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SOME COHOMOTOPY GROUPS OF SUSPENDED QUATERNIONIC PROJECTIVE PLANES

Jin Ho Lee and Kee Young Lee

Abstract. In this paper we present the computation of two kinds of cohomotopy groups $[\Sigma^{n+4}\mathbb{H}P^2, S^n]$ and $[\Sigma^{n+5}\mathbb{H}P^2, S^n]$ for a non-negative integer n, where $\Sigma^k\mathbb{H}P^2$ is the k-fold suspension of quaternionic projective plane $\mathbb{H}P^2$.

1. Introduction

Let X and Y be based topological spaces, and let $[X, Y]$ denote the set of homotopy classes of base point preserving continuous maps from X to Y. Given a space X and an n-dimensional sphere S^n, the set $[X, S^n]$ has been studied by many authors [1, 2, 3, 5, 6, 7]. This set is known as the n-th cohomotopy set of X, and in particular, the n-th cohomotopy group of X if it has a group structure, which is the case when X is a suspension of a space. The cohomotopy groups $[\Sigma^m X, S^n]$ for the m-fold suspension $\Sigma^m X$ of a projective space X have been studied and computed by many authors [2, 3, 7] using the exact sequence associated with the canonical cofiber sequence and a formula for a multiple of the identity class of the suspended projective plane. The cohomotopy groups $[\Sigma^{n+k}\mathbb{H}P^2, S^n]$ for the quaternionic projective space $\mathbb{H}P^2$, in particular, were computed by Kachi, Mukai, and colleagues on the condition that $|k| \leq 3$ [2].

The purpose of the present paper is to compute the cohomotopy groups $[\Sigma^{n+4}\mathbb{H}P^2, S^n]$ and $[\Sigma^{n+5}\mathbb{H}P^2, S^n]$ for each $n \geq 2$. The computation will be done as follows. As is well-known, the quaternionic projective plane $\mathbb{H}P^2$ is defined by the mapping cone $S^4 \cup_{\nu_4} e^8$, where $\nu_4 : S^7 \to S^4$ is the Hopf fibering. Consider a Puppe sequence

$$S^7 \xrightarrow{\nu_4} S^4 \xrightarrow{i} \mathbb{H}P^2 \xrightarrow{p} S^8 \xrightarrow{\nu_5} S^5 \xrightarrow{\Sigma i} \cdots,$$

where $i : S^4 \to \mathbb{H}P^2$ is the inclusion map, $p : \mathbb{H}P^2 \to S^8$ is the collapsing map of S^4 to a point $*$, and $\nu_k = \Sigma^{k-4}\nu_4$ for $k \geq 4$. This gives a long exact sequence

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of homotopy sets

$$\pi_{m+5}(S^n) \xrightarrow{\nu^*_{m+5}} \pi_{m+8}(S^n) \xrightarrow{\Sigma^m p^*} [\Sigma^m \mathbb{H}P^2, S^n]$$

$$\xrightarrow{\Sigma^n i^*} \pi_{m+4}(S^n) \xrightarrow{\nu^*_{m+4}} \pi_{m+7}(S^n)$$

and gives rise to the short exact sequence

$$0 \to \mathrm{Coker}\nu^*_{m+5} \xrightarrow{\Sigma^m p^*} [\Sigma^m \mathbb{H}P^2, S^n] \xrightarrow{\Sigma^m i^*} \mathrm{Ker}\nu^*_{m+4} \to 0.$$

This sequence is referred as *the* (m, n)*-type short exact sequence* throughout this paper.

For $n \geq 2$ and $s = 9$ or 10, we determine $\mathrm{Coker}\nu^*_{n+s}$ and $\mathrm{Ker}\nu^*_{n+s-1}$ using the formulas of Toda brackets [8], some results in [2], and the Frudenthal suspension theorem. We also investigate the splitting properties of the $(n+k, n)$-type short exact sequences for $k = 4$ or 5. As a result, we obtain the following theorem. We prove this theorem by way of several propositions in Sections 4 and 5.

Theorem 1. *For each* $n \geq 2$, $[\Sigma^{n+4}\mathbb{H}P^2, S^n]$ *and* $[\Sigma^{n+5}\mathbb{H}P^2, S^n]$ *are determined as follows*:

case n	2	3	4	5	6
$[\Sigma^{n+4}\mathbb{H}P^2, S^n]$	$4+2+3+35$	2	$(2)^4$	$(2)^3+63$	$16+2+(3)^2+5$
case n	7	8	9	10	11
$[\Sigma^{n+4}\mathbb{H}P^2, S^n]$	$(2)^2$	$(2)^3$	$(2)^2$	$(2)^2+3$	$(2)^3$
case n	12	13	$n \geq 14$		
$[\Sigma^{n+4}\mathbb{H}P^2, S^n]$	$(2)^4$	$(2)^3$	$(2)^2$		

case n	2	3	4	5	6
$[\Sigma^{n+5}\mathbb{H}P^2, S^n]$	*2*	$(2)^2+3$	$(2)^3+(3)^2$	$(2)^3+3$	$(2)^3+3$
case n	7	8	9	10	11
$[\Sigma^{n+5}\mathbb{H}P^2, S^n]$	$(2)^4+3$	$(2)^5+3$	$(2)^4+3$	$\mathbb{Z}+(2)^3+3$	$(2)^4+3$
case n	12	13	14	$n \geq 15$	
$[\Sigma^{n+5}\mathbb{H}P^2, S^n]$	$(4)^2+2+3$	$4+(2)^2+3$	$\mathbb{Z}+(2)^4+3$	$(2)^3+3$	

where the integer " s" denotes the cyclic group $\mathbb{Z}_s$, *"+" denotes the direct sum of abelian groups, and "* $(s)^k$ *" is the k-times direct sum of* $\mathbb{Z}_s$.

Throughout this paper, we follow Toda's notation [8] for elements of homotopy groups of spheres. If G is a finitely generated abelian group generated by $a_1, \ldots, a_n$, then we denote the group G by $G\{a_1, \ldots, a_n\}$. We also denote the t-times direct sum of $\mathbb{Z}_s$ by $\mathbb{Z}_s^t$.

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2. Preliminaries

In this section, we present selected basic principles of composition methods [8].

When G is an abelian group and $p \geq 2$ is a prime number, we denote the p-primary parts of G by $G_{(p)}$.

For $p \geq 5$, we have an isomorphism

$$[\Sigma^n \mathbb{H}P^2, S^k]_{(p)} \cong \pi_{n+4}(S^k)_{(p)} \oplus \pi_{n+8}(S^k)_{(p)},$$

because $\pi_{n+3}(S^n)$ has order 24 for $n \geq 5$ [8, Proposition 5.6].

Moreover, there is an isomorphism [8, (13.1)]

$$\pi_{i-1}(S^{2m-1})_{(p)} \oplus \pi_i(S^{4m-1})_{(p)} \cong \pi_i(S^{2m})_{(p)}$$

given by the correspondence $(\alpha, \beta) \mapsto \Sigma\alpha + [\iota_{2m}, \iota_{2m}] \circ \beta$, where $[\ ,\ ]$ is the Whitehead product. This is known as Serre's isomorphism.

It is well known that the Hopf fibrations $\eta_2 : S^3 \to S^2$, $\nu_4 : S^7 \to S^4$, and $\sigma_8 : S^{15} \to S^8$ induce the isomorphisms

$$[X, S^3] \to [X, S^2], \qquad \alpha \mapsto \eta_2 \circ \alpha, \tag{2.2}$$

$$[X, S^3] \oplus [\Sigma X, S^7] \to [\Sigma X, S^4], \quad (\alpha, \beta) \mapsto \Sigma\alpha + \nu_4 \circ \beta, \tag{2.3}$$

$$[X, S^7] \oplus [\Sigma X, S^{15}] \to [\Sigma X, S^8], \quad (\alpha, \beta) \mapsto \Sigma\alpha + \sigma_8 \circ \beta \tag{2.4}$$

respectively.

Consider elements $\alpha \in [Y, Z]$, $\beta \in [X, Y]$, and $\gamma \in [W, X]$ satisfying $\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$. Let C_β be the mapping cone of β, and $i : Y \to C_\beta$, $p : C_\gamma \to \Sigma X$ be the inclusion and the shrinking map, respectively. We denote an extension of α satisfying $i^*(\overline{\alpha}) = \alpha$ by $\overline{\alpha} \in [C_\beta, Z]$, and a coextension of γ satisfying $p_*(\widetilde{\gamma}) = \Sigma\gamma$ by $\widetilde{\gamma} \in [\Sigma W, C_\beta]$ [8].

We recall some relations between (co)extensions and Toda brackets [8].

Theorem 2. *Let $\alpha \in [Y, Z]$, $\beta \in [X, Y]$, and $\gamma \in [W, X]$ be elements such that $\alpha \circ \beta = 0$ and $\beta \circ \gamma = 0$. Let $\{\alpha, \beta, \gamma\}$ be the Toda bracket, and $i : Y \to C_\beta$, $p : C_\gamma \to \Sigma W$ be the inclusion and the shrinking map, respectively. Then, we have $\overline{\alpha} \circ \widetilde{\gamma} \in \{\alpha, \beta, \gamma\}$ and $\alpha \circ \overline{\beta} \in \{\alpha, \beta, \gamma\} \circ p$.*

The following is useful for determining 3-primary parts of the class $[\Sigma^n \mathbb{H}P^2, S^m]$ [2].

Theorem 3.

$$24\Sigma\iota_{\mathbb{H}} = \Sigma i \circ \overline{24\iota_5} + \widetilde{24\iota_8} \circ \Sigma p$$

on $[\Sigma\mathbb{H}P^2, \Sigma\mathbb{H}P^2]$, where $\iota_{\mathbb{H}} : \mathbb{H}P^2 \to \mathbb{H}P^2$ and $\iota_8 : S^8 \to S^8$ are the identity maps on $\mathbb{H}P^2$ and S^8, respectively.

3. Basic computations

In this section, we describe the basic computation of two 3-dimensional cohomotopy groups, and apply them to the computation of other cohomotopy groups.

By [8, (5.9)], we have $\eta_n \circ \nu_{n+1} = 0$ for $n \geq 5$; thus, there is an extension $\overline{\eta_n} \in [\Sigma^{n-3}\mathbb{H}P^2, S^n]$ of η_n for $n \geq 5$. Moreover, by [4, Proposition (2.2)], we have $\epsilon' \circ \nu_{13} = 0$; thus, there is an extension $\overline{\epsilon'} \in [\Sigma^9\mathbb{H}P^2, S^3]$ of ϵ'.

Proposition 1. (1) $[\Sigma^8\mathbb{H}P^2, S^3] = \mathbb{Z}_2^2\{\nu' \circ \eta_6 \circ \mu_4 \circ \Sigma^8 p, \epsilon_3 \circ \overline{\eta_{11}}\} \oplus \mathbb{Z}_3\{\alpha_1(3) \circ \beta_1(6) \circ \Sigma^8 p\}$.

(2) $[\Sigma^9\mathbb{H}P^2, S^3]_{(2)} \cong \mathbb{Z}_4\{\overline{\epsilon'}\} \oplus \mathbb{Z}_2\{\mu_3 \circ \overline{\eta_{12}}\}$.

