

A STUDY ON THE RECURRENCE RELATIONS OF 5-DIMENSIONAL ES -MANIFOLD

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ABSTRACT. The manifold $*g-ESX_n$ is a generalized n -dimensional Riemannian manifold on which the differential geometric structure is imposed by the unified field tensor $*g^{\lambda\nu}$ through the ES -connection which is both Einstein and semi-symmetric. The purpose of the present paper is to study the algebraic geometric structures of 5-dimensional $*g-ESX_5$. Particularly, in 5-dimensional $*g-ESX_5$, we derive a new set of powerful recurrence relations in the first class.

1. Introduction

In Appendix *II* to his last book Einstein([3], 1950) proposed a new unified field theory that would include both gravitation and electromagnetism. Although the intent of this theory is physical, its exposition is mainly geometrical. It may be characterized as a set of geometrical postulates for the space time X_4 . Characterizing Einstein's unified field theory as a set of geometrical postulates for X_4 , Hlavatý([4], 1957) gave its mathematical foundation for the first time. Since then Hlavatý and number of mathematicians contributed for the development of this theory and obtained many geometrical consequences of these postulates.

The main purpose of the present paper is to study the algebraic geometric properties of 5-dimensional $*g-ESX_5$ in the first class. In

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particular, we derive a powerful recurrence relations in 5-dimensional $*g - ESX_5$.

2. Preliminaries

This section is a brief collection of basic concepts, results, and notations needed in subsequent considerations. They are due to Chung ([1], 1963), and Mishra([5], 1959) mostly due to Datta([2], 1964).

(a) n -dimensional $*g$ -unified field theory

Corresponding to the Einstein's $n - g$ -UFT, our $n - *g$ -UFT is based on the following three principles.

PRINCIPLE A. Let X_n be an n -dimensional generalized Riemannian manifold referred to a real coordinate system x^ν , which obeys the coordinate transformations $x^\nu \rightarrow x^{\nu'}$ for which

$$(2.1) \quad \det\left(\frac{\partial x'}{\partial x}\right) \neq 0$$

In $n - g - UFT$ the manifold X_n is endowed with a real nonsymmetric tensor $g_{\lambda\mu}$, which may be decomposed into its symmetric part $h_{\lambda\mu}$ and skew-symmetric part $k_{\lambda\mu}$:

$$(2.2) \quad g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}$$

where

$$(2.3) \quad \mathfrak{g} = \det(g_{\lambda\mu}) \neq 0, \quad \mathfrak{h} = \det(h_{\lambda\mu}) \neq 0, \quad \mathfrak{k} = \det(k_{\lambda\mu})$$

In $n - *g - UFT$ the algebraic structure on X_n is imposed by the basic real tensor $*g^{\lambda\nu}$ defined by

$$(2.4) \quad g_{\lambda\mu} *g^{\lambda\nu} = g_{\mu\lambda} *g^{\nu\lambda} = \delta_\mu^\nu$$

It may be also decomposed into its symmetric part $*h^{\lambda\nu}$ and skew-symmetric part $*k^{\lambda\nu}$:

$$(2.5) \quad *g^{\lambda\nu} = *h^{\lambda\nu} + *k^{\lambda\nu}$$

Since $\det(*h^{\lambda\nu}) \neq 0$, we may define a unique tensor $*h_{\lambda\mu}$ by

$$(2.6) \quad *h_{\lambda\mu} *h^{\lambda\nu} = \delta_\mu^\nu$$

In $n - *g$ -UFT we use both $*h^{\lambda\nu}$ and $*h_{\lambda\mu}$ as tensors for raising and/or lowering indices of all tensors in X_n in the usual manner. We then have

$$(2.7) \quad *k_{\lambda\mu} = *k^{\rho\sigma} *h_{\lambda\rho} *h_{\mu\sigma}, \quad *g_{\lambda\mu} = *g^{\rho\sigma} *h_{\lambda\rho} *h_{\mu\sigma}$$

so that

$$(2.8) \quad *g_{\lambda\mu} = *h_{\lambda\mu} + *k_{\lambda\mu}$$

PRINCIPLE B The differential geometric structure on X_n is imposed by the tensor $*g^{\lambda\nu}$ by means of a connection $\Gamma_{\lambda}^{\nu\mu}$ defined by a system of equations

$$(2.9) \quad D_{\omega} *g^{\lambda\nu} = -2S_{\omega\alpha}^{\nu} *g^{\lambda\alpha}$$

Here D_{ω} denotes the symbol of the covariant derivative with respect to $\Gamma_{\lambda}^{\nu\mu}$ and $S_{\lambda\mu}^{\nu}$ is the torsion tensor of $\Gamma_{\lambda}^{\nu\mu}$. Under certain conditions the system (2.9) admits a unique solutions $\Gamma_{\lambda}^{\nu\mu}$.

