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QUANTITATIVE ESTIMATES FOR GENERALIZED TWO DIMENSIONAL BASKAKOV OPERATORS

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Abstract. In this paper, we obtain quantitative estimates for generalized double Baskakov operators. We calculate global results for these operators using Lipschitz-type spaces and estimate the error using modulus of continuity.

1. Introduction

The Baskakov operator $V_n(f;x)$ was introduced by V.A. Baskakov [2] given by:

$$V_n(f;x) = \sum_{k=0}^{\infty} p_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $p_{n,k}(x) = \binom{n+k-1}{k} x^k (1+x)^{-(n+k)}$, $f \in C_B[0,\infty)$, $C_B[0,\infty)$ is the set of functions which are bounded on the set.
By now a number of results about the operator have been obtained [1, 3–7]. In this paper, we address the investigation for the multivariate Baskakov operator defined as follows.

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Let $P_n(f;x,y)$ are the following two variate Baskakov operators:

$$P_n(f;x,y) = \sum_{k,l=0}^{\infty} p_{n,k}(x)\, p_{n,l}(y) f\left(\frac{k}{n}, \frac{l}{n}\right), \tag{1.2}$$

where $0 \leq k \leq n, 0 \leq l \leq n, f(x,y) \in C([0,\infty) \times [0,\infty))$.

Ozarslan and Duman [9] have introduced a different approach in order to get a faster approximation without preserving the test functions. In [8], Özarslan and Aktuğlu have calculated quantitative global estimates for double Szasz-Mirakjan operators. In this paper, same method is used for generalized double Baskakov operators.

Consider the classical Baskakov operators defined by (1.1). Since for $f_i = t^i, i = 0,1,2$

$$V_n(f_0;x) = 1, V_n(f_1;x) = x, V_n(f_2;x) = \left(1 + \frac{1}{n}\right) x^2 + \frac{x}{n}.$$

Following the similar arguments as used in [9], the best error estimation among all the general double Baskakov operators can be obtained from the case by taking

$$a_n = 1, b_n = e_n = 0, c_n = 1 + \frac{1}{n}, d_n = \frac{1}{n}$$

for all $n \in N$ where $(a_n), (b_n), (c_n), (d_n)$ and (e_n) are sequences of non-negative real numbers satisfying the conditions given in [9].

Now observe that

$$u_n(x) = \frac{2a_n x - d_n}{2c_n} = \frac{2nx - 1}{2(n+1)} \in [0,\infty),$$

where u_n is a functional sequence, $u_n : I \to A$ where A denote $\mathbb{R}^+$ and assume that I be subinterval of A.

So, $u_n(x) \in \mathbb{R}^+$ if and only if $x \geq \frac{1}{2n}$ and $n \geq 1$. Hence, choosing

$$I = \left[\frac{1}{2}, \infty\right) \subset \mathbb{R}^+.$$

The best error estimation among all the general double Baskakov operators can be obtained from the case

$$u_n(x) = \frac{2nx - 1}{2(n+1)}, v_n(y) = \frac{2ny - 1}{2(n+1)}; n \in \mathbb{N}$$

for all $f \in C_B([0,\infty) \times C_B[0,\infty))$ and $x, y \in \left[\frac{1}{2}, \infty\right)$. Hence, (1.2) becomes

$$
\begin{aligned}
(1.3)\qquad Q_n(f;x,y) &: P_n(f;u_n(x), v_n(y)) \\
&= \sum_{k,l=0}^{h} \binom{n+k-1}{k}\binom{n+l-1}{l} f\left(\frac{k}{n}, \frac{l}{n}\right)(u_n(x))^k \\
&\qquad (1+u_n(x))^{-(n+k)}(v_n(y))^l(1+v_n(y))^{-(n+l)}, \\
&f \in C_B([0,\infty)\times[0,\infty)).
\end{aligned}
$$

For the operators $Q_n(f;x,y)$, we have following Lemma:

Lemma 1.1. *Let* $\mathbf{x} = (x,y), \boldsymbol{t} = (t,s); e_{i,j}(x) = x^i y^j, i,j = 0,1,2$ *and* $\psi_x^2(t) = \|t-x\|^2$. *Then, for each* $x, y \geq 0$ *and* $n > 1$, *we have*

(i) $Q_n(e_{0,0};x,y) = 1$

(ii) $Q_n(e_{1,0};x,y) = u_n(x)$

(iii) $Q_n(e_{0,1};x,y) = v_n(y)$

(iv) $Q_n(e_{2,0}+e_{0,2};x,y) = \left(1+\frac{1}{n}\right)(u_n^2(x)+v_n^2(y)) + \frac{u_n(x)+v_n(y)}{n}$

(v) $Q_n(\psi_x^2(t);x,y) = (u_n(x)-x)^2+(v_n(y)-y)^2+\frac{1}{n}(u_n^2(x)+v_n^2(y)+u_n(x)+v_n(y))$.

