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SOLUTIONS FOR A CLASS OF FRACTIONAL BOUNDARY VALUE PROBLEM WITH MIXED NONLINEARITIES

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Abstract. In this paper we investigate the existence of nontrivial solutions for the following fractional boundary value problem

$$\text{(FBVP)} \qquad \begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla W(t,u(t)), & t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

where $\alpha \in (1/2, 1)$, $u \in \mathbb{R}^n$, $W \in C^1([0,T] \times \mathbb{R}^n, \mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u. The novelty of this paper is that, when the nonlinearity $W(t,u)$ involves a combination of superquadratic and subquadratic terms, under some suitable assumptions we show that (FBVP) possesses at least two nontrivial solutions. Recent results in the literature are generalized and significantly improved.

1. Introduction

Fractional order models can be found to be more adequate than integer order models in some real world problems as fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. As a consequence, the subject of fractional differential equations is gaining more importance and attention. There has been significant development in ordinary and partial differential equations involving both Riemann-Liouville and Caputo fractional derivatives. For details and examples, one can see the monographs [2, 6, 11, 13, 14, 16] and the papers [1, 9, 19].

Recently, equations including both left and right fractional derivatives are discussed. Apart from their possible applications, equations with left and right derivatives is an interesting and new field in fractional differential equations

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theory. In this topic, many results are obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by using techniques of nonlinear analysis, such as fixed point theory (including Leray-Schauder nonlinear alternative), topological degree theory (including coincidence degree theory) and comparison method (including upper and lower solutions and monotone iterative method), see [4, 8, 20] and so on.

It should be noted that critical point theory and variational methods have also turned out to be very effective tools in determining the existence of solutions for integer order differential equations. The idea behind them is trying to find solutions of a given boundary value problem by looking for critical points of a suitable energy functional defined on an appropriate function space. In the last 30 years, the critical point theory has become a wonderful tool in studying the existence of solutions to differential equations with variational structures, we refer the reader to the books due to Mawhin and Willem [12], Rabinowitz [15], Schechter [17] and the references listed therein.

Motivated by the above classical works, in recent paper [10], for the first time, the authors showed that critical point theory is an effective approach to deal with the existence of solutions for the following fractional boundary value problem

$$\text{(FBVP)} \qquad \begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = \nabla W(t,u(t)), \quad t \in [0,T], \\ u(0) = u(T), \end{cases}$$

where $\alpha \in (1/2, 1)$, $u \in \mathbb{R}^n$, $W \in C^1([0,T] \times \mathbb{R}^n, \mathbb{R})$ and $\nabla W(t,u)$ is the gradient of $W(t,u)$ at u. Explicitly, under the assumption that

(H_1) $|W(t,u)| \le \bar{a}|u|^2 + \bar{b}(t)|u|^{2-\tau} + \bar{c}(t)$ for all $t \in [0,T]$ and $u \in \mathbb{R}^n$,

where $\bar{a} \in [0, \Gamma^2(\alpha+1)/2T^{2\alpha})$, $\tau \in (0,2)$, $\bar{b} \in L^{2/\tau}([0,T],\mathbb{R})$ and $\bar{c} \in L^1([0,T],\mathbb{R})$, combining with some other reasonable hypotheses on $W(t,u)$, the authors showed that (FBVP) has at least one nontrivial solution. In addition, assuming that the potential $W(t,u)$ satisfies the following superquadratic condition:

(H_2) there exist $\mu > 2$ and $R > 0$ such that

$$0 < \mu W(t,u) \le (\nabla W(t,u), u)$$

for all $t \in [0,T]$ and $u \in \mathbb{R}^n$ with $|u| \ge R$,

and some other assumptions on $W(t,u)$, they also obtained the existence of at least one nontrivial solution for (FBVP). Inspired by this work, in [18] the author considered the following fractional boundary value problem

$$\text{(1.1)} \qquad \begin{cases} {}_tD_T^{\alpha}({}_0D_t^{\alpha}u(t)) = f(t,u(t)), \quad t \in [0,T], \\ u(0) = u(T) = 0, \end{cases}$$

with $\alpha \in (1/2, 1)$, $u \in \mathbb{R}$, $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ satisfying the following hypotheses:

(f_1) $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$;

(f_2) there is a constant $\mu > 2$ such that

$$0 < \mu F(t,u) \leq uf(t,u) \quad \text{for all } t \in [0,T] \text{ and } u \in \mathbb{R}\backslash\{0\},$$

the author showed that (1.1) possesses at least one nontrivial solution via Mountain Pass Theorem.

