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CONVERGENCE RATE OF EXTREMES FOR THE GENERALIZED SHORT-TAILED SYMMETRIC DISTRIBUTION

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Abstract. Denote M_n the maximum of n independent and identically distributed variables from the generalized short-tailed symmetric distribution. This paper shows the pointwise convergence rate of the distribution of M_n to $\exp(-e^{-x})$ and the supremum-metric-based convergence rate as well.

1. Introduction

Systematic introductions and reviews of the extreme value theory (EVT) can be found in Leadbetter et al. [11], Resnick [20] and recently in Embrechts et al. [5], Kotz and Nadarajah [10], Falk et al. [6] and De Haan and Ferreira [4], amongst others. Many fields of science, with the field of finance as the most important proponent, have found non-normal data and more than seventy years after the Extremal Types Theorem was proved in complete generality by Gnedenko [7], the exploration of limit distributions of non-normal extremes is as momentous and timely as ever. Peng et al. [9] considered the convergence of extremes of the general error distribution and for more results of non-normal extremes, we refer the reader to Lin and Peng [16], Lin and Jiang [15] and Liao et al. [13]. Since asymptotic results are used in most EVT applications, such as estimating the VaR in modern risk management, and the estimation precision need to be assessed, we have to face another interesting problem, i.e., examining the convergence rate of extremes. With regard to general sequences, several authors considered the uniform convergence rate of their extremes; for instance, De Haan and Resnick [4] and Cheng and Jiang [2]. Particularly, Hall and Wellner [9] and Hall [8] studied the cases of exponential and normal distributions extremes, respectively and for other related results, we refer the

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reader to Peng et al. [18], Lin et al. [17], Chen et al. [1], Liao and Peng [12] and Liao et al. [14].

Recently, Tiku and Vaughan [21] introduced a short-tailed symmetric distribution (STSD for short) family whose density is given by

$$s_r(x) = \frac{D}{\sqrt{2\pi}}\Big\{1+\frac{\lambda}{2r}x^2\Big\}^r \exp\Big\{-\frac{x^2}{2}\Big\},$$

where r is positive, $\lambda = r/(r-a)$ with $a<r$, and constant

$$D^{-1} = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\Big\{1+\frac{\lambda}{2r}x^2\Big\}^r \exp\Big\{-\frac{x^2}{2}\Big\}dx.$$

Lin and Jiang [15] generalized the STSD to the version of the family of 4-parameters, the generalized short-tailed symmetric distribution (GSTSD for short), with the probability density function

$$F'(x) = \frac{A}{\sqrt{2\pi}}\Big\{1+\frac{\lambda}{2h}|x|^p\Big\}^h \exp\Big\{-\frac{|x|^r}{q}\Big\}. \tag{1.1}$$

Here, p,q,r and h are positive, $\lambda = h/(h-a)$ with $a<h$, $F(x)$ is the cumulative distribution function of GSTSD and constant

$$A^{-1} = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}\Big\{1+\frac{\lambda}{2h}|x|^p\Big\}^h \exp\Big\{-\frac{|x|^r}{q}\Big\}dx.$$

Throughout the paper, let $\{X_k, k\geq 1\}$ be a sequence of independent and identically distributed random variables from GSTSD and $M_n = \max\{X_k, 1\leq k\leq n\}$. Lin and Jiang [15] proved that

$$\lim_{n\to\infty} P(M_n \leq a_n x + b_n) = \exp(-e^{-x}) =: \Lambda(x), \tag{1.2}$$

where norming constants $a_n = \alpha_n$ and $b_n = \beta_n$ with

$$\alpha_n = \frac{q^{1/r}}{r(\log n)^{1-1/r}},$$

$$\beta_n = q^{1/r}\Big((\log n)^{1/r} - \frac{\Delta}{r(\log n)^{1-1/r}}\Big), \tag{1.3}$$

where (and in the sequel) $\Delta = \frac{r-1}{r}\log(q\log n) - h\log(1+\frac{\lambda}{2h}(q\log n)^{p/r}) - \log(\frac{Aq}{r\sqrt{2\pi}})$. They presented that the upper tail of GSTSD is exponentially delaying and satisfies

$$1-F(x) = c(x)\exp\Big(-\int_1^x \frac{g(t)}{f(t)}dt\Big)$$

for $1<p\leq r$ and sufficiently large x, where

$$c(x) \to \frac{Aq}{r\sqrt{2\pi}} e^{-1/2}(2/q)^{1/2} \cdot \exp\left(-h\ln(1+\lambda/2h) + \int_1^{(q/2)^{1/r}} \frac{r}{q}y^{r-1}\Big(1+\frac{q}{2y^r}\Big)dy\right) \text{ as } x\to\infty,$$