Proof. (1) Consider the $(8,3)$-type short exact sequence

$$0 \to \mathrm{Coker}\nu_{13}^* \xrightarrow{\Sigma^8 p^*} [\Sigma^8\mathbb{H}P^2, S^3] \xrightarrow{\Sigma^8 i^*} \mathrm{Ker}\nu_{12}^* \to 0$$

where $\nu_{13}^* : \pi_{13}(S^3) \to \pi_{16}(S^3)$ and $\nu_{12}^* : \pi_{12}(S^3) \to \pi_{15}(S^3)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{13}^* : \mathbb{Z}_4\{\epsilon'\} \oplus \mathbb{Z}_2\{\eta_3 \circ \mu_4\} \oplus \mathbb{Z}_3 \to \mathbb{Z}_2\{\nu' \circ \eta_6 \circ \mu_7\} \oplus \mathbb{Z}_3\{\alpha_1(3) \circ \beta_1(6)\},$$

and

$$\nu_{12}^* : \mathbb{Z}_2^2\{\mu_3, \eta_3 \circ \epsilon_4\} \to \mathbb{Z}_2^2\{\nu' \circ \mu_6, \nu' \circ \eta_6 \circ \epsilon_7\},$$

respectively.

Then, we have $\nu_{13}^*(\epsilon') = 0$ by [4, (2.2)], and $\nu_{13}^*(\eta_3 \circ \mu_4) = 0$ by [8, (5.9)] and [4, (2.2)]. Thus, we have $\nu_{12}^*(\mu_3) = \nu' \circ \eta_6 \circ \epsilon_7$ by [4, (2.2)], and $\nu_{12}^*(\eta_3 \circ \epsilon_4) = 0$ by [8, (5.9)] and [4, (2.1)]. Thus, we have $\mathrm{Coker}\nu_{13}^* = \mathbb{Z}_2\{\nu' \circ \eta_6 \circ \mu_7\} \oplus \mathbb{Z}_3\{\alpha_1(3) \circ \beta_1(6)\}$ and $\mathrm{Ker}\nu_{12}^* = \mathbb{Z}_2\{\eta_3 \circ \epsilon_4\}$. This gives the following short exact sequence

$$0 \to \mathbb{Z}_2\{\nu' \circ \eta_6 \circ \mu_7\} \oplus \mathbb{Z}_3\{\alpha_1(3) \circ \beta_1(6)\} \xrightarrow{\Sigma^8 p^*} [\Sigma^8\mathbb{H}P^2, S^3] \xrightarrow{\Sigma^8 i^*} \mathbb{Z}_2\{\eta_3 \circ \epsilon_4\} \to 0.$$

By [4, (2.1)], we know that $\eta_3 \circ \epsilon_4 = \epsilon_3 \circ \eta_{11}$. Consider the extension $\epsilon_3 \circ \overline{\eta_{11}}$ of $\epsilon_3 \circ \eta_{11}$. By [2, Proposition 4.1], the order of $\overline{\eta_{11}}$ is two; thus, the order of $\epsilon_3 \circ \overline{\eta_{11}}$ is two, and therefore the short exact sequence is split.

(2) Consider the $(9,3)$-type short exact sequence

$$0 \to \mathrm{Coker}\nu_{14}^* \xrightarrow{\Sigma^9 p^*} [\Sigma^9\mathbb{H}P^2, S^3] \xrightarrow{\Sigma^9 i^*} \mathrm{Ker}\nu_{13}^* \to 0,$$

where $\nu_{14}^* : \pi_{14}(S^3) \to \pi_{17}(S^3)$ is the homomorphism induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. This homomorphism can be restated as follows:

$$\nu_{14}^* : \mathbb{Z}_4\{\mu'\} \oplus \mathbb{Z}_2^2\{\epsilon_3 \circ \nu_{11}, \nu' \circ \epsilon_6\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7 \to \mathbb{Z}_2\{\epsilon_3 \circ \nu_{11}^2\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5.$$

Then, we have $\nu_{14}^*(\mu') = 0$ by [4, (2.4)], and $\nu_{14}^*(\nu' \circ \epsilon_6) = \nu' \circ \epsilon_6 \circ \nu_{14} = 0$ since $\epsilon_6 \circ \nu_{14} = (E^3\nu') \circ \overline{\nu}_9 = (2\nu_6) \circ \overline{\nu}_9 = 0$ by [4, (2.1)]. Thus, we have $\mathrm{Coker}\nu_{14}^* = \mathbb{Z}_2\{\epsilon_3 \circ \nu_{11}^2\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ and $\mathrm{Ker}\nu_{13}^* = \mathbb{Z}_4\{\epsilon'\} \oplus \mathbb{Z}_2\{\eta_3 \circ \mu_4\} \oplus \mathbb{Z}_3$ by (1). This gives the following two-primary short exact sequence

$$0 \xrightarrow{\Sigma^9 p^*} [\Sigma^9\mathbb{H}P^2, S^3]_{(2)} \xrightarrow{\Sigma^9 i^*} \mathbb{Z}_4\{\epsilon'\} \oplus \mathbb{Z}_2\{\eta_3 \circ \mu_4\} \to 0.$$

□

4. Computation of $[\Sigma^{n+4}\mathbb{H}P^2, S^n]$ for $n \geq 2$

In this section, we compute the n-th cohomotopy groups of $(n+4)$-fold suspended quaternionic projective planes.

Proposition 2. $[\Sigma^6\mathbb{H}P^2, S^2] = \mathbb{Z}_4\{\eta_2 \circ \mu' \circ \Sigma^6 p\} \oplus \mathbb{Z}_2\{\eta_2 \circ \nu' \circ \epsilon_6 \circ \Sigma^6 p\} \oplus \mathbb{Z}_3\{\eta_2 \circ \alpha_3(3) \circ \Sigma^6 p\} \oplus \mathbb{Z}_{35}$.

Proof. Since $\eta_2 : S^3 \to S^2$ is a fibration with fiber S^1, $\eta_{2*} : [\Sigma^6\mathbb{H}P^2, S^3] \to [\Sigma^6\mathbb{H}P^2, S^2]$ is an isomorphism. Thus, by [2, Theorem 4.7 (2)], this completes the proof. □

Proposition 3. $[\Sigma^7\mathbb{H}P^2, S^3] \cong \mathbb{Z}_2\{\nu' \circ \mu_6 \circ \Sigma^7 p\}$.

Proof. Consider the $(7,3)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{12}^* \xrightarrow{\Sigma^7 p^*} [\Sigma^7\mathbb{H}P^2, S^3] \xrightarrow{\Sigma^7 i^*} \mathrm{Ker}\nu_{11}^* \to 0,$$

where $\nu_{12}^* : \pi_{12}(S^3) \to \pi_{15}(S^3)$ and $\nu_{11}^* : \pi_{11}(S^3) \to \pi_{14}(S^3)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{12}^* : \mathbb{Z}_2\{\mu_3\} \oplus \mathbb{Z}_2\{\eta_3 \circ \epsilon_4\} \to \mathbb{Z}_2\{\nu' \circ \mu_6\} \oplus \mathbb{Z}_2\{\nu' \circ \eta_6 \circ \epsilon_7\},$$

and

$$\nu_{11}^* : \mathbb{Z}_2\{\epsilon_3\} \to \mathbb{Z}_4\{\mu'\} \oplus \mathbb{Z}_2\{\nu' \circ \epsilon_6\} \oplus \mathbb{Z}_2\{\epsilon_3 \circ \nu_{11}\} \oplus \mathbb{Z}_{21},$$

respectively. Then, we have $\nu_{12}^*(\mu_3) = \nu' \circ \eta_6 \circ \epsilon_7$ [4, Proposition 2.2], $\nu_{12}^*(\eta_3 \circ \epsilon_4) = \eta_3 \circ \epsilon_4 \circ \nu_{12} = \epsilon_3 \circ \eta_{11} \circ \nu_{12} = 0$ [4, (2.1)], [8, (5.9)], and $\nu_{11}^*(\epsilon_3) = \epsilon_3 \circ \nu_{11}$. Thus, we have $\mathrm{Coker}\nu_{12}^* = \mathbb{Z}_2\{\nu' \circ \mu_6\}$ and $\mathrm{Ker}\nu_{11}^* = 0$. From the short exact sequence, we have

$$[\Sigma^7\mathbb{H}P^2, S^3] = \mathbb{Z}_2\{\nu' \circ \mu_6 \circ \Sigma^7 p\}.$$ □

Proposition 4. $[\Sigma^8\mathbb{H}P^2, S^4] = \mathbb{Z}_2^4\{\nu_4 \circ \sigma' \circ \eta_{14}^2 \circ \Sigma^8 p, \nu_4 \circ \mu_7 \circ \Sigma^8 p, \nu_4 \circ \eta_7 \circ \epsilon_8 \circ \Sigma^8 p, (E\nu') \circ \mu_7 \circ \Sigma^8 p\}$.

Proof. Consider the $(8,4)$-type short exact sequence

$$0 \to \mathrm{Coker}\nu_{13}^* \xrightarrow{\Sigma^8 p^*} [\Sigma^8\mathbb{H}P^2, S^4] \xrightarrow{\Sigma^8 i^*} \mathrm{Ker}\nu_{12}^* \to 0,$$

where $\nu_{13}^* : \pi_{13}(S^4) \to \pi_{16}(S^4)$ and $\nu_{12}^* : \pi_{12}(S^4) \to \pi_{15}(S^4)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{13}^* : \mathbb{Z}_2^3\{\nu_4^3, \mu_4, \eta_4 \circ \epsilon_5\} \to$$
$$\mathbb{Z}_2^6\{\nu_4 \circ \sigma' \circ \eta_{14}^2, \nu_4^4, \nu_4 \circ \mu_7, \nu_4 \circ \eta_7 \circ \epsilon_8, (E\nu') \circ \mu_7, (E\nu') \circ \eta_7 \circ \epsilon_8\}$$

and

$$\nu_{12}^* : \mathbb{Z}_2\{\epsilon_4\} \to \mathbb{Z}_4 \oplus \mathbb{Z}_2^5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_7,$$

respectively. Then, we have $\nu_{13}^*(\nu_4^3) = \nu_4^4$,

$$\nu_{13}^*(\mu_4) = \mu_4 \circ \nu_{13} = (E\nu') \circ \eta_7 \circ \epsilon_8$$

[4, Proposition (2.2)(4)], and

$$\nu_{13}^*(\eta_4 \circ \epsilon_5) = \eta_4 \circ \epsilon_5 \circ \nu_{13} = \epsilon_4 \circ \eta_{12} \circ \nu_{13} = 0$$

by [4, (2.1)]. Thus, it follows that

$$\mathrm{Coker}\nu_{13}^* = \mathbb{Z}_2^4\{\nu_4 \circ \sigma' \circ \eta_{14}^2, \nu_4 \circ \mu_7, \nu_4 \circ \eta_7 \circ \epsilon_8, (E\nu') \circ \mu_7\}.$$

Since the element $\nu_{12}^*(\epsilon_4) = \epsilon_4 \circ \nu_{12}$ has order 2 in $\pi_{15}(S^4)$, ν_{12}^* is injective. Hence, we have the short exact sequence

$$0 \to \mathbb{Z}_2^4\{\nu_4 \circ \sigma' \circ \eta_{14}^2, \nu_4 \circ \mu_7, \nu_4 \circ \eta_7 \circ \epsilon_8, (E\nu') \circ \mu_7\} \xrightarrow{\Sigma^8 p^*} [\Sigma^8 \mathbb{H}P^2, S^4] \xrightarrow{\Sigma^8 i^*} 0.$$

□

By [4, (1.1) (7)], $H(\epsilon') = \epsilon_5$, and from [4, (2.2) (7)], we have $\epsilon' \circ \nu_{13} = 0$. Thus, there is an extension $\overline{\epsilon'} \in [\Sigma^9 \mathbb{H}P^2, S^3]$ of ϵ'. Denote $\overline{\epsilon_5} = H(\overline{\epsilon'})$ to be an extension of ϵ_5. We denote $\overline{\epsilon_n} = \Sigma^{n-5}\overline{\epsilon_5}$ for $n \geq 5$.

Proposition 5. $[\Sigma^9 \mathbb{H}P^2, S^5] = \mathbb{Z}_2^3\{\nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p, \nu_5 \circ \mu_8 \circ \Sigma^9 p, \overline{\epsilon_5}\} \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7$.