Note that all the papers mentioned above showed that (FBVP) has at least one nontrivial solution. As far as the multiplicity of solutions for (FBVP) is concerned, to the best of our knowledge, only the authors in recent paper [21], using the genus properties of critical point theory, established some new criterion to guarantee the existence of infinitely many solutions of (FBVP) for the case that $W(t,u)$ is subquadratic as $|u| \to +\infty$. Explicitly, the potential $W(t,u)$ is supposed to satisfy the following conditions:

(W_1) $W(t,0) = 0$ for all $t \in [0,T]$, $W(t,u) \geq a(t)|u|^{\vartheta}$ and $|\nabla W(t,u)| \leq b(t)|u|^{\vartheta-1}$ for all $(t,u) \in [0,T] \times \mathbb{R}^n$, where $1 < \vartheta < 2$ is a constant, $a : [0,T] \to \mathbb{R}^+$ is a continuous function and $b : [0,T] \to \mathbb{R}^+$ is a continuous function;

(W_2) there is a constant $1 < \sigma \leq \vartheta < 2$ such that

$$(\nabla W(t,u), u) \leq \sigma W(t,u) \quad \text{for all } t \in [0,T] \text{ and } u \in \mathbb{R}^n.$$

Suppose that (W_1) and (W_2) are satisfied. Moreover, assuming that $W(t,u)$ is even in u, i.e.,

(W_3) $W(t,u) = W(t,-u) \quad$ for all $\;t \in [0,T]\;$ and $\;u \in \mathbb{R}^n$,

then the authors showed that (FBVP) has infinitely many nontrivial solutions.

As is well known, (H_2) is the so-called Ambrosetti-Rabinowitz condition due to Ambrosetti and Rabinowitz (see e.g., [3]), which implies that $W(t,u)$ is superquadratic as $|u| \to +\infty$. On the other hand, from (W_1), it is easy to check that $W(t,u)$ is subquadratic as $|u| \to +\infty$. In fact, in view of (W_1), we have

$$W(t,u) = \int_0^1 (\nabla W(t,su), u)ds \leq \frac{b(t)}{\vartheta}|u|^{\vartheta}, \tag{1.2}$$

which implies that $W(t,u)$ is of subquadratic growth as $|u| \to +\infty$. Therefore, it is natural to find the existence of solutions for (FBVP) when the potential $W(t,u)$ is of the form:

$$W(t,u) = W_1(t,u) + W_2(t,u),$$

that is, $W(t,u)$ is a mixed nonlinearity, where $W_1(t,u)$ is superquadratic as $|u| \to +\infty$ and $W_2(t,u)$ is of subquadratic growth at infinity. To the best our knowledge, there is no literature to consider the mixed nonlinearity associated with (FBVP). Motivated by [18] and [21], in the present paper, we focus our attention on this problem and give some reasonable assumptions on $W_1(t,u)$ and $W_2(t,u)$ to guarantee the existence of at least two nontrivial solutions for (FBVP). For the statement of our main result, the potential $W(t,u)$ is assumed to satisfy the following hypothesis:

(F)$_1$ $W_1 \in C^1([0,T]\times\mathbb{R}^n,\mathbb{R})$ and there exists some constant $\theta > 2$ such that

$$0 < \theta W_1(t,u) \le (\nabla W_1(t,u),u) \quad \text{for all } t\in[0,T] \text{ and } u\in\mathbb{R}^n\backslash\{0\};$$

(F)$_2$ there exists a positive continuous function $a:[0,T]\to\mathbb{R}^+$ such that

$$|\nabla W_1(t,u)| \le a(t)|u|^{\theta-1} \quad \text{for all } (t,u)\in[0,T]\times\mathbb{R}^n;$$