$f(t) = t/(rt^r/q - 1 + r)$

and

$$g(t) = 1 - \frac{\frac{\lambda p t^{p-1}/2}{1+\lambda t^p/2h}}{(rt^r/q - 1 + r)/t}.$$

For convenience, we give instead the following expressions of $f(t)$ and $g(t)$:

$$f(t) = qt^{1-r}/r,$$
$$g(t) = 1 + \frac{(r-1)/t - (\frac{\lambda p t^{p-1}}{2})/(1 + \frac{\lambda}{2h}t^p)}{rt^{r-1}/q}.$$

By Proposition 1.1(a) and Corollary 1.7 in Resnick [20], a natural way to gain the expressions of norming constants a_n and b_n is to solve the following equations

$$ra_n b_n^{r-1}/q = 1, \quad \text{and} \quad \frac{Aqn}{r\sqrt{2\pi}}\Big(1 + \frac{\lambda}{2h}b_n^p\Big)^h \exp\Big(-\frac{b_n^r}{q}\Big) = b_n^{r-1}. \tag{1.4}$$

But exact solutions of equations in (1.4) are impossible to derive, so approximations are necessary, such as those in (1.3). One can easily check that

$$\alpha_n/a_n \to 1 \quad \text{and} \quad (\beta_n - b_n)/a_n \to 0.$$

In the following, a_n and b_n are defined by (1.4) unless otherwise specified. This paper is concerned with considering the convergence rates of (1.2) in two norming constants cases, separately. Like the case of normal extremes in Hall [8], the different norming constants can lead to the different convergence rate of GSTSD extremes, which will be explained later.

The remainder of the paper is organized as follows. In Section 2, main results and some remarks are presented. Section 3 contains some lemmas and proofs of main results are postponed in Section 4. In the sequel, Ç and $Ç_i$, $i = 1, 2, \ldots, 12$ are positive functions of h, λ, r, p and n_i, $i = 1, 2, \ldots, 6$ stand for positive integers whose value may be associated with parameters in (1.1).

2. Main results

The convergence rate of (1.2) may be considered generally by uniform, total variation and Lévy metrics. In fact uniform and Lévy metrics can be replaced by each other, see Remarks in Hall [8]. Here we mainly focus our minds on uniform metrics and the key results are as followed.

Theorem 2.1. *Let $\{\xi_n\}$ be a sequence of independent and identically distributed random variables from GSTSD with parameters satisfying $r > 2$, $r - 1 > (ph) \vee (2p)$ and $r > p > 1$. Then there exist positive constants $C_1(h, \lambda, r, p) < C_2(h, \lambda, r, p)$, independent of n, such that, for $n > n_1$,*

$$C_1(h, \lambda, r, p)/\log n < \sup_{-\infty < x < \infty} |F^n(a_n x + b_n) - \exp(-e^{-x})| < C_2(h, \lambda, r, p)/\log n. \tag{2.1}$$

Theorem 2.2. *Let α_n, β_n be defined in* (1.3)*. Then*

$$F^n(\alpha_n x + \beta_n) - \exp(-e^{-x}) \sim \exp(-e^{-x})e^{-x}\frac{(r-1)(r-1-hp)^2}{2r^3}\frac{(\log\log n)^2}{\log n}$$

for large n, where F is the distribution function of GSTSD with parameters satisfying $r > 2$, $r-1 > ph$ and $r \geq p > 1$.

Remark 2.1. Theorem 2.2 shows that with the norming constants in (1.3) the rate of convergence is no better than $(\log\log n)^2/\log n$. In fact, for $a_n^* = a_n r_n$ and $b_n^* = b_n + \delta_n a_n$ in equation (3.7), Lemma 3.2 and the proof of Theorem 2.2 together imply that the pointwise convergence rate of $F^n(a_n^* x + b_n^*)$ to the corresponding limit is slower than $1/\log n$ and approaches to it as the expansions for a_n^* and b_n^* are made indefinitely.

Remark 2.2. For $h = 0, r = 2$ and $q = 2$, GSTSD reduces to the standard normal distribution and hence a better result can be obtained, i.e., $C_2(h, \lambda, r, p)$ is less than 3 in (2.1), see Hall [8].

3. Some technical lemmas

The following lemmas are used in the proofs of main results.