Proof. Consider the $(9, 5)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{14}^* \xrightarrow{\Sigma^9 p^*} [\Sigma^9 \mathbb{H}P^2, S^5] \xrightarrow{\Sigma^9 i^*} \mathrm{Ker}\nu_{13}^* \to 0,$$

where $\nu_{14}^* : \pi_{14}(S^5) \to \pi_{17}(S^5)$ and $\nu_{13}^* : \pi_{13}(S^5) \to \pi_{16}(S^5)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{14}^* : \mathbb{Z}_2^3\{\nu_5^3, \mu_5, \eta_5 \circ \epsilon_6\} \to \mathbb{Z}_2^3\{\nu_5^4, \nu_5 \circ \mu_8, \nu_5 \circ \eta_8 \circ \epsilon_9\},$$

and

$$\nu_{13}^* : \mathbb{Z}_2\{\epsilon_5\} \to \mathbb{Z}_8\{\zeta_8\} \oplus \mathbb{Z}_2^2\{\nu_5 \circ \overline{\nu}_8, \nu \circ \epsilon_9\} \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7,$$

respectively. Then, we have $\nu_{14}^*(\nu_5^3) = \nu_5^4$, $\nu_{14}^*(\eta_5 \circ \epsilon_6) = 0$, and

$$\nu_{14}^*(\mu_5) = \mu_5 \circ \nu_{14} = \Sigma^2(\nu' \circ \eta_6 \circ \epsilon_7) = (\Sigma^2 \nu') \circ \eta_8 \circ \epsilon_9 = (2\nu_5) \circ \eta_8 \circ \epsilon_9 = 0$$

by [4, Proposition 2.2(4)]. Thus, we have

$$\mathrm{Coker}\nu_{14}^* = \mathbb{Z}_2^2\{\nu_5 \circ \eta_8 \circ \epsilon_9, \nu_5 \circ \mu_8\} + \mathbb{Z}_9 \oplus \mathbb{Z}_7.$$

Additionally, we have

$$\nu_{13}^*(\epsilon_5) = \Sigma^2(\nu' \circ \overline{\nu}_6) = (\Sigma^2 \nu') \circ \overline{\nu}_8 = (2\nu_5) \circ \overline{\nu}_8 = 2(\nu_5 \circ \overline{\nu}_8) = 0$$

since $\nu_5 \circ \overline{\nu}_8 \in \pi_{16}(S^5)$ has order 2 [8]. Thus, we have $\mathrm{Ker}\nu_{13}^* = \mathbb{Z}_2\{\epsilon_5\}$ and therefore, we have the short exact sequence

$$(*) \qquad 0 \to \mathbb{Z}_2^2\{\nu_5 \circ \eta_8 \circ \epsilon_9, \nu_5 \circ \mu_8\} \xrightarrow{\Sigma^9 p^*} [\Sigma^9 \mathbb{H}P^2, S^5]_{(2)} \xrightarrow{\Sigma^9 i^*} \mathbb{Z}_2\{\epsilon_5\} \to 0.$$

If $[\Sigma^9 \mathbb{H}P^2, S^5]$ is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$, we only have three cases:

$$2\overline{\epsilon_5} = \begin{cases} \nu_5 \circ \mu_8 \circ \Sigma^9 p, \\ \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p, \\ \nu_5 \circ \mu_8 \circ \Sigma^9 p + \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p. \end{cases}$$

First, we assume that $2\overline{\epsilon_5} = \nu_5 \circ \mu_8 \circ \Sigma^9 p$. Then, we have $0 = \triangle \circ H(2(\overline{\epsilon'})) = \triangle(2H(\overline{\epsilon'})) = \triangle(2\overline{\epsilon_5}) = \triangle(\nu_5 \circ \mu_8 \circ \Sigma^9 p) = \triangle(\nu_5) \circ \mu_6 \circ \Sigma^7 p = \eta_2 \circ \nu' \circ \mu_6 \circ \Sigma^7 p$, since $\triangle(\nu_5) = \eta_2 \circ \nu'$ [8]. However, we know that

$$[\Sigma^7 \mathbb{H}P^2, S^2] = \mathbb{Z}_2\{\eta_2 \circ \nu' \circ \mu_6 \circ \Sigma^7 p\}.$$

This is a contradiction. Thus, $2\overline{\epsilon_5} \neq \nu_5 \circ \mu_8 \circ \Sigma^9 p$. Next, we assume that $2\overline{\epsilon_5} = \nu_5 \circ \mu_8 \circ \Sigma^9 p + \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p$. Then, we have $0 = \triangle \circ H(2\overline{\epsilon'}) = \triangle(2H(\overline{\epsilon'})) = \triangle(2\overline{\epsilon_5}) = \triangle(\nu_5 \circ \mu_8 \circ \Sigma^9 p + \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p) = \triangle(\nu_5) \circ \mu_6 \circ \Sigma^7 p + \triangle(\nu_5) \circ \eta_6 \circ \epsilon_7 \circ \Sigma^7 p = \eta_2 \circ \nu' \circ \mu_6 \circ \Sigma^7 p + \eta_2 \circ \nu' \circ \eta_6 \circ \epsilon_7 \circ \Sigma^7 p = \eta_2 \circ \nu' \circ \mu_6 \circ \Sigma^7 p$, since $\triangle(\nu_5) = \eta_2 \circ \nu'$ [8] and $\nu' \circ \eta_6 \circ \epsilon_7 \circ \Sigma^7 p = 0 \in [\Sigma^7 \mathbb{H}P^2, S^3]$. As this is also a contradiction, we have $2\overline{\epsilon_5} = \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p$.

Consider a suspension homomorphism $E : [\Sigma^8 \mathbb{H}P^2, S^2] \to [\Sigma^9 \mathbb{H}P^2, S^3]$ where

$$[\Sigma^8 \mathbb{H}P^2, S^2] = \mathbb{Z}_2^2\{\eta_2 \circ \nu' \circ \eta_6 \circ \mu_7 \circ \Sigma^8 p, \eta_2 \circ \epsilon_3 \circ \overline{\eta_{11}}\}$$

and

$$[\Sigma^9 \mathbb{H}P^2, S^3] = \mathbb{Z}_4\{\overline{\epsilon'}\} \oplus \mathbb{Z}_2\{\mu_3 \circ \overline{\eta_{12}}\}.$$

Now, we consider an element $E(\eta_2 \circ \epsilon_3 \circ \overline{\eta_{11}}) \in [\Sigma^9 \mathbb{H}P^2, S^3]$. We have $\Sigma^9 i^*(E(\eta_2 \circ \epsilon_3 \circ \overline{\eta_{11}}) = \Sigma^9 i^*(\eta_3 \circ \epsilon_4 \circ \overline{\eta_{12}}) = \eta_3 \circ \epsilon_4 \circ \eta_{12} = \eta_3^2 \circ \epsilon_5 = 2\epsilon'$. Since $\Sigma^9 i^* : [\Sigma^9 \mathbb{H}P^2, S^3]_{(2)} \to Ker\nu_{13}{}^*_{(2)}$ is an isomorphism, we have $E(\eta_2 \circ \epsilon_3 \circ \overline{\eta_{11}}) = 2\overline{\epsilon'}$. By exactness and $E(\eta_2 \circ \epsilon_3 \circ \overline{\eta_{11}}) = 2\overline{\epsilon'}$, we have $0 = H \circ E(\eta_2 \circ \epsilon_3 \circ \overline{\eta_{11}}) = H(2\overline{\epsilon'}) = 2 \cdot \overline{\epsilon_5} = \nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p$. This contradicts to the statement that $\nu_5 \circ \eta_8 \circ \epsilon_9 \circ \Sigma^9 p$ has order 2. Thus, the short exact sequence $(*)$ splits. □

Proposition 6. $[\Sigma^{10} \mathbb{H}P^2, S^6] = \mathbb{Z}_{16}\{\triangle(\sigma_{13}) \circ \Sigma^{10} p\} \oplus \mathbb{Z}_2\{\overline{\epsilon_6}\} \oplus \mathbb{Z}_3^2\{\overline{\triangle(\alpha_1(13))}, \triangle(\alpha_2(13)) \circ \Sigma^{10} p\} \oplus \mathbb{Z}_5$.

Proof. Consider the $(10, 6)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{15}^* \xrightarrow{\Sigma^{10} p^*} [\Sigma^{10} \mathbb{H}P^2, S^6] \xrightarrow{\Sigma^{10} i^*} \mathrm{Ker}\nu_{14}^* \to 0,$$

where $\nu_{15}^* : \pi_{15}(S^6) \to \pi_{18}(S^6)$ and $\nu_{14}^* : \pi_{14}(S^6) \to \pi_{17}(S^6)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{15}^* : \mathbb{Z}_2^3\{\nu_6^3, \mu_6, \eta_6 \circ \epsilon_7\} \to \mathbb{Z}_{16}\{\triangle(\sigma_{13})\} \oplus \mathbb{Z}_3\{[\iota_6, \iota_6] \circ \alpha_2(11)\} \oplus \mathbb{Z}_5,$$

and

$$\nu_{14}^* : \mathbb{Z}_8\{\overline{\nu}_6\} \oplus \mathbb{Z}_2\{\epsilon_6\} \oplus \mathbb{Z}_3\{[\iota_6, \iota_6] \circ \alpha_1(11)\} \to \mathbb{Z}_8\{\zeta_6\} \oplus \mathbb{Z}_4\{\overline{\nu}_6 \circ \nu_{14}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9$$

by [8, p. 61, p. 66, p. 74, p. 186], respectively. Then, we have

$$\nu_{15}^*(\nu_6^3) = \nu_6^4 = E(\nu_5^4) = E\triangle(\overline{\nu}_{11}) = 0$$

by [8, p. 77],

$$\nu_{15}^*(\mu_6) = E(\mu_5 \circ \nu_{14}) = 0$$

by Proposition 5, and $\nu_{15}^*(\eta_6 \circ \epsilon_7) = 0$. Thus, we have $\mathrm{Coker}\nu_{15}^* = \pi_{18}(S^6)$. In addition, we have $\nu_{14}^*(\overline{\nu}_6) = \overline{\nu}_6 \circ \nu_{14}$, $\nu_{14}^*(\epsilon_6) = \epsilon_6 \circ \nu_{14} = E^3(\nu' \circ \overline{\nu}_6) = (E^3\nu') \circ \overline{\nu}_9 = (2\nu_6) \circ \overline{\nu}_9 = 2(\nu_6 \circ \overline{\nu}_9) = 2(2\overline{\nu}_6 \circ \nu_{14}) = 4(\overline{\nu}_6 \circ \nu_{14}) = 0$ by [4, (2.1)] and [8, Theorem 7.4]. Thus, we have $\mathrm{Ker}\nu_{14}^* = \mathbb{Z}_2^2\{4\overline{\nu}_6, \epsilon_6\} \oplus \mathbb{Z}_3\{[\iota_6,\iota_6]\circ\alpha_1(11)\}$. This gives the short exact sequence

$$0 \to \mathbb{Z}_{16}\{\triangle(\sigma_{13})\} \oplus \mathbb{Z}_3\{[\iota_6,\iota_6]\circ\alpha_2(11)\} \oplus \mathbb{Z}_5 \xrightarrow{\Sigma^{10}p^*}$$
$$\xrightarrow{\Sigma^{10}p^*} [\Sigma^{10}\mathbb{H}P^2, S^6] \xrightarrow{\Sigma^{10}i^*} \mathbb{Z}_2^2\{4\overline{\nu}_6, \epsilon_6\} \oplus \mathbb{Z}_3\{[\iota_6,\iota_6]\circ\alpha_1(11)\} \to 0.$$