2. Global Results

We have used following definitions in this paper for global results of the operators $Q_n(f;x,y)$.
Otto Szasz [10] earlier considered this space of bivariate extension of Lipschitz-type space, given as:

$$
Lip_M^*(\alpha) := \left\{ f \in C([0,\infty)\times[0,\infty)) : |f(\mathbf{t}) - f(\mathbf{x})| \leq M \frac{\|\mathbf{t}-\mathbf{x}\|^{\alpha}}{(\|\mathbf{t}\|+x+y)^{\alpha/2}} \right.
$$
$$
\left. ; t,s;x,y \in (0,\infty) \right\},
$$

where $\mathbf{t} = (t,s), \mathbf{x} = (x,y)$ and M is any positive constant and $0 < \alpha \leq 1$.

For all $f \in C([0,\infty)\times[0,\infty))$, the modulus of f denoted by $\omega(f;\delta)$ is defined as

$$\omega(f;\delta) := \sup\left\{ |f(t,s)-f(x,y)| : \sqrt{(t-x)^2+(s-y)^2} < \delta, (t,s), \right. \\ \left. (x,y) \in [0,\infty)\times[0,\infty) \right\}.$$

Now, for the space $Lip_M^*(\alpha)$ with $0 < \alpha \leq 1$, we have the following approximation result.

THEOREM 2.1. *For any* $f \in Lip_M^*(\alpha), \alpha \in (0,1]$, *and for each* $x, y \in (0,\infty), n \in N$, *we have*

$$\begin{aligned} |Q_n(f;x,y)-f(x,y)| \leq \frac{M}{(x+y)^{\alpha/2}} \Big[& (u_n(x)-x)^2 + (v_n(y)-y)^2 \\ & + \frac{1}{n}\left(u_n^2(x)+v_n^2(y)+u_n(x)+v_n(y)\right) \Big]^{\alpha/2}. \end{aligned} \tag{2.1}$$

Proof. Let $\alpha = 1$. For each $x, y \in (0,\infty)$ and for $f \in Lip_M^*(1)$, we have

$$\begin{aligned} |Q_n(f;x,y)-f(x,y)| &\leq Q_n(|f(t,s)-f(x,y)|;x,y) \\ &\leq MQ_n\left(\frac{\|\mathbf{t}-\mathbf{x}\|}{(\|\mathbf{t}\|+x+y)^{1/2}}; x,y \right) \\ &\leq \frac{M}{(x+y)^{1/2}} Q_n(\|\mathbf{t}-\mathbf{x}\|;x,y). \end{aligned}$$

Applying Cauchy-Schwarz inequality, we get

$$|Q_n(f;x,y)-f(x,y)| \leq \frac{M}{(x+y)^{1/2}} \sqrt{Q_n(\psi_x^2(t);x,y)}$$

$$= \frac{M}{(x+y)^{1/2}} \sqrt{(u_n(x)-x)^2+(v_n(y)-y)^2+\frac{1}{n}(u_n^2(x)+v_n^2(y)+u_n(x)+v_n(y))}.$$

Now, let $0 < \alpha < 1$. Then for each $x, y \in (0, \infty)$ and for $f \in Lip_M^*(\alpha)$, we obtain

$$\begin{aligned}|Q_n(f;x,y) - f(x,y)| &\le Q_n(|f(t,s) - f(x,y)|;x,y) \\ &\le MQ_n\left(\frac{\|\mathbf{t}-\mathbf{x}\|^\alpha}{(\|\mathbf{t}\| + x + y)^{\alpha/2}};x,y\right) \\ &\le \frac{M}{(x+y)^{\alpha/2}}Q_n(\|\mathbf{t}-\mathbf{x}\|^\alpha;x,y).\end{aligned}$$

