2.3 (1))} \qquad &= 0.
\end{aligned}$$

Hence, $x \in \mathrm{Ker}_d(A)$. □

Corollary 2.19. *If d is a derivation of a meet-commutative UP-algebra A and for any $x,y \in A$ is such that $y \leq x$ and $y \in \mathrm{Ker}_d(A)$, then $x \in \mathrm{Ker}_d(A)$.*

Theorem 2.20. *In a UP-algebra A, the following statements hold:*
(1) if d is an (l,r)-derivation of A, then $y \cdot x \in \mathrm{Ker}_d(A)$ for all $x \in \mathrm{Ker}_d(A)$ and $y \in A$, and
(2) if d is an (r,l)-derivation of A, then $y \cdot x \in \mathrm{Ker}_d(A)$ for all $x \in \mathrm{Ker}_d(A)$ and $y \in A$.

Proof. (1) Assume that d is an (l,r)-derivation of A. Let $x \in \mathrm{Ker}_d(A)$ and $y \in A$. Then $d(x) = 0$. Thus

$$\begin{aligned} d(y \cdot x) &= (d(y) \cdot x) \wedge (y \cdot d(x)) \\ &= (d(y) \cdot x) \wedge (y \cdot 0) \\ \text{(By UP-3)} \quad &= (d(y) \cdot x) \wedge 0 \\ \text{(By Proposition 2.3 (2))} \quad &= 0. \end{aligned}$$

Hence, $y \cdot x \in \mathrm{Ker}_d(A)$.
(2) Assume that d is an (r,l)-derivation of A. Let $x \in \mathrm{Ker}_d(A)$ and $y \in A$. Then $d(x) = 0$. Thus

$$\begin{aligned} d(y \cdot x) &= (y \cdot d(x)) \wedge (d(y) \cdot x) \\ &= (y \cdot 0) \wedge (d(y) \cdot x) \\ \text{(By UP-3)} \quad &= 0 \wedge (d(y) \cdot x) \\ \text{(By Proposition 2.3 (1))} \quad &= 0. \end{aligned}$$

Hence, $y \cdot x \in \mathrm{Ker}_d(A)$. □

Corollary 2.21. *If d is a derivation of A, then $y \cdot x \in \mathrm{Ker}_d(A)$ for all $x \in \mathrm{Ker}_d(A)$ and $y \in A$.*

Theorem 2.22. *In a UP-algebra A, the following statements hold:*
(1) if d is an (l,r)-derivation of A, then $\mathrm{Ker}_d(A)$ is a UP-subalgebra of A, and
(2) if d is an (r,l)-derivation of A, then $\mathrm{Ker}_d(A)$ is a UP-subalgebra of A.

Proof. (1) Assume that d is an (l,r)-derivation of A. By Theorem 2.5 (1), we have $d(0) = 0$ and so $0 \in \mathrm{Ker}_d(A) \neq \emptyset$. Let $x, y \in \mathrm{Ker}_d(A)$. Then $d(x) = 0$ and $d(y) = 0$. Thus

$$\begin{aligned} d(x \cdot y) &= (d(x) \cdot y) \wedge (x \cdot d(y)) \\ &= (0 \cdot y) \wedge (x \cdot 0) \\ \text{(By UP-2 and UP-3)} \quad &= y \wedge 0 \\ \text{(By Proposition 2.3 (2))} \quad &= 0. \end{aligned}$$

Hence, $x \cdot y \in \mathrm{Ker}_d(A)$, so $\mathrm{Ker}_d(A)$ is a UP-subalgebra of A.
(2) Assume that d is an (r,l)-derivation of A. By Theorem 2.5 (2), we have $d(0) = 0$ and so $0 \in \mathrm{Ker}_d(A) \neq \emptyset$. Let $x, y \in \mathrm{Ker}_d(A)$. Then

$d(x) = 0$ and $d(y) = 0$. Thus

$$\begin{aligned} d(x \cdot y) &= (x \cdot d(y)) \wedge (d(x) \cdot y) \\ &= (x \cdot 0) \wedge (0 \cdot y) \\ \text{(By UP-2 and UP-3)} \quad &= 0 \wedge y \\ \text{(By Proposition 2.3 (1))} \quad &= 0. \end{aligned}$$

Hence, $x \cdot y \in \mathrm{Ker}_d(A)$, so $\mathrm{Ker}_d(A)$ is a UP-subalgebra of A. □

Corollary 2.23. *If d is a derivation of A, then $\mathrm{Ker}_d(A)$ is a UP-subalgebra of A.*

Definition 2.24. Let d be an (l, r)-derivation (resp. (r, l)-derivation, derivation) of A. We define a subset $\mathrm{Fix}_d(A)$ of A by

$$\mathrm{Fix}_d(A) = \{x \in A \mid d(x) = x\}.$$

Theorem 2.25. *In a UP-algebra A, the following statements hold:*

(1) if d is an (l, r)-derivation of A, then $\mathrm{Fix}_d(A)$ is a UP-subalgebra of A, and
(2) if d is an (r, l)-derivation of A, then $\mathrm{Fix}_d(A)$ is a UP-subalgebra of A.