(F)$_3$ $W_2(t,0)=0$ for all $t\in[0,T]$, $W_2\in C^1([0,T]\times\mathbb{R}^n,\mathbb{R})$ and there exist a constant $1<\varrho<2$ and a positive continuous function $b:[0,T]\to\mathbb{R}^+$ such that

$$W_2(t,u)\ge b(t)|u|^{\varrho}$$

for all $(t,u)\in[0,T]\times\mathbb{R}^n$;

(F)$_4$ for all $t\in[0,T]$ and $u\in\mathbb{R}^n$,

$$|\nabla W_2(t,u)|\le c(t)|u|^{\varrho-1},$$

where $c:[0,T]\to\mathbb{R}^+$ is a positive continuous function.

To guarantee the existence of at least two nontrivial solutions of (FBVP), we also need the following estimation on a and c:

(F)$_5$ $\left(\frac{2\bar{c}C_\varrho^\varrho}{\varrho}\frac{\theta-\varrho}{\theta-2}\right)^{\theta-2} < \left(\frac{\theta}{2\bar{a}C_\theta^\theta}\frac{2-\varrho}{\theta-\varrho}\right)^{2-\varrho}$, where

$$\bar{a}=\max_{t\in[0,T]}a(t), \quad \bar{c}=\max_{t\in[0,T]}c(t),$$

$\theta>2$ and $1<\varrho<2$ are defined in (F)$_1$ and (F)$_3$, respectively, C_ϱ and C_θ are defined in (2.4) below.

Now we are in the position to state our main result.

Theorem 1.1. *If* (F)$_1$-(F)$_5$ *are satisfied, then* (FBVP) *possesses at least two nontrivial solutions.*

Remark 1.2. In view of (F)$_1$, we deduce that (see [7, Fact 2.1])

$$W_1(t,u)\le W_1\big(t,\frac{u}{|u|}\big)|u|^{\theta} \quad \text{for } t\in[0,T] \text{ and } 0<|u|\le 1 \tag{1.3}$$

and

$$W_1(t,u)\ge W_1\big(t,\frac{u}{|u|}\big)|u|^{\theta} \quad \text{for } t\in[0,T] \text{ and } |u|\ge 1. \tag{1.4}$$

Moreover, according to (F)$_3$ and (F)$_4$, it is obvious that

$$W_2(t,u)\le\frac{c(t)}{\varrho}|u|^{\varrho} \quad \text{for all } t\in[0,T] \text{ and } u\in\mathbb{R}^n. \tag{1.5}$$

In addition, from (F)$_1$-(F)$_4$, it is easy to obtain that

(1.6)
$$\begin{aligned} W(t,u) &= \int_0^1(\nabla W(t,su),u)ds \le \frac{a(t)}{\theta}|u|^{\theta}+\frac{c(t)}{\varrho}|u|^{\varrho} \\ &\le \frac{\bar{a}}{\theta}|u|^{\theta}+\frac{\bar{c}}{\varrho}|u|^{\varrho} =: d_1|u|^{\theta}+d_2|u|^{\varrho} \quad \text{for all } t\in[0,T] \text{ and } u\in\mathbb{R}^n. \end{aligned}$$

Although, the assumption $(\mathrm{F})_5$ looks some cumbersome, it plays an essential role in checking the Mountain Pass Theorem, see Step 2 in Section 3. In addition, we must point pout that the assumption $(\mathrm{F})_5$ is only used in Step 2 in the proof of Theorem 1.1 and is easy to be verified. In what follows, for the reader's convenience, we present one example to illustrate our main result. Let

$$W(t,u)=\frac{a(t)}{3}|u|^3+\frac{2c(t)}{3}|u|^{\frac{3}{2}},$$

where $a, c:[0,T]\to\mathbb{R}^+$ are positive continuous functions, then it is easy to check that $W(t,u)$ satisfies $(\mathrm{F})_1$-$(\mathrm{F})_4$. Meanwhile, the additional assumption $2\bar{c}C_\varrho^\varrho\sqrt{3\bar{a}C_\theta^\theta}<1$ is sufficient to guarantee that $(\mathrm{F})_5$ holds with $\theta=3$ and $\varrho=\frac{3}{2}$.