For Holder inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, for any $f \in Lip_M^*(\alpha)$, we have

$$\begin{aligned}|Q_n(f;x,y) - f(x,y)| &\le \frac{M}{(x+y)^{\alpha/2}}\left[Q_n\left(\psi_x^2(t);x,y\right)\right]^{\alpha/2} \\ &= \frac{M}{(x+y)^{\alpha/2}}\Big[(u_n(x)-x)^2 + (v_n(y)-y)^2 \\ &\quad + \frac{1}{n}\left(u_n^2(x) + v_n^2(y) + u_n(x) + v_n(y)\right)\Big]^{\alpha/2}\end{aligned}$$

which is the required result. □

Lemma 2.2. *For each $x, y > 0$,*

$$\begin{aligned}Q_n&\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2 + \left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right) \\ &\le \frac{1}{\sqrt{x}}\sqrt{(u_n(x)-x)^2 + \frac{u_n^2(x)+u_n(x)}{n}} \\ &\quad + \frac{1}{\sqrt{y}}\sqrt{(v_n(y)-y)^2 + \frac{v_n^2(y)+v_n(y)}{n}}.\end{aligned} \tag{2.2}$$

Proof. We have $\sqrt{c+d} \leq \sqrt{c} + \sqrt{d}\,(c, d \geq 0)$, therefore

$$Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2 + \left(\sqrt{s}-\sqrt{y}\right)^2}; x, y\right)$$
$$= \sum_{k,l=0}^{\infty} \binom{n+k-1}{k} \binom{n+l-1}{l} \sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2 + \left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2}$$
$$\times (u_n(x))^k (1+u_n(x))^{-(n+k)} (v_n(y))^l (1+v_n(y))^{-(n+l)}$$
$$\leq \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left|\sqrt{\frac{k}{n}}-\sqrt{x}\right| (u_n(x))^k (1+u_n(x))^{-(n+k)}$$
$$+ \sum_{l=0}^{\infty} \binom{n+l-1}{l} \left|\sqrt{\frac{l}{n}}-\sqrt{y}\right| (v_n(y))^l (1+v_n(y))^{-(n+l)}$$
$$= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \frac{\left|\frac{k}{n}-x\right|}{\sqrt{\frac{k}{n}}+\sqrt{x}} (u_n(x))^k (1+u_n(x))^{-(n+k)}$$
$$+ \sum_{l=0}^{\infty} \binom{n+l-1}{l} \frac{\left|\frac{l}{n}-y\right|}{\sqrt{\frac{l}{n}}+\sqrt{y}} (v_n(y))^l (1+v_n(y))^{-(n+l)}$$
$$= \frac{1}{\sqrt{x}} \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left|\frac{k}{n}-x\right| (u_n(x))^k (1+u_n(x))^{-(n+k)}$$
$$+ \frac{1}{\sqrt{y}} \sum_{l=0}^{\infty} \binom{n+l-1}{l} \left|\frac{l}{n}-y\right| (v_n(y))^l (1+v_n(y))^{-(n+l)}.$$

Using the Cauchy-Schwarz inequality,

$$Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2 + \left(\sqrt{s}-\sqrt{y}\right)^2}; x, y\right)$$
$$\leq \frac{1}{\sqrt{x}} \sqrt{\sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{k}{n}-x\right)^2 (u_n(x))^k (1+u_n(x))^{-(n+k)}}$$
$$+ \frac{1}{\sqrt{y}} \sqrt{\sum_{l=0}^{\infty} \binom{n+l-1}{l} \left(\frac{l}{n}-y\right)^2 (v_n(y))^l (1+v_n(y))^{-(n+l)}}.$$

Using Lemma 1.1,

$$Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)$$
$$\leq \frac{1}{\sqrt{x}}\sqrt{\left(u_n\left(x\right)-x\right)^2+\frac{u_n^2\left(x\right)+u_n\left(x\right)}{n}}+\frac{1}{\sqrt{y}}\sqrt{\left(v_n\left(y\right)-y\right)^2+\frac{v_n^2\left(y\right)+v_n\left(y\right)}{n}}$$

which is the desired result. □

THEOREM 2.3. *Let $g\left(x,y\right)=f\left(x^2,y^2\right)$. Then we have for each $x,y>0$,*

$$\left|Q_n\left(f;x,y\right)-f\left(x,y\right)\right|\leq 2\omega\left(g;\delta_n\left(x,y\right)\right),$$

where $\delta_n\left(x,y\right)=\frac{1}{\sqrt{x}}\sqrt{\left(u_n\left(x\right)-x\right)^2+\frac{u_n^2(x)+u_n(x)}{n}}+\frac{1}{\sqrt{y}}\sqrt{\left(v_n\left(y\right)-y\right)^2+\frac{v_n^2(y)+v_n(y)}{n}}$.