PRINCIPLE C In order to obtain $*g^{\lambda\nu}$ involved in the solution for $\Gamma_{\lambda}^{\nu\mu}$ certain conditions are imposed. These conditions may be condensed to

$$(2.10) \quad S_{\lambda} = S_{\lambda\alpha}^{\alpha} = 0, \quad R_{[\mu\lambda]} = \partial_{[\mu} Y_{\lambda]}, \quad R_{(\mu\lambda)} = 0$$

where Y_{λ} is an arbitrary vector, and $R_{\omega\mu\lambda}^{\nu}$ are the curvature tensors of X_n defined by

$$(2.11) \quad R_{\omega\mu\lambda}^{\nu} = 2(\partial_{[\mu} \Gamma_{|\lambda|}^{\nu\omega]} + \Gamma_{\alpha}^{\nu}{}_{[\mu} \Gamma_{|\lambda|}^{\alpha\omega]}), \quad R_{\mu\lambda} = R_{\alpha\mu\lambda}^{\alpha}$$

(b) Some notations and results

The following quantities are frequently used in our further considerations:

$$(2.12) \quad *g = \det(*g_{\lambda\mu}), \quad *h = \det(*h_{\lambda\mu}), \quad *k = \det(*k_{\lambda\mu})$$

$$(2.13) \quad *g = \frac{*g}{*h}, \quad *k = \frac{*k}{*h}.$$

$$(2.14) \quad K_p = *k_{[\alpha_1}^{\alpha_1} *k_{\alpha_2}^{\alpha_2} \dots *k_{\alpha_p]}^{\alpha_p}, \quad (p = 0, 1, 2, \dots).$$

$$(2.15) \quad {}^{(0)}*k_{\lambda}^{\nu} = \delta_{\lambda}^{\nu}, \quad {}^{(p)}*k_{\lambda}^{\nu} = *k_{\lambda}^{\alpha} {}^{(p-1)}*k_{\alpha}^{\nu} \quad (p = 1, 2, \dots).$$

In X_n it was proved in [1] that

$$(2.16) \quad K_0 = 1, \quad K_n = {}^*k \text{ if } n \text{ is even, and } K_n = 0 \text{ if } n \text{ is odd.}$$

$$(2.17) \quad {}^*g = 1 + K_2 + \cdots + K_{n-\sigma}.$$

$$(2.18) \quad \sum_{s=0}^{n-\sigma} K_s ({}^{n-s}k)_\lambda{}^\nu = 0 \quad (p = 0, 1, 2, \dots).$$

We also use the following useful abbreviations for an arbitrary tensor T_{\dots} for $p = 1, 2, 3, \dots$:

$$(2.19) \quad ({}^pT)_{\dots} = ({}^{p-1}){}^*k_\alpha{}^\nu T_{\dots}.$$

(c) n -dimensional ES manifold $n - {}^*g$ -UFT

In this subsection, we display an useful representation of the ES connection in $n - {}^*g$ -UFT.

DEFINITION 2.1. A connection $\Gamma_{\lambda}{}^\nu{}_\mu$ is said to be *semi-symmetric* if its torsion tensor $S_{\lambda\mu}{}^\nu$ is of the form

$$(2.20) \quad S_{\lambda\mu}{}^\nu = 2\delta_{[\lambda}{}^\nu X_{\mu]}.$$

for an arbitrary non-null vector X_μ .

A connection which is both semi-symmetric and Einstein is called an ES connection. An n -dimensional generalized Riemannian manifold X_n , on which the differential geometric structure is imposed by ${}^*g^{\lambda\nu}$ by means of an ES connection, is called an n -dimensional *g - ES manifold. We denote this manifold by *g - ESX_n in our further considerations.