Lemma 3.1. *If $r \geq p > 1$ and $r - 1 > ph$ in* (1.1)*, then for $x > Ç$*

$$\begin{aligned} 1 - F(x) &= \frac{Aqx^{1-r}}{r\sqrt{2\pi}}\Big\{1 + \frac{\lambda}{2h}x^p\Big\}^h \exp\Big\{-\frac{x^r}{q}\Big\} - r(x) \\ &= \frac{Aq}{r\sqrt{2\pi}}x^{1-r}\Big(1 + \frac{\lambda}{2h}x^p\Big)^{h-1}\exp(-x^r/q) \\ &\quad \times \Big(\Big(1 + \frac{\lambda}{2h}x^p\Big)\Big(1 - \frac{q(r-1)}{r}x^{-r}\Big) + \frac{\lambda pq}{2r}x^{p-r}\Big) + s(x), \end{aligned} \tag{3.1}$$

where

$$\begin{aligned} 0 < r(x) < &\frac{Aq^2}{r^2\sqrt{2\pi}}\Big((r-1)x^{1-2r} + \frac{\lambda(r-1) - \lambda ph}{2h}x^{1+p-2r}\Big) \\ &\times \Big(1 + \frac{\lambda}{2h}x^p\Big)^{h-1}\exp(-x^r/q), \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} 0 < s(x) < &\frac{Aq^3}{r^3\sqrt{2\pi}}(1 + \frac{\lambda}{2h}x^p)^{h-2}\exp(-x^r/q) \\ &\times \Big(\frac{\lambda^2(r-1-ph)(2r-1-ph)}{4h^2}x^{2p-3r+1} \\ &+ \Big(\frac{2\lambda r^2 - 3\lambda r + \lambda}{h} - \frac{3\lambda pr}{2} + \lambda p + \frac{\lambda p^2}{2}\Big)x^{p-3r+1} \\ &+ (r-1)(2r-1)x^{1-3r}\Big). \end{aligned} \tag{3.3}$$

Proof. By integration by parts, we have

$$1-F(x) = \int_x^{+\infty} \frac{A}{\sqrt{2\pi}}\Big\{1+\frac{\lambda}{2h}t^p\Big\}^h \exp\Big\{-t^r/q\Big\}dt$$
$$= \frac{Aq}{r\sqrt{2\pi}}x^{1-r}\Big\{1+\frac{\lambda}{2h}x^p\Big\}^h \exp\Big\{-\frac{x^r}{q}\Big\} - r(x). \tag{3.4}$$

Similarly,

$$r(x) = \int_x^{+\infty} \frac{Aqt^{-r}}{r\sqrt{2\pi}}\Big((r-1)+\frac{\lambda(r-1)-\lambda ph}{2h}t^p\Big)\Big(1+\frac{\lambda}{2h}t^p\Big)^{h-1}\exp(-t^r/q)dt$$
$$= \frac{Aq^2}{r^2\sqrt{2\pi}}\Big((r-1)x^{1-2r}+\frac{\lambda(r-1)-\lambda ph}{2h}x^{1+p-2r}\Big)$$
$$\Big(1+\frac{\lambda}{2h}x^p\Big)^{h-1}\exp(-x^r/q) - s(x). \tag{3.5}$$

Substituting the expression for $r(x)$ in (3.5) into equation (3.4) yields equation (3.1). Noting that $\frac{\lambda(r-1)-\lambda ph}{2h} > 0$ (the coefficient of t^p) in the integration in equation (3.5), there exists a positive constant $Ç_1(p,q,r,h)$ such that for $x > Ç_1(p,q,r,h)$, we have $r(x) > 0$. Similarly to arguments for $r(x)$ in equation (3.5), we furthermore have

$$s(x) = \int_x^{+\infty} \frac{Aq^2}{r^2\sqrt{2\pi}}\Big(1+\frac{\lambda}{2h}t^p\Big)^{h-2}\exp(-t^r/q)$$
$$\times\Big(\frac{\lambda^2(r-1-ph)(2r-1-ph)}{4h^2}t^{2p-2r}$$
$$+\Big(\frac{2\lambda r^2-3\lambda r+\lambda}{h}-\frac{3\lambda pr}{2}+\lambda p+\frac{\lambda p^2}{2}\Big)t^{p-2r}$$
$$+(r-1)(2r-1)t^{-2r}\Big)dt$$
$$= \frac{Aq^3}{r^3\sqrt{2\pi}}\Big(1+\frac{\lambda}{2h}x^p\Big)^{h-2}\exp(-x^r/q)$$
$$\times\Big(\frac{\lambda^2(r-1-ph)(2r-1-ph)}{4h^2}x^{2p-3r+1}$$
$$+\Big(\frac{2\lambda r^2-3\lambda r+\lambda}{h}-\frac{3\lambda pr}{2}+\lambda p+\frac{\lambda p^2}{2}\Big)x^{p-3r+1}$$
$$+(r-1)(2r-1)x^{1-3r}\Big)$$
$$+\int_x^{+\infty}\frac{Aq^3}{r^3\sqrt{2\pi}}\exp(-t^r/q)B(t)dt, \tag{3.6}$$