Here we must point out that, in our Theorem 1.1, for the first time we obtain that (FBVP) has at least two nontrivial solutions for the case that $W(t,u)$ is a mixed nonlinearity. Therefore, the previous results [10, 18, 21] are generalized and improved significantly. However, we do not know whether (FBVP) also possesses infinitely solutions if the potential $W(t,u)$ is even with respect to u as usual.

The remaining part of this paper is organized as follows. Some preliminary results are presented in Section 2. In Section 3, we are devoted to accomplishing the proof of Theorem 1.1.

2. Preliminary results

2.1. Fractional calculus

In this subsection, for the reader's convenience, we introduce some basic definitions of fractional calculus which are used further in this paper, see [11].

Definition 2.1 (Left and Right Riemann-Liouville fractional integrals)**.** Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional integrals of order $\alpha>0$ for function u are defined by

$${}_aI_t^\alpha u(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(t-s)^{\alpha-1}u(s)ds,\quad t\in[a,b]$$

and

$${}_tI_b^\alpha u(t)=\frac{1}{\Gamma(\alpha)}\int_t^b(s-t)^{\alpha-1}u(s)ds,\quad t\in[a,b].$$

Definition 2.2 (Left and Right Riemann-Liouville fractional derivatives)**.** Let u be a function defined on $[a,b]$. The left and right Riemann-Liouville fractional derivatives of order $\alpha>0$ for function u denoted by ${}_aD_t^\alpha u(t)$ and ${}_tD_b^\alpha u(t)$, respectively, are defined by

$${}_aD_t^\alpha u(t)=\frac{d^n}{dt^n}{}_aI_t^{n-\alpha}u(t)$$

and

$$ {}_tD_b^\alpha u(t) = (-1)^n \frac{d^n}{dt^n}\, {}_tI_b^{n-\alpha} u(t), $$

where $t \in [a,b]$, $n-1 \le \alpha < n$ and $n \in \mathbb{N}$.

2.2. Fractional derivative spaces

In order to establish the variational structure which enables us to reduce the existence of solutions for (FBVP) to find critical points of the corresponding functional, it is necessary to construct appropriate function spaces.

Firstly, we recall some fractional spaces, for more details see [9, 10]. To this end, denote by $L^p([0,T],\mathbb{R}^n)$ $(1 < p < +\infty)$ the Banach spaces of functions on $[0,T]$ with values in $\mathbb{R}^n$ under the norms

$$ \|u\|_p = \Big(\int_0^T |u(t)|^p dt \Big)^{1/p}, $$

and $L^\infty([0,T],\mathbb{R}^n)$ is the Banach space of essentially bounded functions from $[0,T]$ into $\mathbb{R}^n$ equipped with the norm

$$ \|u\|_\infty = \operatorname{ess\,sup} \left\{ |u(t)| : t \in [0,T] \right\}. $$

For $0 < \alpha \le 1$ and $1 < p < +\infty$, the fractional derivative space $E_0^{\alpha,p}$ is defined by

$$ \begin{aligned} E_0^{\alpha,p} &= \{ u \in L^p([0,T],\mathbb{R}^n) : {}_0D_t^\alpha u \in L^p([0,T],\mathbb{R}^n) \text{ and } u(0) = u(T) = 0 \} \\ &= \overline{C_0^\infty([0,T],\mathbb{R}^n)}^{\|\cdot\|_{\alpha,p}}, \end{aligned} $$

where $\|\cdot\|_{\alpha,p}$ is defined as follows

$$ \|u\|_{\alpha,p} = \Big(\int_0^T |u(t)|^p dt + \int_0^T |{}_0D_t^\alpha u(t)|^p dt \Big)^{1/p}. \tag{2.1} $$

Then $E_0^{\alpha,p}$ is a reflexive and separable Banach space. Moreover, $E_0^{\alpha,p} \in C([0,T],\mathbb{R}^n)$, see [10].