THEOREM 2.2. Under the condition (2.20), the system of equations (2.9) is equivalent to

$$(2.21) \quad \Gamma_{\lambda}{}^\nu{}_\mu = {}^* \left\{ \begin{array}{c} \nu \\ \lambda\mu \end{array} \right\} + U^\nu{}_{\lambda\mu} + 2\delta_{[\lambda}{}^\nu X_{\mu]}.$$

where

$$(2.22) \quad U^\nu{}_{\lambda\mu} = -{}^*h_{\lambda\mu}{}^{(2)}X^\nu$$

Proof. Substituting (2.20) for $S_{\lambda\mu}{}^\nu$ into (2.9), we have the representation (2.21). \square

3. Recurrence relations in ${}^*g - ESX_5$

In this section we derive several powerful recurrence relations, establishing a non-holonomic frame in ${}^*g - ESX_5$.

DEFINITION 3.1. The tensors ${}^*g_{\lambda\mu}$ is said to be

- (1) of the first class, if $K_{n-\sigma} \neq 0$
 - (2) of the second class with j th category ($j \geq 1$), if
- $$(3.1) \quad K_{2j} \neq 0, \quad K_{2j+2} = K_{2j+4} = \dots = K_{n-\sigma} = 0$$
- (3) of the third class, if $K_2 = K_4 = \dots = K_{n-\sigma} = 0$

In $5 - {}^*g$ -UFT, we have three classes; namely the first class when $K_4 \neq 0$, the second class when $K_4 = 0, K_2 \neq 0$, and the third class when $K_4 = K_2 = 0$. In $5 - {}^*g$ -UFT, the relation (2.17) gives

$$(3.2) \quad {}^*g = 1 + K_2 + K_4$$

In this chapter we investigate only the first class of $5 - {}^*g$ -UFT. Hence all considerations in this chapter are restricted to $n = 5$.

A. Basic vectors in the first class

REMARK 3.2. For the simplicity of our discussion, we assume in this and in what follows that

$$(3.3) \quad K_4 < 0$$

The eigenvalues M and the corresponding eigenvectors A^ν in ${}^*g - ESX_n$, defined by

$$(3.4) \quad MA^\nu = {}^*k_\mu{}^\nu A^\mu, \quad (M : \text{a scalar}).$$

are called *basic scalars* and *basic vectors*, respectively.

THEOREM 3.3. The basic scalars in ${}^*g - ESX_5$ may be given by

$$(3.5) \quad \begin{aligned} M_1 = -M_2 &= \sqrt{-L - K} \neq 0 \\ M_3 = -M_4 &= \sqrt{L - K} \neq 0, \quad M_5 = 0 \end{aligned}$$

where

$$(3.6) \quad K = \frac{K_2}{2}, \quad L = \sqrt{\left(\frac{K_2}{2}\right)^2 - K_4}$$

Proof. In 5- *g -UFT, the characteristic equation is reduced to

$$(3.7) \quad M(M^4 + K_2M^2 + K_4) = 0$$

from which our assertion follows in virtue of (3.3) and (3.6). \square

THEOREM 3.4. *There are five linearly independent basic vectors A'_1, \dots, A'_5 and they have the following properties:*

(a) *They are defined up to an arbitrary factor of proportionality.*

(b) A'_1, \dots, A'_4 are null vectors, while A'_5 is non-null.

(c) A'_1, A'_2 are perpendicular to A'_3, A'_4 and A'_5 is also perpendicular to A'_1, \dots, A'_4 .

(d) *They satisfy the conditions*

$$(3.8) \quad {}^*h_{\lambda\mu}A'_1A'_2 \neq 0, \quad {}^*h_{\lambda\mu}A'_3A'_4 \neq 0$$

Proof. Since the basic scalars M_i are all distinct, (3.4) admits five linearly independent basic vectors A'_i which are defined up to an arbitrary factor of proportionality. The first half of statement (b) is a consequence of (3.4), (3.5), and

$$(3.9) \quad M_x^*h_{\lambda\mu}A'_x A'_x = {}^*k_{\lambda\mu}A'_x A'_x = 0, \quad (x = 1, \dots, 4)$$

Since $M_x + M_x \neq 0$, ($x = 1, \dots, 4$), statement (c) follows from (3.5) as in the following way:

$$(3.10) \quad M_x^*h_{\lambda\mu}A'_x A'_y = {}^*k_{\lambda\mu}A'_x A'_y = -M_y^*h_{\mu\lambda}A'_x A'_y, \quad (y = 3, 4)$$

In order to prove statement (d), consider a conic C with equation ${}^*h_{\lambda\mu}A^\lambda A^\mu = 0$ on a projective plane P_2 . In virtue of statement (b), A'_1 and A'_2 are two different points on C while ${}^*h_{\lambda\mu}A^\lambda = A_{1\mu}$ is the tangent line to C at A'_1 . Since $\det({}^*h_{\lambda\mu}) \neq 0$, C is non-degenerate. Consequently ${}^*h_{\lambda\mu}A^\lambda = A_{1\mu}$ and A'_2 are not incident; that is, ${}^*h_{\lambda\mu}A'_1 A'_2 \neq 0$. \square