In Proposition 5, we showed that $\overline{\epsilon_5}$ has order 2. Thus $\overline{\epsilon_6}$ has order 2. By [4, (2.1)] and [2, Proposition 4.1], we have $4\overline{\nu}_6 = \sigma''\circ\eta_{13}$ and $\overline{\eta_{13}}$ has order 2. Therefore, we conclude that $\overline{4\overline{\nu}_6} = \sigma''\circ\overline{\eta_{13}}$ has order 2. It implies that the 2-primary exact sequence

$$0 \to \mathbb{Z}_{16}\{\triangle(\sigma_{13})\} \xrightarrow{\Sigma^{10}p^*} [\Sigma^{10}\mathbb{H}P^2, S^6]_{(2)} \xrightarrow{\Sigma^{10}i^*} \mathbb{Z}_2^2\{4\overline{\nu}_6, \epsilon_6\} \to 0$$

is split. Now consider the 3-primary parts

$$0 \to \mathbb{Z}_3\{[\iota_6,\iota_6]\circ\alpha_2(11)\} \xrightarrow{\Sigma^{10}p^*} [\Sigma^{10}\mathbb{H}P^2, S^6]_{(3)} \xrightarrow{\Sigma^{10}i^*} \mathbb{Z}_3\{[\iota_6,\iota_6]\circ\alpha_1(11)\} \to 0.$$

By Theorem 3, we have $3\overline{\alpha_1(11)} = \overline{\alpha_1(11)}\circ 24\Sigma^{10}\iota_H = \overline{\alpha_1(11)}\circ\Sigma^{10}i\circ\overline{24\iota_{14}} + \overline{\alpha_1(11)}\circ\widetilde{24\iota_{17}}\circ\Sigma^{10}p = \alpha_1(11)\circ\overline{24\iota_{14}} + \overline{\alpha_1(11)}\circ\widetilde{24\iota_{17}}\circ\Sigma^{10}p$.

By [2, Theorem 2.7], we have $\alpha_1(11)\circ\overline{24\iota_{14}} \in \{\alpha_1(11), 24\iota_{14}, \alpha_1(14)\}\circ\Sigma^{10}p$. Since $\alpha_2(11)$ has order 3, we have

$$-\alpha_2(11) = 8\alpha_2(11) \in 8\{\alpha_1(11), 3\iota_{14}, \alpha_1(14)\} = \{\alpha_1(11), 24\iota_{14}, \alpha_1(14)\}$$

by [8, Lemma 13.5]. Then, we have

$$\alpha_1(11)\circ\overline{24\iota_{14}} \in \{\alpha_1(11), 24\iota_{14}, \alpha_1(14)\}\circ\Sigma^{10}p \ni -\alpha_2(11)\circ\Sigma^{10}p \mod 0,$$

that is, $\alpha_1(11)\circ\overline{24\iota_{14}} = -\alpha_2(11)\circ\Sigma^{10}p$.

By Theorem 2, we have $\alpha_1(11)\circ\widetilde{24\iota_{17}} \in \{\alpha_1(11), \alpha_1(14), 24\iota_{17}\}$. From [8, (13.8)], we have $(1/2)\alpha_2(11) \in \{\alpha_1(11), \alpha_1(14), 3\iota_{17}\}$ and so, we have

$$\alpha_2(11) = 4\alpha_2(11) \in 8\{\alpha_1(11), \alpha_1(14), 3\iota_{17}\} = \{\alpha_1(11), \alpha_1(14), 24\iota_{17}\}.$$

Thus, we have

$$\overline{\alpha_1(11)}\circ\widetilde{24\iota_{17}} \in \{\alpha_1(11), \alpha_1(14), 24\iota_{17}\} \ni \alpha_1(11) \mod 0,$$

that is, $\overline{\alpha_1(11)}\circ\widetilde{24\iota_{17}} = \alpha_2(11)$. Thus, we have

$$\begin{aligned} 3\overline{\alpha_1(11)} &= \alpha_1(11)\circ\overline{24\iota_{14}} + \overline{\alpha_1(11)}\circ\widetilde{24\iota_{17}}\circ\Sigma^{10}p \\ &= -\alpha_2(11)\circ\Sigma^{10}p + \alpha_2(11)\circ\Sigma^{10}p \\ &= 0. \end{aligned}$$

Consequently, $\overline{\alpha_1(11)}$ has order 3, and so has $[\iota_6,\iota_6]\circ\overline{\alpha_1(11)}$. Therefore, we have

$$[\Sigma^{10}\mathbb{H}P^2, S^6]_{(3)} = \mathbb{Z}_3^2\{[\iota_6,\iota_6]\circ\overline{\alpha_1(11)}, [\iota_6,\iota_6]\circ\alpha_2(11)\circ\Sigma^{10}p\}. \qquad \square$$

Proposition 7. (1) $[\Sigma^{11}\mathbb{H}P^2, S^7] = \mathbb{Z}_2^2\{\sigma'\circ\overline{\eta_{14}}, \overline{\epsilon_7}\}$.
(2) $[\Sigma^{12}\mathbb{H}P^2, S^8] = \mathbb{Z}_2^3\{\sigma_8\circ\overline{\eta_{15}}, (E\sigma')\circ\overline{\eta_{15}}, \overline{\epsilon_8}\}$.
(3) $[\Sigma^{13}\mathbb{H}P^2, S^9] = \mathbb{Z}_2^2\{\sigma_9\circ\overline{\eta_{16}}, \overline{\epsilon_9}\}$.

Proof. (1) Consider the homomorphisms $\nu_{16}^* : \pi_{16}(S^7) \to \pi_{19}(S^7)$ and $\nu_{15}^* : \pi_{15}(S^7) \to \pi_{18}(S^7)$ related to the $(11, 7)$-type short exact sequence. Since $\pi_{19}(S^7) = 0$, $\mathrm{Coker}\nu_{16}^* = 0$. Moreover, for the homomorphism

$$\nu_{15}^* : \mathbb{Z}_2^3\{\sigma' \circ \eta_{14}, \overline{\nu}_7, \epsilon_7\} \to \mathbb{Z}_8\{\zeta_7\} \oplus \mathbb{Z}_2\{\overline{\nu}_7 \circ \nu_{15}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9,$$

we have $\nu_{15}^*(\sigma' \circ \eta_{14}) = 0$, $\nu_{15}^*(\overline{\nu}_7) = \overline{\nu}_7 \circ \nu_{15}$, and $\nu_{15}^*(\epsilon_7) = E(\epsilon_6 \circ \nu_{14}) = E0 = 0$ from Proposition 6. Therefore, we have $\mathrm{Ker}\nu_{15}^* = \mathbb{Z}_2^2\{\sigma' \circ \eta_{14}, \epsilon_7\}$. Thus, we obtain the following short exact sequence form the $(11, 7)$-type short exact sequence

$$0 \to [\Sigma^{11}\mathbb{H}P^2, S^7] \xrightarrow{\Sigma^{11}i^*} \mathbb{Z}_2^2\{\sigma' \circ \eta_{14}, \epsilon_7\} \to 0.$$

(2) Consider the homomorphisms $\nu_{17}^* : \pi_{17}(S^8) \to \pi_{20}(S^8)$ and $\nu_{16}^* : \pi_{16} \to \pi_{19}(S^8)$ related to the $(12, 8)$-type short exact sequence. Since $\pi_{20}(S^8) = 0$, $\mathrm{Coker}\nu_{17}^* = 0$. For the homomorphism

$$\nu_{16}^* : \mathbb{Z}_2^4\{\sigma_8 \circ \eta_{15}, (E\sigma') \circ \eta_{15}, \overline{\nu}_8, \epsilon_8\} \to \mathbb{Z}_8\{\zeta_8\} \oplus \mathbb{Z}_2\{\overline{\nu}_8 \circ \nu_{16}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9,$$

we have $\nu_{16}^*(\sigma_8 \circ \eta_{15}) = 0$, $\nu_{16}^*((E\sigma') \circ \eta_{15}) = 0$, $\nu_{16}^*(\overline{\nu}_8) = \overline{\nu}_8 \circ \nu_{16}$, and $\nu_{16}^*(\epsilon_8) = \epsilon_8 \circ \nu_{16} = E^2(\epsilon_6 \circ \nu_{14}) = 0$ from Proposition 6. Thus, we have $\mathrm{Ker}\nu_{16}^* = \mathbb{Z}_2^3\{\sigma_8 \circ \eta_{15}, (E\sigma') \circ \eta_{15}, \epsilon_8\}$, which leads to the short exact sequence

$$0 \to [\Sigma^{12}\mathbb{H}P^2, S^8] \xrightarrow{\Sigma^{12}i^*} \mathbb{Z}_2^3\{\sigma_8 \circ \eta_{15}, (E\sigma') \circ \eta_{15}, \epsilon_8\} \to 0.$$

(3) Consider the homomorphisms $\nu_{18}^* : \pi_{18}(S^9) \to \pi_{21}(S^9)$ and $\nu_{17}^* : \pi_{17}(S^9) \to \pi_{20}(S^9)$ related to the $(13, 9)$-type short exact sequence. Since $\pi_{21}(S^9) = 0$, $\mathrm{Coker}\nu_{18}^* = 0$. Moreover, for the homomorphism

$$\nu_{17}^* : \mathbb{Z}_2^3\{\sigma_9 \circ \eta_{16}, \overline{\nu}_9, \epsilon_9\} \to \mathbb{Z}_8\{\zeta_9\} \oplus \mathbb{Z}_2\{\overline{\nu}_9 \circ \nu_{17}\} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_9,$$

we have $\nu_{17}^*(\sigma_9 \circ \eta_{16}) = 0$, $\nu_{17}^*(\overline{\nu}_9) = \overline{\nu}_9 \circ \nu_{17}$, and $\nu_{17}^*(\epsilon_9) = \epsilon_9 \circ \nu_{17} = E(\epsilon_8 \circ \nu_{16}) = E0 = 0$ from Proposition 6. Thus, we have $\mathrm{Ker}\nu_{17}^* = \mathbb{Z}_2^2\{\sigma_9 \circ \eta_{16}, \epsilon_9\}$, and consequently, we obtain the following short exact sequence from the $(13, 9)$-type short exact sequence:

$$0 \to [\Sigma^{13}\mathbb{H}P^2, S^9] \xrightarrow{\Sigma^{13}i^*} \mathbb{Z}_2^2\{\sigma_9 \circ \eta_{16}, \epsilon_9\} \to 0. \qquad \square$$

Proposition 8. (1) $[\Sigma^{14}\mathbb{H}P^2, S^{10}] = \mathbb{Z}_2^2\{\overline{\overline{\nu}_{10}}, \overline{\epsilon_{10}}\} \oplus \mathbb{Z}_3\{\triangle(\alpha_1(21)) \circ \Sigma^{14}p\}$.
(2) $[\Sigma^{15}\mathbb{H}P^2, S^{11}] = \mathbb{Z}_2^3\{\theta' \circ \Sigma^{15}p, \overline{\overline{\nu}_{11}}, \overline{\epsilon_{11}}\}$.
(3) $[\Sigma^{16}\mathbb{H}P^2, S^{12}] = \mathbb{Z}_2^4\{(E\theta') \circ \Sigma^{16}p, \theta \circ \Sigma^{16}p, \overline{\overline{\nu}_{12}}, \overline{\epsilon_{12}}\}$.
(4) $[\Sigma^{17}\mathbb{H}P^2, S^{13}] = \mathbb{Z}_2^3\{(E\theta) \circ \Sigma^{17}p, \overline{\overline{\nu}_{13}}, \overline{\epsilon_{13}}\}$.
(5) $[\Sigma^{n+4}\mathbb{H}P^2, S^n] = \mathbb{Z}_2^2\{\overline{\overline{\nu}_n}, \overline{\epsilon_n}\}$ *for* $n \geq 14$.