Lemma 2.3. *Let $0 < \alpha \le 1$ and $1 < p < +\infty$. For all $u \in E_0^{\alpha,p}$, if $\alpha > \frac{1}{p}$, we have*

$$ {}_0I_t^\alpha ({}_0D_t^\alpha u(t)) = u(t) $$

and

$$ \|u\|_p \le \frac{T^\alpha}{\Gamma(\alpha+1)} \|{}_0D_t^\alpha u\|_p. \tag{2.2} $$

In addition, if $\alpha > \frac{1}{p}$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ \|u\|_\infty \le \frac{T^{\alpha - \frac{1}{p}}}{\Gamma(\alpha)((\alpha-1)q+1)^{\frac{1}{q}}} \|{}_0D_t^\alpha u\|_p. $$

Remark 2.4. According to (2.1) and (2.2), we can consider in $E_0^{\alpha,p}$ the following norm

$$\|u\|_{\alpha,p} = \|{}_0D_t^\alpha u\|_p, \tag{2.3}$$

which is equivalent to (2.1).

In what follows we denote by $E^\alpha = E_0^{\alpha,2}$. Then it is a Hilbert space with respect to the norm $\|u\|_{\alpha,2}$ given by (2.3). Moreover, from Lemma 2.3 and Remark 2.4, we have

Proposition 2.5. *Let $\alpha \in (1/2, 1)$. Then, for any $p \in (1, +\infty)$, there exists some constant $C_p > 0$ such that*

$$\|u\|_p \leq C_p \|u\|_{\alpha,2}, \quad \forall u \in E^\alpha. \tag{2.4}$$

To verify that the functional corresponding to (FBVP) satisfies the (PS) condition, we need the following proposition.

Proposition 2.6. *Let $0 < \alpha \leq 1$ and $1 < p < +\infty$. Assume that $\alpha > \frac{1}{p}$ and $u_k \rightharpoonup u$ in $E_0^{\alpha,p}$, then $u_k \to u$ in $C([0,T], \mathbb{R}^n)$, i.e.,*

$$\|u_k - u\|_\infty \to 0$$

as $k \to +\infty$.

Now we introduce more notations and some necessary definitions. Let $\mathcal{B}$ be a real Banach space, $I \in C^1(\mathcal{B}, \mathbb{R})$ means that I is a continuously Fréchet-differentiable functional defined on $\mathcal{B}$.

Definition 2.7. $I \in C^1(\mathcal{B}, \mathbb{R})$ is said to satisfy the (PS) condition if any sequence $\{u_k\}_{k\in\mathbb{N}} \subset \mathcal{B}$, for which $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$, possesses a convergent subsequence in $\mathcal{B}$.

Moreover, let B_r be the open ball in $\mathcal{B}$ with the radius r and centered at 0 and ∂B_r denotes its boundary. Under the conditions of Theorem 1.1, we obtain the existence of the first solution of (FBVP) by using of the following well-known Mountain Pass Theorem, see [15].

Lemma 2.8 ([15, Theorem 2.2])**.** *Let $\mathcal{B}$ be a real Banach space and $I \in C^1(\mathcal{B}, \mathbb{R})$ satisfying the* (PS) *condition. Suppose that $I(0) = 0$ and*

(A1) *there are constants $\rho, \eta > 0$ such that $I|_{\partial B_\rho} \geq \eta$, and*

(A2) *there is an $e \in \mathcal{B} \setminus \overline{B}_\rho$ such that $I(e) \leq 0$.*

Then I possesses a critical value $c \geq \eta$. Moreover c can be characterized as

$$c = \inf_{g\in\Gamma} \max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], \mathcal{B}) : g(0) = 0, g(1) = e\}.$$

As far as the second one is concerned, we obtain it by minimizing method, which is contained in a small ball centered at 0, see Step 4 in the proof of Theorem 1.1.