B. Nonholonomic frame of reference in the first class

In the first class, we have a set of 5 linearly independent basic vectors $A^{\nu}_i, (i = 1, \dots, 5)$ and a unique reciprocal set $A_{\lambda}^i, (i = 1, \dots, 5)$ satisfying

$$(3.11) \quad A_{\lambda}^j A^{\lambda}_i = \delta^j_i, \quad A_{\lambda}^i A^{\nu}_i = \delta^{\nu}_{\lambda}$$

With these two set of vectors, we may construct a nonholonomic frame of reference as follows;

DEFINITION 3.5. If $T_{\lambda \dots}^{\nu \dots}$ are holonomic components of a tensor, then its nonholonomic components are defined by

$$(3.12) \quad T_{j \dots}^{i \dots} = T_{\lambda \dots}^{\nu \dots} A_{\nu}^i \dots A^{\lambda}_j \dots$$

An easy inspection shows that

$$(3.13) \quad T_{\lambda \dots}^{\nu \dots} = T_{j \dots}^{i \dots} A^{\nu}_i \dots A_{\lambda}^j \dots$$

THEOREM 3.6. The nonholonomic components $*h_{ij}$ and $*h^{ij}$ are given by the matrix equation

$$(3.14) \quad (*h_{ij}) = (*h^{ij}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Proof. (3.14) is a direct result of (3.5) and theorem (3.4) □

THEOREM 3.7. We have

$$(3.15) \quad A^{\nu}_i = A_{\lambda}^j *h_{ij} *h^{\lambda\nu}, \quad A_{\lambda}^j = A^{\nu}_i *h^{ij} *h_{\lambda\nu}$$

so that

$$A^{\nu}_1 = A^{2\nu}, \quad A^{\nu}_2 = A^{1\nu}, \quad A^{\nu}_3 = A^{4\nu}, \quad A^{\nu}_4 = A^{3\nu}, \quad A^{\nu}_5 = A^{5\nu}$$

$$(3.16) \quad \overset{1}{A}_\lambda = \overset{2}{A}_{2\lambda}, \quad \overset{2}{A}_\lambda = \overset{1}{A}_{1\lambda}, \quad \overset{3}{A}_\lambda = \overset{4}{A}_{4\lambda}, \quad \overset{4}{A}_\lambda = \overset{3}{A}_{3\lambda}, \quad \overset{5}{A}_\lambda = \overset{5}{A}_{5\lambda}$$

Proof. In virtue of (2.6), (3.11) and (3.12), the first relation of (3.15) follows as in the following way;

$$(3.17) \quad \overset{j}{A}_\lambda {}^*h_{ij} {}^*h^{\lambda\nu} = \overset{j}{A}_\lambda ({}^*h_{\alpha\beta} \overset{\alpha}{A}_i \overset{\beta}{A}_j) {}^*h^{\lambda\nu} = \overset{\alpha}{A}_i ({}^*h_{\alpha\beta} {}^*h^{\lambda\nu}) \delta_\lambda^\beta = \overset{\nu}{A}_i.$$

(3.16) follows from (3.15) and (3.14). \square

THEOREM 3.8. *The nonholonomic components of ${}^{(p)*}k_\lambda{}^\nu$, ${}^{(p)*}k_{\lambda\mu}$ and ${}^{(p)*}k^{\lambda\nu}$ are given by*

$$(3.18) \quad {}^{(p)*}k_x{}^i = M^p \delta_x^i$$

$$(3.19) \quad {}^{(p)*}k_{xi} = M^p {}^*h_{xi}$$

$$(3.20) \quad {}^{(p)*}k^{xi} = M^p {}^*h^{xi}$$

Proof. Let $\overset{x}{A}^\nu$ be the basic vector corresponding to the basic scalar M_x . Then from (3.4), we have

$$(3.21) \quad {}^{(p)*}k_\lambda{}^\nu \overset{x}{A}^\lambda = M_x^p \overset{x}{A}^\nu \quad (p = 0, 1, 2, \dots)$$

(3.18) follows immediately by multiplying $\overset{i}{A}_\nu$ to both sides of (3.21). The remaining relations may be obtained from (3.18) by lowering and/or raising indices. \square