where

$$B(t) = \Big(1+\frac{\lambda}{2h}t^p\Big)^{h-3}\Big(\frac{\lambda^3(r-1-ph)(2r-1-hp)(ph-3r+1)}{8h^3}t^{3p-3r}$$
$$+\Big(\frac{\lambda^2(4r^2-6r+2-3prh+2ph+p^2h)(ph-p-3r+1)}{4h^2}$$

$$+\frac{\lambda^2(r-1-ph)(2r-1-hp)(2p-3r+1)}{4h^2}\Big)t^{2p-3r}$$
$$+\Big(\frac{\lambda(4r^2-6r+2-3prh+2ph+p^2h)(p-3r+1)}{2h}$$
$$+\frac{\lambda(ph-2p+1-3r)(r-1)(2r-1)}{2h}\Big)t^{p-3r}$$
$$-(r-1)(2r-1)(3r-1)t^{-3r}\Big).$$

Noting $\frac{\lambda^2(r-1-ph)(2r-1-ph)}{4h^2}>0$ (the coefficient of t^{2p-2r} in the integration in equation (3.6)) there also exists a positive constant $Ç_2(p,q,r,h)$ such that, for $x>Ç_2(p,q,r,h)$, we have $s(x)>0$. Hence the second inequality in (3.2) holds. Noting $\frac{\lambda^3(r-1-ph)(2r-1-hp)(ph-3r+1)}{8h^3}<0$, thus for $x>Ç_3(p,q,r,h)$ (the positive constant certainly exists), we obtain the second inequality in (3.3). Consequently, for $x>Ç=:\max(Ç_1(p,q,r,h),\ Ç_2(p,q,r,h),Ç_3(p,q,r,h))$, the desired results hold. □

Define new norming constants a_n^* and b_n^* such that

$$a_n^*=a_nr_n,\ b_n^*=b_n+\delta_na_n, \tag{3.7}$$

with $r_n\to1$ and $\delta_n\to0$ as $n\to\infty$. Write $a_n^*x+b_n^*=:t_n^*(x)$ in this section and we have the following result.

Lemma 3.2. *Suppose that parameters r, p and h satisfy Lemma 3.1. For fixed $x\in R$ and sufficiently large n such that $t_n^*(x)>Ç$, where $Ç$ satisfies Lemma 3.1, we have*

$$F^n(t_n^*(x))-\Lambda(x)$$
$$=\Lambda(x)e^{-x}(a_nb_n^{-1}((r-1)x^2/2-(ph-(r-1))x+r-1-ph)+(r_n-1)x+\delta_n$$
$$+O((a_nb_n^{-1})^2+(r_n-1)^2+\delta_n^2)).$$

Proof. According to Lemma 3.1, we have

$$\begin{aligned}&\frac{Aq}{r\sqrt{2\pi}}\Big(1+\frac{\lambda}{2h}(t_n^*(x))^p\Big)^h\exp\Big(-\frac{(t_n^*(x))^r}{q}\Big)(t_n^*(x))^{1-r}\\
&=\frac{Aq}{r\sqrt{2\pi}}\Big(1+\frac{\lambda}{2h}b_n^p\Big)^h\exp(-b_n^r/q)b_n^{1-r}(1+a_nb_n^{-1}(r_nx+\delta_n))^{1-r}\\
&\quad\times\Big(1+\frac{\lambda}{2h}(t_n^*(x))^p\Big)^h\exp\Big(-\frac{(t_n^*(x))^r}{q}\Big)(1+\frac{\lambda}{2h}b_n^p)^{-h}\exp(b_n^r/q)\\
&=n^{-1}e^{-x}\Big(1+(1-r)a_nb_n^{-1}((r_n-1)x+\delta_n)+(1-r)a_nb_n^{-1}x\\
&\quad-\frac{r(1-r)}{2}(a_nb_n^{-1}(r_nx+\delta_n))^2+O((a_nb_n^{-1}(r_nx+\delta_n))^3)\Big)\\
&\quad\times\Big(1+\frac{p\lambda a_nb_n^{p-1}((r_n-1)x+\delta_n)}{2(1+\frac{\lambda}{2h}b_n^p)}+\frac{p\lambda a_nb_n^{p-1}x}{2(1+\frac{\lambda}{2h}b_n^p)}+O((a_nb_n^{-1}(r_nx+\delta_n))^2)\Big)\end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&\times \exp\Big(- (r_n-1)x - \delta_n - \frac{r(r-1)}{2q} b_n^r (a_n b_n^{-1}(r_n x+\delta_n))^2 \\
&- \frac{b_n^r}{q} O((a_n b_n^{-1}(r_n x+\delta_n))^3)\Big) \\
= {}& n^{-1} e^{-x} \Big(1 + (1-r) a_n b_n^{-1}\Big(\frac{x^2}{2} + \Big(1+\frac{hp}{1-r}\Big)x\Big) - (r_n-1)x - \delta_n \\
&+ O(b_n^{-r-p} + (r_n-1)^2 + \delta_n^2)\Big).
\end{aligned}
$$