3. Proof of Theorem 1.1

The aim of this section is to give the proof of Theorem 1.1. To do this, we are going to establish the corresponding variational framework of (FBVP). Define the functional $I : \mathcal{B} = E^\alpha \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int_0^T [\frac{1}{2}|{}_0D_t^\alpha u(t)|^2 - W(t, u(t))]dt \\ &= \frac{1}{2}\|u\|^2_{\alpha,2} - \int_0^T W(t, u(t))dt. \end{aligned} \tag{3.1}$$

Lemma 3.1 ([10, Corollary 3.1])**.** *Under the conditions of Theorem 1.1, I is a continuously Fréchet-differentiable functional defined on E^α, i.e., $I \in C^1(E^\alpha, \mathbb{R})$. Moreover, we have*

$$I'(u)v = \int_0^T [({}_0D_t^\alpha u(t), {}_0D_t^\alpha v(t)) - (\nabla W(t, u(t)), v(t))]dt$$

for all u, $v \in E^\alpha$, which yields that

$$\begin{aligned} I'(u)u &= \int_0^T |{}_0D_t^\alpha u(t)|^2 dt - \int_0^T (\nabla W(t, u(t)), u(t))dt \\ &= \|u\|^2_{\alpha,2} - \int_0^T (\nabla W(t, u(t)), u(t))dt. \end{aligned} \tag{3.2}$$

Furthermore, any critical point of I is a weak solution of (FBVP)*.*

In order to check that the corresponding functional I satisfies the condition (A1) of Lemma 2.8, the following lemma plays an essential role.

Lemma 3.2. *Let $1 < \varrho < 2 < \theta$, $A, B > 0$, and consider the function*

$$\Phi_{A,B}(t) := t^2 - At^\varrho - Bt^\theta, \quad t \geq 0.$$

Then $\max_{t\geq 0} \Phi_{A,B}(t) > 0$ if and only if

$$A^{\theta-2}B^{2-\varrho} < d(\varrho, \theta) := \frac{(\theta-2)^{\theta-2}(2-\varrho)^{2-\varrho}}{(\theta-\varrho)^{\theta-\varrho}}.$$

Furthermore, for $t = t_B := [(2-\varrho)/B(\theta-\varrho)]^{1/(\theta-2)}$, one has

$$\max_{t\geq 0} \Phi_{A,B}(t) = \Phi_{A,B}(t_B) = t_B^2\Big[\frac{\theta-2}{\theta-\varrho} - AB^{\frac{2-\varrho}{\theta-2}}\Big(\frac{\theta-\varrho}{2-\varrho}\Big)^{\frac{2-\varrho}{\theta-2}}\Big] > 0. \tag{3.3}$$

Proof. The proof is essentially the same as that in [5, Lemma 3.2], so we omit its details. □

In what follows, we verify that I satisfies the (PS) condition.

Lemma 3.3. *If* $(F)_1$-$(F)_4$ *hold, then I satisfies the* (PS) *condition.*

Proof. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E^\alpha$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $M > 0$ such that

$$|I(u_k)| \le M \quad \text{and} \quad \|I'(u_k)\|_{(E^\alpha)^*} \le M \tag{3.4}$$

for every $k \in \mathbb{N}$, where $(E^\alpha)^*$ is the dual space of E^α.

We firstly prove that $\{u_k\}_{k\in\mathbb{N}}$ is bounded in E^α. From (3.1), (3.2), (1.5), $(F)_1$, $(F)_3$, $(F)_4$ and Proposition 2.5, we obtain that

$$\begin{aligned} M + \frac{M}{\theta}\|u_k\| &\ge I(u_k) - \frac{1}{\theta}I'(u_k)u_k \\ &= \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_k\|^2_{\alpha,2} - \int_0^T [W(t,u_k(t)) - \frac{1}{\theta}(\nabla W(t,u_k(t)),u_k(t))]dt \\ &\ge \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_k\|^2_{\alpha,2} - \Big(d_2 + \frac{\bar{c}}{\theta}\Big)\|u_k\|^\varrho_\varrho \\ &\ge \Big(\frac{1}{2} - \frac{1}{\theta}\Big)\|u_k\|^2_{\alpha,2} - C^\varrho_\varrho\Big(d_2 + \frac{\bar{c}}{\theta}\Big)\|u_k\|^\varrho_{\alpha,2}. \end{aligned} \tag{3.5}$$