In the following theorem, we express the components of tensors ${}^*h_{\lambda\mu}$, ${}^{(p)*}k_\lambda{}^\nu$, ${}^{(p)*}k_{\lambda\mu}$, ${}^{(p)*}k^{\lambda\nu}$ in terms of basic vectors:

THEOREM 3.9. *The representation of ${}^*h_{\lambda\mu}$, ${}^{(p)*}k_\lambda{}^\nu$, ${}^{(p)*}k_{\lambda\mu}$, ${}^{(p)*}k^{\lambda\nu}$ in terms of basic vectors are given by*

$$(3.22) \quad {}^*h_{\lambda\mu} = 2\overset{1}{A}_{(\lambda} \overset{2}{A}_{\mu)} + \overset{3}{A}_{(\lambda} \overset{4}{A}_{\mu)} + \overset{5}{A}_\lambda \overset{5}{A}_\mu$$

$$(3.23) \quad {}^{(p)*}k_\lambda{}^\nu = M_1^p (\overset{1}{A}_\lambda \overset{1}{A}^\nu + (-1)^p \overset{2}{A}_\lambda \overset{2}{A}^\nu) + M_3^p (\overset{3}{A}_\lambda \overset{3}{A}^\nu + (-1)^p \overset{4}{A}_\lambda \overset{4}{A}^\nu)$$

$$(3.24) \quad {}^{(p)*}k_{\lambda\mu} = \begin{cases} 2M_1^p A_{(\lambda}^1 A_{\mu)}^2 + 2M_3^p A_{(\lambda}^3 A_{\mu)}^4, & \text{if } p \text{ is even} \\ 2M_1^p A_{[\lambda}^1 A_{\mu]}^2 + 2M_3^p A_{[\lambda}^3 A_{\mu]}^4, & \text{if } p \text{ is odd} \end{cases}$$

$$(3.25) \quad {}^{(p)*}k^{\lambda\nu} = \begin{cases} 2M_1^p A^{(\lambda} A^{\nu)} + 2M_3^p A^{(\lambda} A^{\nu)}, & \text{if } p \text{ is even} \\ 2M_1^p A^{[\lambda} A^{\nu]} + 2M_3^p A^{[\lambda} A^{\nu]}, & \text{if } p \text{ is odd} \end{cases}$$

Proof. The representations (3.22) – (3.25) follow from (3.13) in virtue of (3.5), (3.14) and (3.16). \square

C. Recurrence relations in the first class

In this subsection we derive several recurrence relations.

THEOREM 3.10. *In the first class, the tensor $T_{\omega\mu\nu}$, skew-symmetric in the first two indices, satisfies*

$$(3.26) \quad T_{\omega\mu\nu}^{(pq)r} = \sum_{x,y,z} T_{xyz} M_x^{(p} M_y^{q)} M_z^r A_\omega^x A_\mu^y A_\nu^z$$

$$(3.27) \quad T_{\nu[\omega\mu]}^{r(pq)} = \sum_{x,y,z} T_{x[yz]} M_x^{(p} M_y^{q)} M_z^r A_\nu^x A_\omega^y A_\mu^z$$

Proof. In virtue of (3.13) and (3.18), our assertion (3.26) may be derived as

$$(3.28) \quad \begin{aligned} T_{\omega\mu\nu}^{(pq)r} &= \sum_{x,y,z} T_{xyz}^{(pq)r} A_\omega^x A_\mu^y A_\nu^z \\ &= \sum_{x,y,z} \frac{1}{2} ({}^{(p)*}k_x^i {}^{i(q)*}k_y^j + {}^{(q)*}k_x^i {}^{i(p)*}k_y^j) {}^{(r)*}k_z^k A_\omega^x A_\mu^y A_\nu^z \\ &= \frac{1}{2} \sum_{x,y,z} T_{xyz} (M_x^p M_y^q + M_x^q M_y^p) M_z^r A_\omega^x A_\mu^y A_\nu^z \end{aligned}$$

The second relation may be proved similarly. \square

THEOREM 3.11. *The main recurrence relation in the first class is*

$$(3.29) \quad {}^{(p+5)*}k_\lambda^\nu = -K_2 {}^{(p+3)*}k_\lambda^\nu - K_4 {}^{(p+1)*}k_\lambda^\nu, \quad (p = 0, 1, 2, \dots)$$