Proof. (1) Consider the $(14, 10)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{19}^* \xrightarrow{\Sigma^{14}p^*} [\Sigma^{14}\mathbb{H}P^2, S^{10}] \xrightarrow{\Sigma^{14}i^*} \mathrm{Ker}\nu_{18}^* \to 0,$$

where $\nu_{19}^* : \pi_{19}(S^{10}) \to \pi_{22}(S^{10})$ and $\nu_{18}^* : \pi_{18}(S^{10}) \to \pi_{21}(S^{10})$ are homomorphisms induced by ν_{19} and ν_{18}, respectively. These homomorphisms can be restated as follows:

$$\nu_{19}^* : \mathbb{Z}\{\triangle(\iota_{21})\} \oplus \mathbb{Z}_2^3\{\nu_{10}^3, \mu_{10}, \eta_{10} \circ \epsilon_{11}\} \to \mathbb{Z}_4\{\triangle(\nu_{21})\} \oplus \mathbb{Z}_3\{\triangle(\alpha_1(21))\}$$

by [8, Proposition 2.5, Theorem 7.2, Proposition 13.6] and

$$\nu_{18}^* : \mathbb{Z}_2^2\{\overline{\nu}_{10}, \epsilon_{10}\} \to \mathbb{Z}_8\{\zeta_{10}\} \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_7,$$

respectively. Then, we have $\nu_{19}^*(\triangle(\iota_{21})) = \triangle(\iota_{21}) \circ \nu_{19} = \triangle(\iota_{21} \circ \nu_{21}) = \triangle(\nu_{21})$ by [8, Propostion 2.5]. Moreover, since $\nu_{10}^3 \circ \nu_{19} = 0$, $\mu_{10} \circ \nu_{19} = 0$, and $\eta_{10} \circ \epsilon_{11} \circ \nu_{19} = 0$, we have $\mathrm{Coker}\nu_{19}^* = \mathbb{Z}_3\{\triangle(\alpha_1(21))\}$. Since $\triangle(\nu_{19}) = \overline{\nu}_9 \circ \nu_{17}$ by [8, (7.22)], $\overline{\nu}_{10} \circ \nu_{18} = E\triangle(\nu_{19}) = 0$, and $\nu_{18}^*(\epsilon_{10}) = 0$, we have $\mathrm{Ker}\nu_{18}^* = \mathbb{Z}_2^2\{\overline{\nu}_{10}, \epsilon_{10}\}$. Consequently, we have the split short exact sequence

$$0 \to \mathbb{Z}_3\{\triangle(\alpha_1(21))\} \xrightarrow{\Sigma^{14}p^*} [\Sigma^{14}\mathbb{H}P^2, S^{10}] \xrightarrow{\Sigma^{14}i^*} \mathbb{Z}_2^2\{\overline{\nu}_{10}, \epsilon_{10}\} \to 0.$$

(2) We can show that $\nu_{20}^* : \pi_{20}(S^{11}) \to \pi_{23}(S^{11})$ and $\nu_{19}^* : \pi_{19}(S^{11}) \to \pi_{22}(S^{11})$ are trivial by using approaches similar to those used for (1). Hence, we obtain the following short exact sequence from the $(15, 11)$-type short exact sequence:

$$0 \to \mathbb{Z}_2\{\theta'\} \xrightarrow{\Sigma^{15}p^*} [\Sigma^{15}\mathbb{H}P^2, S^{11}] \xrightarrow{\Sigma^{15}i^*} \mathbb{Z}_2^2\{\overline{\nu}_{11}, \epsilon_{11}\} \to 0.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z}_3\{\triangle(\alpha_1(21))\}_{(2)} & \xrightarrow{\Sigma^{14}p^*} & [\Sigma^{14}\mathbb{H}P^2, S^{10}]_{(2)} & \xrightarrow{\Sigma^{14}i^*} & \mathbb{Z}_2^2\{\overline{\nu}_{10}, \epsilon_{10}\} & \longrightarrow & 0 \\ & & \downarrow \Sigma_1 & & \downarrow \Sigma_2 & & \downarrow \Sigma_3 & & \\ 0 & \longrightarrow & \mathbb{Z}_2\{\theta'\} & \xrightarrow{\Sigma^{15}p^*} & [\Sigma^{15}\mathbb{H}P^2, S^{11}] & \xrightarrow{\Sigma^{15}i^*} & \mathbb{Z}_2^2\{\overline{\nu}_{11}, \epsilon_{11}\} & \longrightarrow & 0, \end{array}$$

where Σ_1, Σ_2, and Σ_3 are homomorphisms induced by Freudenthal's homomorphisms.

Since the first row splits and Σ_3 is an isomorphism, the second row also splits.

(3) By using the same approach as in (1) and (2), we obtain the following split short exact sequence from the $(16, 12)$-type short exact sequence:

$$0 \to \mathbb{Z}_2^2\{\theta, E\theta'\} \xrightarrow{\Sigma^{16}p^*} [\Sigma^{16}\mathbb{H}P^2, S^{12}] \xrightarrow{\Sigma^{16}i^*} \mathbb{Z}_2^2\{\overline{\nu}_{12}, \epsilon_{12}\} \to 0.$$

(4) By using the same approach as in (1)$\sim$(3), we obtain the following split short exact sequence from the $(17, 13)$-type short exact sequence:

$$0 \to \mathbb{Z}_2\{E\theta\} \xrightarrow{\Sigma^{17}p^*} [\Sigma^{17}\mathbb{H}P^2, S^{13}] \xrightarrow{\Sigma^{17}i^*} \mathbb{Z}_2^2\{\overline{\nu}_{13}, \epsilon_{13}\} \to 0.$$

(5) Consider the homomorphisms $\nu_{23}^* : \pi_{23}(S^{14}) \to \pi_{26}(S^{14})$ and $\nu_{22}^* : \pi_{22}(S^{14}) \to \pi_{25}(S^{14})$ related to the $(18, 14)$-type short exact sequence. Since $\mathrm{Coker}\nu_{23}^* = 0$ and ν_{22}^* is a trivial homomorphism, we obtain the following short exact sequence from the $(18, 14)$-type short exact sequence:

$$0 \to [\Sigma^{18}\mathbb{H}P^2, S^{14}] \xrightarrow{\Sigma^{18}i^*} \mathbb{Z}_2^2\{\overline{\nu}_{14}, \epsilon_{14}\} \to 0.$$

The suspension homomorphism

$$\Sigma : [\Sigma^{n+4}\mathbb{H}P^2, S^n] \to [\Sigma^{n+5}\mathbb{H}P^2, S^{n+1}]$$

is isomorphic for $n + 12 < 2n - 1$ (that is, $13 < n$), since $\Sigma^{n+4}\mathbb{H}P^2$ is an $(n + 12)$-dimensional CW-complex and S^n is $(n - 1)$-connected. Thus, the proof is complete according to above fact. $\square$

5. Computation of $[\Sigma^{n+5}\mathbb{H}P^2, S^n]$ for $n \geq 2$

In this section, we compute the n-th cohomotopy groups of $(n + 5)$-fold suspended quaternionic projective planes.

Proposition 9. $[\Sigma^7\mathbb{H}P^2, S^2] = \mathbb{Z}_2\{\eta_2 \circ \nu' \circ \mu_6 \circ \Sigma^7 p\}$.

Proof. By Proposition 1(1) and (2.2), the proof is complete. $\square$

Proposition 10. (1) $[\Sigma^8\mathbb{H}P^2, S^3] = \mathbb{Z}_2\{\nu' \circ \eta_6 \circ \mu_7 \circ \Sigma^8 p\} \oplus \mathbb{Z}_2\{\epsilon_3 \circ \overline{\eta_{11}}\} \oplus \mathbb{Z}_3\{\alpha_2(3) \circ \Sigma^8 p\}$.

(2) $[\Sigma^9\mathbb{H}P^2, S^4] = \mathbb{Z}_2^3\{\nu_4 \circ \eta_7 \circ \mu_8 \circ \Sigma^9 p, (E\nu') \circ \eta_7 \circ \mu_8 \circ \Sigma^9 p, \epsilon_4 \circ \overline{\eta_{12}}\} \oplus \mathbb{Z}_3^2\{\alpha_1(4) \circ \beta_1(7) \circ \Sigma^9 p, [\iota_4, \iota_4] \circ \beta_1(7) \circ \Sigma^9 p\}$.

Proof. Now, we consider the $(8, 3)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{13}^* \xrightarrow{\Sigma^8 p^*} [\Sigma^8\mathbb{H}P^2, S^3] \xrightarrow{\Sigma^8 i^*} \mathrm{Ker}\nu_{12}^* \to 0,$$

where $\nu_{13}^* : \pi_{13}(S^3) \to \pi_{16}(S^3)$ and $\nu_{12}^* : \pi_{12}(S^3) \to \pi_{15}(S^3)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{13}^* : \mathbb{Z}_4\{\epsilon'\} \oplus \mathbb{Z}_2\{\eta_3 \circ \mu_4\} \oplus \mathbb{Z}_3\{\alpha_2(3)\} \to \mathbb{Z}_2\{\nu' \circ \eta_6 \circ \mu_7\} \oplus \mathbb{Z}_3\{\alpha_1(3) \circ \beta_1(6)\}$$

and

$$\nu_{12}^* : \mathbb{Z}_2\{\mu_3\} \oplus \mathbb{Z}_2\{\eta_3 \circ \epsilon_4\} \to \mathbb{Z}_2\{\nu' \circ \mu_6\} \oplus \mathbb{Z}_2\{\nu' \circ \eta_6 \circ \epsilon_7\},$$

respectively. Then, we have $\nu_{13}^*(\nu') = 0$ by [4, Proposition (2.2)] and $\nu_{13}^*(\eta_3 \circ \mu_4) = \eta_3 \circ \mu_4 \circ \nu_{13} = \mu_3 \circ \eta_{12} \circ \nu_{13} = 0$ by [4, (2.1)] and [8, (5.9)], $\nu_{13}^*(\alpha_2(3)) = 0$. Moreover, $\nu_{12}^*(\mu_3) = \nu' \circ \eta_6 \circ \epsilon_7$ by [4, Proposition 2.2], and $\nu_{12}^*(\eta_3 \circ \epsilon_4) = \eta_3 \circ \epsilon_4 \circ \nu_{12} = \epsilon_3 \circ \eta_{11} \circ \nu_{12} = 0$ by [4, (2.1)] and [8, (5.9)]. Thus, we have $\mathrm{Coker}\nu_{13}^* = \mathbb{Z}_2\{\nu' \circ \eta_6 \circ \mu_7\} \oplus \mathbb{Z}_3\{\alpha_1(3) \circ \beta_1(6)\}$ and $\mathrm{Ker}\nu_{12}^* = \mathbb{Z}_2\{\eta_3 \circ \epsilon_4\}$. Then, we have the short exact sequence

$$0 \to \mathbb{Z}_2\{\nu' \circ \eta_6 \circ \mu_7\} \oplus \mathbb{Z}_3\{\alpha_1(3) \circ \beta_1(6)\} \xrightarrow{\Sigma^8 p^*} [\Sigma^8\mathbb{H}P^2, S^3] \xrightarrow{\Sigma^8 i^*} \mathbb{Z}_2\{\eta_3 \circ \epsilon_4\} \to 0.$$

Consider an extension $\epsilon_3 \circ \overline{\eta_{11}}$ of $\epsilon_3 \circ \eta_{11} = \eta_3 \circ \epsilon_4$ [4, (2.1)]. Since $\overline{\eta_{11}}$ has order 2, $\epsilon_3 \circ \overline{\eta_{11}}$ has order 2 [2, Proposition 4.1]. Therefore, the short exact sequence splits.