Since $1 < \varrho < 2$, the boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in E^α follows directly. Then the sequence $\{u_k\}_{k\in\mathbb{N}}$ has a subsequence, again denoted by $\{u_k\}_{k\in\mathbb{N}}$, and there exists $u \in E^\alpha$ such that

$$u_k \rightharpoonup u \text{ weakly in } E^\alpha,$$

which yields that

$$(I'(u_k) - I'(u))(u_k - u) \to 0. \tag{3.6}$$

Moreover, due to the fact that $W \in C^1([0,T]\times\mathbb{R}^n,\mathbb{R})$, according to Proposition 2.6, we have

$$\int_0^T (\nabla W(t,u_k(t)) - \nabla W(t,u(t)), u_k(t) - u(t))dt \to 0 \tag{3.7}$$

as $k \to +\infty$. Consequently, combining (3.6), (3.7) with the following equality

$$\begin{aligned} &(I'(u_k) - I'(u))(u_k - u) \\ &= \|u_k - u\|^2_{\alpha,2} - \int_0^T (\nabla W(t,u_k(t)) - \nabla W(t,u(t)), u_k(t) - u(t))dt, \end{aligned}$$

we deduce that $\|u_k - u\|_{\alpha,2} \to 0$ as $k \to +\infty$. That is, I satisfies the (PS) condition. □

Now we are in the position to complete the proof of Theorem 1.1. For illumining the ideas, we divide its proof into four steps.

Proof. **Step 1.** It is clear that $I(0) = 0$ and $I \in C^1(E^\alpha,\mathbb{R})$ satisfies the (PS) condition by Lemma 3.3.

Step 2. To show that there exist constants $\rho > 0$ and $\eta > 0$ such that I satisfies $I|_{\partial B_\rho} \geq \eta > 0$, that is, the condition (A1) of Lemma 2.8 holds. To this end, in view of (1.6), we have

$$\begin{aligned}\int_0^T W(t,u)dt &\leq d_1 \int_0^T |u|^\theta dt + d_2 \int_0^T |u|^\varrho dt \\ &= d_1\|u\|_\theta^\theta + d_2\|u\|_\varrho^\varrho,\end{aligned} \tag{3.8}$$

which, combining with (2.4), yields that

$$\begin{aligned}I(u) &= \frac{1}{2}\|u\|_{\alpha,2}^2 - \int_0^T W(t,u)dt \\ &\geq \frac{1}{2}\|u\|_{\alpha,2}^2 - d_1 C_\theta^\theta \|u\|_\theta^\theta - d_2 C_\varrho^\varrho \|u\|_\varrho^\varrho \quad \text{for all} \quad u \in E^\alpha.\end{aligned} \tag{3.9}$$

Applying Lemma 3.2 with

$$A = 2d_2 C_\varrho^\varrho \quad \text{and} \quad B = 2d_1 C_\theta^\theta,$$

we obtain that

$$I(u) \geq \frac{1}{2}\Phi_{A,B}(t_B) > 0,$$

provided that $A^{\theta-2}B^{2-\varrho} < d(\varrho,\theta)$, that is, provided that

$$\Big(\frac{2\bar{c}C_\varrho^\varrho}{\varrho}\frac{\theta-\varrho}{\theta-2}\Big)^{\theta-2} < \Big(\frac{\theta}{2\bar{a}C_\theta^\theta}\frac{2-\varrho}{\theta-\varrho}\Big)^{\varrho-2}.$$

Let $\rho = t_B = \Big[\frac{2-\varrho}{B(\theta-\varrho)}\Big]^{\frac{1}{\theta-2}}$ and $\eta = \frac{1}{2}\Phi_{A,B}(t_B)$, then we have $I|_{\partial B_\rho} \geq \eta > 0$.