Proof. (1) Assume that d is an (l, r)-derivation of A. By Theorem 2.5 (1), we have $d(0) = 0$ and so $0 \in \mathrm{Fix}_d(A) \neq \emptyset$. Let $x, y \in \mathrm{Fix}_d(A)$. Then $d(x) = x$ and $d(y) = y$. Thus

$$\begin{aligned} d(x \cdot y) &= (d(x) \cdot y) \wedge (x \cdot d(y)) \\ &= (x \cdot y) \wedge (x \cdot y) \\ \text{(By Proposition 2.3 (3))} \quad &= x \cdot y. \end{aligned}$$

Hence, $x \cdot y \in \mathrm{Fix}_d(A)$, so $\mathrm{Fix}_d(A)$ is a UP-subalgebra of A.
(2) Assume that d is an (r, l)-derivation of A. By Theorem 2.5 (2), we have $d(0) = 0$ and so $0 \in \mathrm{Fix}_d(A) \neq \emptyset$. Let $x, y \in \mathrm{Fix}_d(A)$. Then $d(x) = x$ and $d(y) = y$. Thus

$$\begin{aligned} d(x \cdot y) &= (x \cdot d(y)) \wedge (d(x) \cdot y) \\ &= (x \cdot y) \wedge (x \cdot y) \\ \text{(By Proposition 2.3 (3))} \quad &= x \cdot y. \end{aligned}$$

Hence, $x \cdot y \in \mathrm{Fix}_d(A)$, so $\mathrm{Fix}_d(A)$ is a UP-subalgebra of A. □

COROLLARY 2.26. *If d is a derivation of A, then* $\mathrm{Fix}_d(A)$ *is a UP-subalgebra of A.*

THEOREM 2.27. *In a UP-algebra A, the following statements hold:*

(1) if d is an (l, r)-derivation of A, then $x \wedge y \in \mathrm{Fix}_d(A)$ for all $x, y \in \mathrm{Fix}_d(A)$, and
(2) if d is an (r, l)-derivation of A, then $x \wedge y \in \mathrm{Fix}_d(A)$ for all $x, y \in \mathrm{Fix}_d(A)$.

Proof. (1) Assume that d is an (l, r)-derivation of A. Let $x, y \in \mathrm{Fix}_d(A)$. Then $d(x) = x$ and $d(y) = y$. By Theorem 2.25 (1), we get $d(y \cdot x) = y \cdot x$. Thus

$$\begin{aligned} d(x \wedge y) &= d((y \cdot x) \cdot x) \\ &= (d(y \cdot x) \cdot x) \wedge ((y \cdot x) \cdot d(x)) \\ &= ((y \cdot x) \cdot x) \wedge ((y \cdot x) \cdot x) \\ \text{(By Proposition 2.3 (3))} \quad &= (y \cdot x) \cdot x \\ &= x \wedge y. \end{aligned}$$

Hence, $x \wedge y \in \mathrm{Fix}_d(A)$.
(2) Assume that d is an (r, l)-derivation of A. Let $x, y \in \mathrm{Fix}_d(A)$. Then $d(x) = x$ and $d(y) = y$. By Theorem 2.25 (2), we get $d(y \cdot x) = y \cdot x$. Thus

$$\begin{aligned} d(x \wedge y) &= d((y \cdot x) \cdot x) \\ &= ((y \cdot x) \cdot d(x)) \wedge (d(y \cdot x) \cdot x) \\ &= ((y \cdot x) \cdot x) \wedge ((y \cdot x) \cdot x) \\ \text{(By Proposition 2.3 (3))} \quad &= (y \cdot x) \cdot x \\ &= x \wedge y. \end{aligned}$$

Hence, $x \wedge y \in \mathrm{Fix}_d(A)$. □

COROLLARY 2.28. *If d is a derivation of A, then $x \wedge y \in \mathrm{Fix}_d(A)$ for all $x, y \in \mathrm{Fix}_d(A)$.*

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Kaewta Sawika
Department of Mathematics
School of Science
University of Phayao
Phayao 56000, Thailand
E-mail: kaewta.ska@gmail.com

Rossukon Intasan
Department of Mathematics
School of Science
University of Phayao
Phayao 56000, Thailand
E-mail: rossukon.indi@gmail.com

Arocha Kaewwasri
Department of Mathematics
School of Science
University of Phayao
Phayao 56000, Thailand
E-mail: arocha.kws@gmail.com

Aiyared Iampan
Department of Mathematics
School of Science
University of Phayao
Phayao 56000, Thailand
E-mail: aiyared.ia@up.ac.th