(2) Consider the $(9, 4)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{14}^* \xrightarrow{\Sigma^9 p^*} [\Sigma^9\mathbb{H}P^2, S^4] \xrightarrow{\Sigma^9 i^*} \mathrm{Ker}\nu_{13}^* \to 0,$$

where $\nu_{14}^* : \pi_{14}(S^4) \to \pi_{17}(S^4)$ and $\nu_{13}^* : \pi_{13}(S^4) \to \pi_{16}(S^4)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{14}^* : \mathbb{Z}_8\{\nu_4 \circ \sigma'\} \oplus \mathbb{Z}_4\{E\epsilon'\} \oplus \mathbb{Z}_2\{\eta_4 \circ \mu_5\} \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_5 \to$$

$\mathbb{Z}_8\{\nu_4^2\circ\sigma_{10}\}\oplus\mathbb{Z}_2^2\{\nu_4\circ\eta_7\circ\mu_8,(E\nu')\circ\eta_7\circ\mu_8\}\oplus\mathbb{Z}_3^2\{\alpha_1(4)\circ\beta_1(7),[\iota_4,\iota_4]\circ\beta_1(7)\}$, and

$$\nu_{13}^*:\mathbb{Z}_2^3\{\nu_4^3,\mu_4,\eta_4\circ\epsilon_5\}\to$$
$$\mathbb{Z}_2^6\{\nu_4\circ\sigma'\circ\eta_{14}^2,\nu_4^4,\nu_4\circ\mu_7,\nu_4\circ\eta_7\circ\epsilon_8,(E\nu')\circ\mu_7,(E\nu')\circ\eta_7\circ\epsilon_8\},$$

respectively. Then, we have

$$\nu_{14}^*(\nu_4\circ\sigma')=\nu_4\circ\sigma'\circ\nu_{14}=\nu_4\circ(x\nu_7\circ\sigma_{10})=x(\nu_4^2\circ\sigma_{10}),$$

where x is odd by [4, (2.1)]; thus, it has order 8. Moreover,

$$\nu_{14}^*(E\epsilon')=E(\epsilon'\circ\nu_{13})=0$$

by [4, (2.2) (7)] and $\nu_{14}^*(\eta_4\circ\epsilon_5)=0$. Thus, we have

$\mathrm{Coker}\nu_{14}^*=\mathbb{Z}_2^2\{\nu_4\circ\eta_7\circ\mu_8,(E\nu')\circ\eta_7\circ\mu_8\}\oplus\mathbb{Z}_3^2\{\alpha_1(4)\circ\beta_1(7),[\iota_4,\iota_4]\circ\beta_1(7)\}$.

Since we also have that $\nu_{13}^*(\nu_4^3)=\nu_4^4$,

$$\nu_{13}^*(\mu_4)=E(\mu_3\circ\nu_{12})=E(\nu'\circ\eta_6\circ\epsilon_7)=(E\nu')\circ\eta_7\circ\epsilon_8,$$

and $\nu_{13}^*(\eta_4\circ\epsilon_5)=0$, we have $\mathrm{Ker}\nu_{13}^*=\mathbb{Z}_2\{\eta_4\circ\epsilon_5\}$. Thus, we have the short exact sequence

$$0\to\mathbb{Z}_2^2\{\nu_4\circ\eta_7\circ\mu_8,(E\nu')\circ\eta_7\circ\mu_8\}\oplus\mathbb{Z}_3^2\{\alpha_1(4)\circ\beta_1(7),[\iota_4,\iota_4]\circ\beta_1(7)\}\xrightarrow{\Sigma^9p^*}$$
$$\xrightarrow{\Sigma^9p^*}[\Sigma^9\mathbb{H}P^2,S^4]\xrightarrow{\Sigma^9i^*}\mathbb{Z}_2\{\eta_4\circ\epsilon_5\}\to 0.$$

Since $\eta_4\circ\epsilon_5=\epsilon_4\circ\eta_{12}$ and $\overline{\eta_{12}}$ has order 2, the extension $\epsilon_4\circ\overline{\eta_{12}}$ also has order 2 by [4, (2.1)] and [2, Proposition 4.1]. Consequently, the short exact sequence splits. □

Proposition 11.

$[\Sigma^{10}\mathbb{H}P^2,S^5]=\mathbb{Z}_2^3\{\nu_5\circ\eta_8\circ\mu_9\circ\Sigma^{10}p,\overline{\mu_5},\epsilon_5\circ\overline{\eta_{13}}\}\oplus\mathbb{Z}_3\{\alpha_1(5)\circ\beta_1(8)\circ\Sigma^{10}p\}$.

Proof. Consider the $(10,5)$-type short exact sequence:

$$0\to\mathrm{Coker}\nu_{15}^*\xrightarrow{\Sigma^{10}p^*}[\Sigma^{10}\mathbb{H}P^2,S^5]\xrightarrow{\Sigma^{10}i^*}\mathrm{Ker}\nu_{14}^*\to 0,$$

where $\nu_{15}^*:\pi_{15}(S^5)\to\pi_{18}(S^5)$ and $\nu_{14}^*:\pi_{14}(S^5)\to\pi_{17}(S^5)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4:S^7\to S^3$. These homomorphisms can be restated as follows:

$$\nu_{15}^*:\mathbb{Z}_8\{\nu_5\circ\sigma_8\}\oplus\mathbb{Z}_2\{\eta_5\circ\mu_6\}\oplus\mathbb{Z}_9\{\beta_1(5)\}\to$$
$$\mathbb{Z}_2^2\{\nu_5\circ\sigma_8\circ\nu_{15},\nu_5\circ\eta_8\circ\mu_9\}\oplus\mathbb{Z}_3\{\alpha_1(5)\circ\beta_1(8)\},$$

and

$$\nu_{14}^*:\mathbb{Z}_2^3\{\nu_5^3,\mu_5,\eta_5\circ\epsilon_6\}\to\mathbb{Z}_2^3\{\nu_5^4,\nu_5\circ\mu_8,\nu_5\circ\eta_8\circ\epsilon_9\},$$

respectively. Then, we have $\nu_{15}^*(\nu_5\circ\sigma_8)=\nu_5\circ\sigma_8\circ\nu_{15}$ and $\nu_{15}^*(\eta_5\circ\mu_6)=0$. Thus, $\mathrm{Coker}\nu_{15}^*=\mathbb{Z}_2\{\nu_5\circ\eta_8\circ\mu_9\}\oplus\mathbb{Z}_3\{\alpha_1(5)\circ\beta_1(8)\}$. In addition, we have $\nu_{14}^*(\nu_5^3)=\nu_5^4$, $\nu_{14}^*(\mu_5)=E^2(\mu_3\circ\nu_{12})=E^2(\nu'\circ\eta_6\circ\epsilon_7)=(E^2\nu')\circ\eta_8\circ\epsilon_9=$

$2\nu_5 \circ \eta_8 \circ \epsilon_9 = 0$ and $\nu_{14}^*(\eta_5 \circ \epsilon_6) = 0$. Thus, $\mathrm{Ker}\nu_{14}^* = \mathbb{Z}_2^2\{\mu_5, \eta_5 \circ \epsilon_6\}$, which gives the short exact sequence

$$0 \to \mathbb{Z}_2\{\nu_5\circ\eta_8\circ\mu_9\}\oplus\mathbb{Z}_3\{\alpha_2(5)\} \xrightarrow{\Sigma^{10}p^*} [\Sigma^{10}\mathbb{H}P^2, S^5] \xrightarrow{\Sigma^{10}i^*} \mathbb{Z}_2^2\{\mu_5, \eta_5\circ\epsilon_6\} \to 0.$$

Since the extension $\overline{\eta_{13}}$ has order 2 [2, Proposition 4.1(4)], the extension $\epsilon_5\circ\overline{\eta_{13}}$ of $\epsilon_5\circ\eta_{13}(=\eta_5\circ\epsilon_6)$ also has order 2. We know that $H(\mu') = \mu_5$ [8] and $\mu'\circ\nu_{14} = 0$ by [4, Proposition (2.4)(1)]. Thus, we have an extension $\overline{\mu'} \in [\Sigma^{10}\mathbb{H}P^2, S^5]$. Denote $\overline{\mu_5} = H(\overline{\mu'})$. Suppose that $[\Sigma^{10}\mathbb{H}P^2, S^5]$ is isomorphic to $\mathbb{Z}_4 \oplus \mathbb{Z}_2$. We know that the extension $\epsilon_5 \circ \overline{\eta_{13}}$ of $\epsilon_5 \circ \eta_{13}$ is of order 2. Thus, we have $\nu_5 \circ \eta_8 \circ \mu_9 \circ \Sigma^{10}p = 2\overline{\mu_5}$. Then, $0 = \triangle \circ H(2\overline{\mu'}) = \triangle(2H(\overline{\mu'})) = \triangle(2\overline{\mu_5}) = \triangle(\nu_5 \circ \eta_8 \circ \mu_9 \circ \Sigma^{10}p) = \triangle(\nu_5) \circ \eta_6 \circ \mu_7 \circ \Sigma^8 p = \eta_2 \circ \nu' \circ \eta_6 \circ \mu_7 \circ \Sigma^8 p$ by [8]. This is a contradiction, since $\eta_2 \circ \nu' \circ \eta_6 \circ \mu_7 \circ \Sigma^8 p$ has order 2. Thus, the short exact sequence splits. □

Proposition 12. (1) $[\Sigma^{11}\mathbb{H}P^2, S^6] = \mathbb{Z}_2^3\{\overline{\nu_6^3}, \overline{\mu_6}, \epsilon_6 \circ \overline{\eta_{14}}\} \oplus \mathbb{Z}_3\{\alpha_1(6) \circ \beta_1(9) \circ \Sigma^{11}p\}$.

(2) $[\Sigma^{12}\mathbb{H}P^2, S^7] = \mathbb{Z}_2^4\{\sigma' \circ \overline{\eta_{14}^2}, \overline{\nu_7^3}, \overline{\mu_7}, \epsilon_7 \circ \overline{\eta_{15}}\} \oplus \mathbb{Z}_3\{\alpha_1(7) \circ \beta_1(10) \circ \Sigma^{12}p\}$.

(3) $[\Sigma^{13}\mathbb{H}P^2, S^8] = \mathbb{Z}_2^5\{\sigma_8 \circ \overline{\eta_{15}^2}, (E\sigma') \circ \overline{\eta_{15}^2}, \overline{\nu_8^3}, \overline{\mu_8}, \epsilon_8 \circ \overline{\eta_{15}}\} \oplus \mathbb{Z}_3\{\alpha_1(8) \circ \beta_1(11) \circ \Sigma^{13}p\}$.

(4) $[\Sigma^{14}\mathbb{H}P^2, S^9] = \mathbb{Z}_2^4\{\sigma_9 \circ \overline{\eta_{16}^2}, \overline{\nu_9^3}, \overline{\mu_9}, \epsilon_9 \circ \overline{\eta_{16}}\} \oplus \mathbb{Z}_3\{\alpha_1(9) \circ \beta_1(12) \circ \Sigma^{14}p\}$.

(5) $[\Sigma^{15}\mathbb{H}P^2, S^{10}] = \mathbb{Z}\{\overline{4\triangle(\iota_{21})}\}\oplus\mathbb{Z}_2^3\{\overline{\nu_{10}^3}, \overline{\mu_{10}}, \epsilon_{10}\circ\overline{\eta_{18}}\}\oplus\mathbb{Z}_3\{\alpha_1(10)\circ\beta_1(13)\circ \Sigma^{10}p\}$.