Step 3. To obtain that there exists an $e \in E^\alpha$ such that $I(e) < 0$ with $\|e\|_{\alpha,2} > \rho$, where ρ is defined in Step 2. For this purpose, take $\psi \in E^\alpha$ such that $\psi(t) > 0$ on some closed subset $\Omega \subset (0,T)$. In view of (3.1), (1.4), $(F)_1$ and $(F)_3$, for $l \in (0,\infty)$ such that $|l\psi(t)| \geq 1$ for all $t \in \Omega$, we deduce that

$$\begin{aligned}I(l\psi) &= \frac{l^2}{2}\|\psi\|_{\alpha,2}^2 - \int_0^T W(t,l\psi(t))dt \\ &\leq \frac{l^2}{2}\|\psi\|_{\alpha,2}^2 - \int_\Omega W_1(t,l\psi(t))dt \\ &\leq \frac{l^2}{2}\|\psi\|_{\alpha,2}^2 - l^\theta \int_\Omega W_1\Big(t,\frac{\psi(t)}{|\psi(t)|}\Big)|\psi(t)|^\theta dt \\ &\leq \frac{l^2}{2}\|\psi\|_{\alpha,2}^2 - ml^\theta \int_\Omega |\psi(t)|^\theta dt,\end{aligned} \tag{3.10}$$

where $m = \min\{W_1(t,u) : t \in \Omega, |u| = 1\}$ (on account of $(F)_3$, it is obvious that $m > 0$). Since $\theta > 2$, (3.10) implies that $I(l\varphi) = I(e) < 0$ for some $l \gg 1$ with $\|l\varphi\|_{\alpha,2} > \rho$, where ρ is defined in Step 2. By Lemma 2.8, I possesses a critical value $c_1 \geq \eta > 0$ given by

$$c_1 = \inf_{g\in\Gamma}\max_{s\in[0,1]} I(g(s)),$$

where

$$\Gamma = \{g \in C([0,1], E^{\alpha}) : g(0) = 0,\, g(1) = e\}.$$

Hence there is $0 \neq u_1 \in E^{\alpha}$ such that

$$I(u_1) = c_1 \quad \text{and} \quad I'(u_1) = 0.$$

That is, the first nontrivial solution of (FBVP) exists.

Step 4 From (3.9), we see that I is bounded from below on $\overline{B_\rho(0)}$. Therefore, we can denote by

$$c_2 = \inf_{\|u\|_{\alpha,2} \leq \rho} I(u),$$

where ρ is defined in Step 1. Then there is a minimizing sequence $\{v_k\}_{k\in\mathbb{N}} \subset \overline{B_\rho(0)}$ such that

$$I(v_k) \to c_2 \quad \text{and} \quad I'(v_k) \to 0$$

as $k \to \infty$. That is, $\{v_k\}_{k\in\mathbb{N}}$ is a (PS) sequence. Furthermore, from Lemma 3.3, I satisfies the (PS) condition. Therefore, c_2 is one critical value of I. In what follows, we show that c_2 is one nontrivial critical point. For $0 \not\equiv \varphi \in E^{\alpha}$, according to $(\mathrm{F})_1$ and $(\mathrm{F})_3$, one deduces that,

$$\begin{aligned} I(l\varphi) &= \frac{l^2}{2}\|\varphi\|_{\alpha,2}^2 - \int_0^T W(t, l\varphi(t))dt \\ &\leq \frac{l^2}{2}\|\varphi\|_{\alpha,2}^2 - \int_0^T W_2(t, l\varphi(t))dt \\ &\leq \frac{l^2}{2}\|\varphi\|_{\alpha,2}^2 - l^{\varrho}\int_0^T b(t)|\varphi(t)|^{\varrho}dt, \quad \forall l \in (0, +\infty). \end{aligned} \tag{3.11}$$

Since $1 < \varrho < 2$, (3.11) implies that $I(l\varphi) < 0$ for l small enough such that $\|l\varphi\|_{\alpha,2} \leq \rho$. Therefore, $c_2 < 0 < c_1$. Consequently, there is $0 \neq u_2 \in E^{\alpha}$ such that

$$I(u_2) = c_2 \quad \text{and} \quad I'(u_2) = 0.$$

That is, (FBVP) has another nontrivial solution. □

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