Proof. Let M_x be a basic scalar. In $5 - *g - ESX_5$, the characteristic equation is

$$(3.30) \quad \sum_{f=0}^4 K_f M_x^{5-f} = 0$$

Multiplying δ_x^i to both sides of (3.30) and making use of (3.18), we have

$$(3.31) \quad \sum_{f=0}^4 K_f {}^{(5-f)*}k_x^i = 0$$

whose holonomic form is

$$(3.32) \quad \sum_{f=0}^4 K_f {}^{(5-f)*}k_\lambda^\alpha = 0$$

The relation (3.29) immediately follows by multiplying ${}^{(p)*}k_\alpha^\nu$ to both sides of (3.32). \square

The following theorem is simple consequences of (5.2) and (5.5).

THEOREM 3.12. *The basic scalars M_x satisfy*

$$(3.33) \quad M_1 + M_2 = M_3 + M_4 = 0$$

$$(3.34) \quad M_1 M_5 = M_2 M_5 = M_3 M_5 = M_4 M_5 = 0$$

$$(3.35) \quad M_1^2 M_3^2 = M_1^2 M_4^2 = M_2^2 M_3^2 = M_2^2 M_4^2 = K_4$$

$$(3.36) \quad M_1^2 + M_3^2 = M_1^2 + M_4^2 = M_2^2 + M_3^2 = M_2^2 + M_4^2 = -K_2$$

Proof. These relations follow easily from (3.5). \square

In virtue of the above theorem, we have

THEOREM 3.13. *In the first class, the following identities hold for all values of x and y when $x \neq y$*

$$(3.37) \quad M_{x \ y}^{(4} M^1) = -M_{x \ y}^{(3} M^2) - K_2 M_{x \ y}^{(2} M^1)$$

$$(3.38) \quad M_{x \ y}^{(4} M^3) = K_4 M_{x \ y}^{(2} M^1)$$

$$(3.39) \quad M_{x \ y}^4 M^4 = K_4^2 M_{x \ y}^2 M^2 + K_2 M_{x \ y}^3 M^3 + 2K_4 M_{x \ y}^{(3} M^1)$$

$$(3.40) \quad 2M_{x \ y}^{(4} M^2) = -M_{x \ y}^3 M^3 - K_2 M_{x \ y}^2 M^2 + K_4 M_{x \ y} M M$$

THEOREM 3.14. *(Recurrence relations in the first class) If $T_{\omega\mu\nu}$ is a tensor skew-symmetric in the first two indices, then the following recurrence relations hold in the first class of $5 - *g - ESX_5$:*

$$(3.41) \quad \begin{matrix} (41)r \\ T \end{matrix} = - \begin{matrix} (32)r \\ T \end{matrix} - K_2 \begin{matrix} (21)r \\ T \end{matrix}$$

$$(3.42) \quad \begin{matrix} (43)r \\ T \end{matrix} = K_4 \begin{matrix} (21)r \\ T \end{matrix}$$

$$(3.43) \quad \begin{matrix} 44r \\ T \end{matrix} = K_4 \begin{matrix} 22r \\ T \end{matrix} + K_2 \begin{matrix} 33r \\ T \end{matrix} + 2K_4 \begin{matrix} (31)r \\ T \end{matrix}$$

$$(3.44) \quad 2 \begin{matrix} (42)r \\ T \end{matrix} = - \begin{matrix} 33r \\ T \end{matrix} - K_2 \begin{matrix} 22r \\ T \end{matrix} + K_4 \begin{matrix} 11r \\ T \end{matrix}$$

Proof. The above relations are consequences of (3.26), (3.27) and (3.37)–(3.40). For example, the relation (3.44) is proved as in the following way:

$$\begin{aligned} 2 \begin{matrix} (42)r \\ T \end{matrix} &= 2 \begin{matrix} (42)r \\ T \end{matrix} \omega_{\mu\nu} = 2 \sum_{x,y,z} T_{xyz} M_x^{(4} M_y^2) M_z^r \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\ &= \sum_{x,y,z} T_{xyz} (-M_x^3 M_y^3 - K_2 M_x^2 M_y^2 + K_4 M_x M_y M_z) M_z^r \overset{x}{A}_\omega \overset{y}{A}_\mu \overset{z}{A}_\nu \\ (3.45) \quad &= - \begin{matrix} 33r \\ T \end{matrix} \omega_{\mu\nu} - K_2 \begin{matrix} 22r \\ T \end{matrix} + K_4 \begin{matrix} 11r \\ T \end{matrix} \end{aligned}$$

□

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