Proof. (1) Consider the $(11,6)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{16}^* \xrightarrow{\Sigma^{11}p^*} [\Sigma^{11}\mathbb{H}P^2, S^6] \xrightarrow{\Sigma^{11}i^*} \mathrm{Ker}\nu_{15}^* \to 0,$$

where $\nu_{16}^* : \pi_{16}(S^6) \to \pi_{19}(S^6)$ and $\nu_{15}^* : \pi_{15}(S^6) \to \pi_{18}(S^6)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{16}^* : \mathbb{Z}_8\{\nu_6\circ\sigma_9\}\oplus\mathbb{Z}_2\{\eta_6\circ\mu_7\}\oplus\mathbb{Z}_9\{\beta_1(6)\} \to \mathbb{Z}_2\{\nu_6\circ\sigma_9\circ\nu_{16}\}\oplus\mathbb{Z}_3\{\alpha_1(6)\circ\beta_1(9)\}$$

and

$$\nu_{15}^* : \mathbb{Z}_2^3\{\nu_6^3, \mu_6, \eta_6 \circ \epsilon_7\} \to \mathbb{Z}_{16}\{\triangle(\sigma_{13})\} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5,$$

respectively. Then, we have $\nu_{16}^*(\nu_6 \circ \sigma_9) = \nu_6 \circ \sigma_9 \circ \nu_{16}$ and $\nu_{16}^*(\eta_6 \circ \mu_7) = 0$. Thus, we have $\mathrm{Coker}\nu_{16}^* = \mathbb{Z}_3\{\alpha_1(6) \circ \beta_1(9)\}$. Moreover, since $\nu_{15}^*(\nu_6^3) = 0$, $\nu_{15}^*(\mu_6) = 0$ and $\nu_{15}^*(\eta_6 \circ \epsilon_7) = 0$, we have $\mathrm{Ker}\nu_{15}^* = \mathbb{Z}_2^3\{\nu_6^3, \mu_6, \eta_6 \circ \epsilon_7\}$. Thus, we have a split short exact sequence

$$0 \to \mathbb{Z}_3\{\alpha_1(6) \circ \beta_1(9)\} \xrightarrow{\Sigma^{11}p^*} [\Sigma^{11}\mathbb{H}P^2, S^6] \xrightarrow{\Sigma^{11}i^*} \mathbb{Z}_2^3\{\nu_6^3, \mu_6, \eta_6 \circ \epsilon_7\} \to 0.$$

(2) Consider the $(12,7)$-type short exact sequence

$$0 \to \mathrm{Coker}\nu_{17}^* \xrightarrow{\Sigma^{12}p^*} [\Sigma^{12}\mathbb{H}P^2, S^7] \xrightarrow{\Sigma^{12}i^*} \mathrm{Ker}\nu_{16}^* \to 0,$$

where $\nu_{17}^* : \pi_{17}(S^7) \to \pi_{20}(S^7)$ and $\nu_{16}^* : \pi_{16}(S^7) \to \pi_{19}(S^7)(\cong 0)$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. ν_{17}^* can be restated as follows:

$$\nu_{17}^* : \mathbb{Z}_8\{\nu_7\circ\sigma_{10}\}\oplus\mathbb{Z}_2\{\eta_7\circ\mu_8\}\oplus\mathbb{Z}_3\{\beta_1(7)\}\to\mathbb{Z}_2\{\nu_7\circ\sigma_{10}\circ\nu_{17}\}\oplus\mathbb{Z}_3\{\alpha_1(7)\circ\beta_1(10)\}.$$

Then, we have $\nu_{17}^*(\nu_7\circ\sigma_{10}) = \nu_7\circ\sigma_{10}\circ\nu_{17}$ and $\nu_{17}^*(\eta_7\circ\mu_8) = 0$. Hence, $\mathrm{Coker}\nu_{17}^* = \mathbb{Z}_3\{\alpha_1(7)\circ\beta_1(10)\}$ and $\mathrm{Ker}\nu_{16}^* = \pi_{16}(S^7) = \mathbb{Z}_2^4\{\sigma'\circ\eta_{14}^2, \nu_7^3, \mu_7, \eta_7\circ\epsilon_8\}$. Thus, we have a split short exact sequence

$$0\to\mathbb{Z}_3\{\alpha_1(7)\circ\beta_1(10)\}\xrightarrow{\Sigma^{11}p^*}[\Sigma^{11}\mathbb{H}P^2, S^6]\xrightarrow{\Sigma^{11}i^*}\mathbb{Z}_2^4\{\sigma'\circ\eta_{14}^2, \nu_7^3, \mu_7, \eta_7\circ\epsilon_8\}\to 0.$$

(3) Consider the $(13, 8)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{18}^* \xrightarrow{\Sigma^{13}p^*} [\Sigma^{13}\mathbb{H}P^2, S^8] \xrightarrow{\Sigma^{13}i^*} \mathrm{Ker}\nu_{17}^* \to 0,$$

where $\nu_{18}^* : \pi_{18}(S^8) \to \pi_{21}(S^8)$ and $\nu_{17}^* : \pi_{17}(S^8) \to \pi_{20}(S^8) \cong 0$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. ν_{18}^* can be restated as follows:

$$\nu_{18}^* : \mathbb{Z}_8^2\{\sigma_8\circ\nu_{15}, \nu_8\circ\sigma_{11}\}\oplus\mathbb{Z}_2\{\eta_8\circ\mu_9\}\oplus\mathbb{Z}_3^2\to$$
$$\mathbb{Z}_2^2\{\sigma_8\circ\nu_{15}^2, \nu_8\circ\sigma_{11}\circ\nu_{18}\}\oplus\mathbb{Z}_3\{\alpha_1(8)\circ\beta_1(11)\}.$$

Then, we have $\nu_{18}^*(\sigma_8\circ\nu_{15}) = \sigma_8\circ\nu_{15}^2$, $\nu_{18}^*(\nu_8\circ\sigma_{11}) = \nu_8\circ\sigma_{11}\circ\nu_{18}$, and $\nu_{18}^*(\eta_8\circ\mu_9) = 0$. Thus, we have $\mathrm{Coker}\nu_{18}^* = \mathbb{Z}_3\{\alpha_1(8)\circ\beta_1(11)\}$ and $\mathrm{Ker}\nu_{17}^* = \pi_{17}(S^8) = \mathbb{Z}_2^5\{\sigma_8\circ\eta_{15}^2, (E\sigma')\circ\eta_{15}^2, \nu_8^3, \mu_8, \eta_8\circ\epsilon_9\}$. Therefore, we have a split short exact sequence

$$0 \to \mathbb{Z}_3\{\alpha_1(8)\circ\beta_1(11)\} \xrightarrow{\Sigma^{11}p^*} [\Sigma^{11}\mathbb{H}P^2, S^6] \xrightarrow{\Sigma^{11}i^*}$$
$$\xrightarrow{\Sigma^{11}i^*} \mathbb{Z}_2^5\{\sigma_8\circ\eta_{15}^2, (E\sigma')\circ\eta_{15}^2, \nu_8^3, \mu_8, \eta_8\circ\epsilon_9\} \to 0.$$

(4) Consider the $(14, 9)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{19}^* \xrightarrow{\Sigma^{14}p^*} [\Sigma^{14}\mathbb{H}P^2, S^9] \xrightarrow{\Sigma^{14}i^*} \mathrm{Ker}\nu_{18}^* \to 0,$$

where $\nu_{19}^* : \pi_{19}(S^9) \to \pi_{22}(S^9)$ and $\nu_{18}^* : \pi_{18}(S^9) \to \pi_{21}(S^9) \cong 0$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. ν_{19}^* can be restated as follows:

$$\nu_{19}^* : \mathbb{Z}_8\{\sigma_9\circ\nu_{16}\}\oplus\mathbb{Z}_2\{\eta_9\circ\mu_{10}\}\to\mathbb{Z}_2\{\sigma_9\circ\nu_{16}^2\}\oplus\mathbb{Z}_3\{\alpha_1(9)\circ\beta_1(12)\}.$$

Then, we have $\nu_{19}^*(\sigma_9\circ\nu_{16}) = \sigma_9\circ\nu_{16}^2$ and $\nu_{18}^*(\eta_9\circ\mu_{10}) = 0$. Hence, $\mathrm{Coker}\nu_{19}^* = \mathbb{Z}_3\{\alpha_1(9)\circ\beta_1(12)\}$ and $\mathrm{Ker}\nu_{18}^* = \pi_{18}(S^9) = \mathbb{Z}_2^4\{\sigma_9\circ\eta_{16}^2, \nu_9^3, \mu_9, \eta_9\circ\epsilon_{10}\}$. Thus, we have a split short exact sequence

$$0 \to \mathbb{Z}_3\{\alpha_1(9)\circ\beta_1(12)\} \xrightarrow{\Sigma^{11}p^*} [\Sigma^{14}\mathbb{H}P^2, S^9] \xrightarrow{\Sigma^{11}i^*}$$
$$\xrightarrow{\Sigma^{14}i^*} \mathbb{Z}_2^4\{\sigma_9\circ\eta_{16}^2, \nu_9^3, \mu_9, \eta_9\circ\epsilon_{10}\} \to 0.$$

(5) Consider the $(15, 9)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{20}^* \xrightarrow{\Sigma^{15}p^*} [\Sigma^{15}\mathbb{H}P^2, S^{10}] \xrightarrow{\Sigma^{15}i^*} \mathrm{Ker}\nu_{19}^* \to 0,$$

where $\nu_{20}^* : \pi_{20}(S^{10}) \to \pi_{23}(S^{10})$ and $\nu_{19}^* : \pi_{19}(S^{10}) \to \pi_{22}(S^{10})$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows: $\nu_{20}^* : \mathbb{Z}_4\{\sigma_{10} \circ \nu_{17}\} \oplus \mathbb{Z}_2\{\eta_{10} \circ \mu_{11}\} \oplus \mathbb{Z}_3\{\beta_1(10)\} \to \mathbb{Z}_2\{\sigma_{10} \circ \nu_{17}^2\} \oplus \mathbb{Z}_3\{\alpha_1(10) \circ \beta_1(13)\}$ and

$$\nu_{19}^* : \mathbb{Z}\{\triangle(\iota_{21})\} \oplus \mathbb{Z}_2^3\{\nu_{10}^3, \mu_{10}, \eta_{10} \circ \epsilon_{11}\} \to \mathbb{Z}_4\{\triangle(\nu_{21})\} \oplus \mathbb{Z}_3.$$

Then, we have $\nu_{20}^*(\sigma_{10} \circ \nu_{17}) = \sigma_{10} \circ \nu_{17}^2$ and $\nu_{20}^*(\eta_{10} \circ \mu_{11}) = 0$. Thus, we have $\mathrm{Coker}\nu_{20}^* = \mathbb{Z}_3\{\alpha_1(10) \circ \beta_1(13)\}$.

In addition, we have $\nu_{19}^*(\triangle(\iota_{21})) = \triangle(\iota_{21}) \circ \nu_{19} = \triangle(\nu_{21})$, $\nu_{19}^*(\nu_{10}^3) = 0$, $\nu_{19}^*(\mu_{10}) = 0$, and $\nu_{19}^*(\eta_{10} \circ \epsilon_{11}) = 0$. Thus, we have $\mathrm{Ker}\nu_{19}^* = \mathbb{Z}\{4\triangle(\iota_{21})\} \oplus \mathbb{Z}_2^3\{\nu_{10}^3, \mu_{10}, \eta_{10} \circ \epsilon_{11}\}$. Therefore, we have a split short exact sequence

$$0 \to \mathbb{Z}_3\{\alpha_1(10) \circ \beta_1(13)\} \xrightarrow{\Sigma^{15} p^*} [\Sigma^{15}\mathbb{H}P^2, S^{10}] \xrightarrow{\Sigma^{15} i^*}$$
$$\xrightarrow{\Sigma^{15} i^*} \mathbb{Z}\{4\triangle(\iota_{21})\} \oplus \mathbb{Z}_2^3\{\nu_{10}^3, \mu_{10}, \eta_{10} \circ \epsilon_{11}\} \to 0. \qquad \square$$

Proposition 13. (1) $[\Sigma^{16}\mathbb{H}P^2, S^{11}] = \mathbb{Z}_2^4\{\theta' \circ \eta_{23} \circ \Sigma^{16}, \overline{\nu_{11}^3}, \overline{\mu_{11}}, \eta_{11} \circ \overline{\epsilon_{12}}\} \oplus \mathbb{Z}_3\{\alpha_1(11) \circ \beta_1(14) \circ \Sigma^{16}p\}$.