Proof. We have

$$\left|Q_n\left(f;x,y\right)-f\left(x,y\right)\right|\leq Q_n\left(\left|f\left(t,s\right)-f\left(x,y\right)\right|;x,y\right)$$
$$=Q_n\left(\left|g\left(\sqrt{t},\sqrt{s}\right)-g\left(\sqrt{x},\sqrt{y}\right)\right|;x,y\right)$$
$$\leq Q_n\left(\omega\left(g;\sqrt{\left(t-x\right)^2+\left(s-y\right)^2}\right);x,y\right)$$
$$=\sum_{k,l=0}^{\infty}\binom{n+k-1}{k}\binom{n+l-1}{l}\omega\left(g;\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2};x,y\right)$$
$$\times\left(u_n\left(x\right)\right)^k\left(1+u_n\left(x\right)\right)^{-(n+k)}\left(v_n\left(y\right)\right)^l\left(1+v_n\left(y\right)\right)^{-(n+l)}$$
$$=\sum_{k,l=0}^{\infty}\binom{n+k-1}{k}\binom{n+l-1}{l}$$
$$\omega\left(g;\frac{\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2}}{Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)}\right.$$
$$\left.\times Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right);x,y\right).$$

Now, we have

$$\omega\left(f;\lambda\delta\right)\leq\left(1+\lambda\right)\omega\left(f;\delta\right).$$

Therefore,

$$\begin{aligned}
&\left|Q_n\left(f;x,y\right)-f\left(x,y\right)\right| \\
&\quad\le \omega\left(g;Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)\right) \\
&\qquad\times\sum_{k,l=0}^{\infty}\binom{n+k-1}{k}\binom{n+l-1}{l} \\
&\qquad\left[1+\frac{\sqrt{\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2}}{Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)}\right] \\
&\qquad\left(u_n\left(x\right)\right)^k\left(1+u_n\left(x\right)\right)^{-(n+k)}\left(v_n\left(y\right)\right)^l\left(1+v_n\left(y\right)\right)^{-(n+l)} \\
&\quad\le 2\omega\left(g;Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)\right).
\end{aligned}$$

Now, using Lemma 2.2, completes the proof. □

THEOREM 2.4. *Let* $g\left(x,y\right)=f\left(x^2,y^2\right)$. *Let*

$$g\in Lip_M\left(\alpha\right):=\left\{g\in C_{\mathbf{B}}\left(\left[0,\infty\right)\times\left[0,\infty\right)\right):\left|g\left(\boldsymbol{t}\right)-g\left(\mathbf{x}\right)\right|\le M\|\boldsymbol{t}-\mathbf{x}\|^{\alpha}\right.$$
$$\left.;t,s;x,y\in\left(0,\infty\right)\right\},$$

where $\boldsymbol{t}=\left(t,s\right),\mathbf{x}=\left(x,y\right)$ *and* M *is any positive constant and* $0<\alpha\le 1$.
Then,

$$\left|Q_n\left(f;x,y\right)-f\left(x,y\right)\right|\le M\delta_n^{\alpha}\left(x,y\right), \tag{2.3}$$

where $\delta_n\left(x,y\right)=\frac{1}{\sqrt{x}}\sqrt{\left(u_n\left(x\right)-x\right)^2+\frac{u_n^2(x)+u_n(x)}{n}}+\frac{1}{\sqrt{y}}\sqrt{\left(v_n\left(y\right)-y\right)^2+\frac{v_n^2(y)+v_n(y)}{n}}$.

Proof. We have

$$\left|Q_n(f;x,y)-f(x,y)\right|\le Q_n\left(\left|f(t,s)-f(x,y)\right|;x,y\right)$$
$$=Q_n\left(\left|g\left(\sqrt{t},\sqrt{s}\right)-g\left(\sqrt{x},\sqrt{y}\right)\right|;x,y\right)$$
$$\le MQ_n\left(\left(\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2\right)^{\alpha/2};x,y\right)$$
$$=M\sum_{k,l=0}^{\infty}\binom{n+k-1}{k}\binom{n+l-1}{l}\left(\left(\sqrt{\frac{k}{n}}-\sqrt{x}\right)^2+\left(\sqrt{\frac{l}{n}}-\sqrt{y}\right)^2\right)^{\alpha/2}$$
$$(u_n(x))^k(1+u_n(x))^{-(n+k)}(v_n(y))^l(1+v_n(y))^{-(n+l)}.$$

For Holder inequality with $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, we have

$$\left|Q_n(f;x,y)-f(x,y)\right|\le M\left[Q_n\left(\sqrt{\left(\sqrt{t}-\sqrt{x}\right)^2+\left(\sqrt{s}-\sqrt{y}\right)^2};x,y\right)\right]^{\alpha}.$$

By using Lemma 2.2, completes the proof. □

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