(2) $[\Sigma^{17}\mathbb{H}P^2, S^{12}] = \mathbb{Z}_4^2\{\overline{\nu_{12}^3}, \overline{\mu_{12}}\} \oplus \mathbb{Z}_2\{\epsilon_{12} \circ \overline{\eta_{13}}\} \oplus \mathbb{Z}_3\{\alpha_1(12) \circ \beta_1(15) \circ \Sigma^{17}p\}$.

(3) $[\Sigma^{18}\mathbb{H}P^2, S^{13}] = \mathbb{Z}_4\{\overline{\nu_{13}^3}\} \oplus \mathbb{Z}_2^2\{\overline{\mu_{13}}, \epsilon_{13} \circ \overline{\eta_{21}}\} \oplus \mathbb{Z}_3\{\alpha_1(13) \circ \beta_1(16) \circ \Sigma^{18}p\}$.

(4) $[\Sigma^{19}\mathbb{H}P^2, S^{14}] = \mathbb{Z}\{\triangle(\iota_{29}) \circ \Sigma^{19}p\} \oplus \mathbb{Z}_2^3\{\overline{\nu_{14}^3}, \overline{\mu_{14}}, \epsilon_{14} \circ \overline{\eta_{22}}\} \oplus \mathbb{Z}_3\{\alpha_1(14) \circ \beta_1(17) \circ \Sigma^{19}p\}$.

(5) $[\Sigma^{n+5}\mathbb{H}P^2, S^n] = \mathbb{Z}_2^3\{\overline{\nu_n^3}, \overline{\mu_n}, \epsilon_n \circ \overline{\eta_{n+8}}\} \oplus \mathbb{Z}_3\{\alpha_1(n) \circ \beta_1(n+3) \circ \Sigma^{n+5}p\}$ *for* $n \geq 15$.

Proof. (1) Consider the $(16, 11)$-type short exact sequence:

$$0 \to \mathrm{Coker}\nu_{21}^* \xrightarrow{\Sigma^{16} p^*} [\Sigma^{16}\mathbb{H}P^2, S^{11}] \xrightarrow{\Sigma^{16} i^*} \mathrm{Ker}\nu_{20}^* \to 0,$$

where $\nu_{21}^* : \pi_{21}(S^{11}) \to \pi_{24}(S^{11})$ and $\nu_{20}^* : \pi_{20}(S^{11}) \to \pi_{23}(S^{11})$ are the homomorphisms induced originally by the Hopf fibration $\nu_4 : S^7 \to S^3$. These homomorphisms can be restated as follows:

$$\nu_{21}^* : \mathbb{Z}_2^2\{\sigma_{11} \circ \nu_{18}, \eta_{11} \circ \mu_{12}\} \to \mathbb{Z}_2^2\{\theta' \circ \eta_{23}, \sigma_{11} \circ \nu_{18}^2\} \oplus \mathbb{Z}_3\{\alpha_1(11) \circ \beta_1(14)\}$$

and

$$\nu_{20}^* : \mathbb{Z}_2^3\{\nu_{11}^3, \mu_{11}, \eta_{11} \circ \epsilon_{12}\} \to \mathbb{Z}_2\{\theta'\},$$

respectively. Then, we have $\nu_{21}^*(\sigma_{11} \circ \nu_{18}) = \sigma_{11} \circ \nu_{18}^2$ and $\nu_{21}^*(\eta_{11} \circ \mu_{12}) = 0$. Thus, we have $\mathrm{Coker}\nu_{21}^* = \mathbb{Z}_2\{\theta' \circ \eta_{23}\} \oplus \mathbb{Z}_3\{\alpha_1(11) \circ \beta_1(14)\}$. Moreover, since $\nu_{20}^*(\nu_{11}^3) = 0$, $\nu_{20}^*(\mu_3) = 0$, and $\nu_{20}^*(\eta_{11} \circ \epsilon_{12}) = 0$, we have $\mathrm{Ker}\nu_{20}^* = \mathbb{Z}_2^3\{\nu_{11}^3, \mu_{11}, \eta_{11} \circ \epsilon_{12}\}$. Therefore, we have the short exact sequence

$$0 \to \mathbb{Z}_2\{\theta' \circ \eta_{23}\} \oplus \mathbb{Z}_3\{\alpha_1(11) \circ \beta_1(14)\} \xrightarrow{\Sigma^{16} p^*} [\Sigma^{16}\mathbb{H}P^2, S^{11}] \xrightarrow{\Sigma^{16} i^*}$$
$$\xrightarrow{\Sigma^{16} i^*} \mathbb{Z}_2^3\{\nu_{11}^3, \mu_{11}, \eta_{11} \circ \epsilon_{12}\} \to 0.$$

We now consider a generalized EHP-sequence

$$[\Sigma^{17}\mathbb{H}P^2, S^{21}]_{(2)} \xrightarrow{\triangle} [\Sigma^{15}\mathbb{H}P^2, S^{10}]_{(2)} \xrightarrow{E} [\Sigma^{16}\mathbb{H}P^2, S^{11}]_{(2)} \xrightarrow{H} [\Sigma^{16}\mathbb{H}P^2, S^{21}]_{(2)},$$

where $[\Sigma^{17}\mathbb{H}P^2, S^{21}]_{(2)} = \mathbb{Z}\{\overline{8\iota_{21}}\}$, $[\Sigma^{16}\mathbb{H}P^2, S^{21}]_{(2)} = 0$, and

$$[\Sigma^{15}\mathbb{H}P^2, S^{10}]_{(2)} = \mathbb{Z}\{\overline{4\triangle(\iota_{21})}\} \oplus \mathbb{Z}_2^3\{\overline{\nu_{10}^3}, \overline{\mu_{10}}, \epsilon_{10} \circ \overline{\eta_{18}}\}$$

by [7, Theorem 4.5, Theorem 4.15] and Proposition 12(5). Clearly, $\triangle(\overline{8\iota_{21}}) = 2 \cdot \overline{4\triangle(\iota_{21})}$. Thus, we have

$$\begin{aligned}[\Sigma^{16}\mathbb{H}P^2, S^{11}]_{(2)} &\cong [\Sigma^{15}\mathbb{H}P^2, S^{10}]_{(2)}/KerE \\ &= [\Sigma^{15}\mathbb{H}P^2, S^{10}]_{(2)}/Im\triangle = \mathbb{Z} \oplus \mathbb{Z}_2^3/2\mathbb{Z} \cong \mathbb{Z}_2^4.\end{aligned}$$

Therefore, the above short exact sequence splits.

(2) Since two homomorphisms $\nu_{22}^* : \pi_{22}(S^{12}) \to \pi_{25}(S^{12})$ and $\nu_{21}^* : \pi_{21}(S^{12}) \to \pi_{24}(S^{12})$ are trivial, we obtain the following short exact sequence from the $(17, 12)$-type short exact sequence:

$$0 \to \mathbb{Z}_2^2\{\theta \circ \eta_{24}, (E\theta') \circ \eta_{24}\} \oplus \mathbb{Z}_3\{\alpha_1(12) \circ \beta_1(15)\} \xrightarrow{\Sigma^{17}p^*} [\Sigma^{17}\mathbb{H}P^2, S^{12}] \xrightarrow{\Sigma^{17}i^*}$$
$$\xrightarrow{\Sigma^{17}i^*} \mathbb{Z}_2^3\{\nu_{12}^3, \mu_{12}, \eta_{12} \circ \epsilon_{13}\} \to 0.$$

Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z}_2\{\theta' \circ \eta_{23}\} & \xrightarrow{\Sigma^{16}p^*} & [\Sigma^{16}\mathbb{H}P^2, S^5]_{(2)} & \xrightarrow{\Sigma^{16}i^*} & \mathbb{Z}_2^3\{\nu_{11}^3, \mu_{11}, \eta_{11} \circ \epsilon_{12}\} & \longrightarrow & 0 \\
 & & \downarrow \Sigma_1 & & \downarrow \Sigma_2 & & \downarrow \Sigma_3 & & \\
0 & \longrightarrow & \mathbb{Z}_2^2\{\theta \circ \eta_{24}, (E\theta') \circ \eta_{24}\} & \xrightarrow{\Sigma^{17}p^*} & [\Sigma^{17}\mathbb{H}P^2, S^{12}]_{(2)} & \xrightarrow{\Sigma^{17}i^*} & \mathbb{Z}_2^3\{\nu_{12}^3, \mu_{12}, \eta_{12} \circ \epsilon_{13}\} & \longrightarrow & 0
\end{array}$$

Since the first row splits, the second row also splits.

(3) By using the same approach as in (2), we obtain the following split short exact sequence from $(18, 13)$-type short exact sequence:

$$0 \to \mathbb{Z}_2\{(E\theta) \circ \eta_{25}\} \oplus \mathbb{Z}_3\{\alpha_1(13) \circ \beta_1(16)\} \xrightarrow{\Sigma^{18}p^*} [\Sigma^{18}\mathbb{H}P^2, S^{13}] \xrightarrow{\Sigma^{18}i^*}$$
$$\xrightarrow{\Sigma^{18}i^*} \mathbb{Z}_2^3\{\nu_{13}^3, \mu_{13}, \eta_{13} \circ \epsilon_{14}\} \to 0.$$

(4) By using the same approach as in (2) and (3), we obtain the following split short exact sequence from the $(19, 14)$-type short exact sequence:

$$0 \to \mathbb{Z}\{\triangle(\iota_{29})\} \oplus \mathbb{Z}_3\{\alpha_1(14) \circ \beta_1(17)\} \xrightarrow{\Sigma^{18}p^*} [\Sigma^{18}\mathbb{H}P^2, S^{13}] \xrightarrow{\Sigma^{18}i^*}$$
$$\xrightarrow{\Sigma^{18}i^*} \mathbb{Z}_2^3\{\nu_{14}^3, \mu_{14}, \eta_{14} \circ \epsilon_{15}\} \to 0.$$

(5) Since two homomorphisms $\nu_{25}^* : \pi_{25}(S^{15}) \to \pi_{28}(S^{15})$ and $\nu_{24}^* : \pi_{24}(S^{15}) \to \pi_{27}(S^{15}) \cong 0$ are trivial, we obtain the following split short exact sequence from the $(20, 15)$-type short exact sequence:

$$0 \to \mathbb{Z}_3\{\alpha_1(15)\circ\beta_1(18)\} \xrightarrow{\Sigma^{18}p^*} [\Sigma^{20}\mathbb{H}P^2, S^{15}] \xrightarrow{\Sigma^{18}i^*} \mathbb{Z}_2^3\{\nu_{15}^3, \mu_{15}, \eta_{15}\circ\epsilon_{16}\} \to 0.$$

The suspension homomorphism $\Sigma : [\Sigma^{n+5}\mathbb{H}P^2, S^n] \to [\Sigma^{n+5}\mathbb{H}P^2, S^{n+1}]$ is isomorphic for $n + 13 < 2n - 1$ (that is, $14 < n$), since $\Sigma^{n+5}\mathbb{H}P^2$ is an $(n + 13)$-dimensional CW complex and S^n is $(n - 1)$-connected. Thus, we have the desired result. □

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Jin Ho Lee
Department of Mathematics
Korea University
Seoul 136-701, Korea
E-mail address: sabforev@korea.ac.kr

Kee Young Lee
Department of Mathematics
Korea University
Sejong City 339-700, Korea
E-mail address: keyolee@korea.ac.kr