The second equality in (3.8) follows from the following arguments. Firstly, using the second equation of (1.4), we have

$$
\frac{Aq}{r\sqrt{2\pi}} \Big(1+\frac{\lambda}{2h} b_n^p\Big)^h \exp\Big(-\frac{b_n^r}{q}\Big) b_n^{1-r} = n^{-1}.
$$

Secondly, noting $a_n b_n^{-1}(r_n x+\delta_n) \to 0$, and using the expansions of $(t_n^*(x)))^r = (a_n b_n^{-1}(r_n x+\delta_n)+1)^r b_n^r$, $(1+a_n b_n^{-1}(r_n x+\delta_n))^{1-r}$ and $\big(\frac{1+\frac{\lambda}{2h} b_n^p (a_n b_n^{-1}(r_n x+\delta_n)+1)^p}{1+\frac{\lambda}{2h} b_n^p}\big)^h$ around 0. So some simple calculation can yield the desired result. The proof of the third equality in (3.8) is based on these expressions: $b_n^{-2r} = O(b_n^{-r-p} + (r_n-1)^2 + \delta_n^2)$, $b_n^{-r}\delta_n = O(b_n^{-r-p} + (r_n-1)^2 + \delta_n^2)$, $b_n^{-r}(r_n-1) = O(b_n^{-r-p} + (r_n-1)^2 + \delta_n^2)$. Write

$$
\begin{aligned}
&1 + (1-r) a_n b_n^{-1}((r_n-1)x+\delta_n) + (1-r) a_n b_n^{-1} x - \tfrac{r(1-r)}{2}(a_n b_n^{-1}(r_n x+\delta_n))^2 \\
&+ O((a_n b_n^{-1}(r_n x+\delta_n))^3) = 1 + (1-r) a_n b_n^{-1} x + O(b_n^{-r-p} + (r_n-1)^2 + \delta_n^2),
\end{aligned}
$$

and

$$
\begin{aligned}
&1 + \frac{p\lambda a_n b_n^{p-1}((r_n-1)x+\delta_n)}{2(1+\frac{\lambda}{2h} b_n^p)} + \frac{p\lambda a_n b_n^{p-1} x}{2(1+\frac{\lambda}{2h} b_n^p)} + O((a_n b_n^{-1}(r_n x+\delta_n))^2) \\
= {}& 1 + hp a_n b_n^{-1} x + O(b_n^{-r-p} + (r_n-1)^2 + \delta_n^2).
\end{aligned}
$$

Using $e^x = 1 + x + x^2/2 + o(x^3)$ as $x \to 0$ we further have

$$
\begin{aligned}
&\exp\Big(- (r_n-1)x - \delta_n - \frac{r(r-1)}{2q} b_n^r (a_n b_n^{-1}(r_n x+\delta_n))^2 \\
&\quad - \frac{b_n^r}{q} O((a_n b_n^{-1}(r_n x+\delta_n))^3)\Big) \\
= {}& 1 - (r_n-1)x - \delta_n - \frac{r-1}{2} a_n b_n^{-1} x^2 + O(b_n^{-r-p} + (r_n-1)^2 + \delta_n^2).
\end{aligned}
$$

So, an easy multiplication can result in the equality.

Similarly to the proof of (3.8),

$$
\begin{aligned}
&1 - \frac{q}{r} \frac{(r-1)(t_n^*(x))^{-r} + \frac{\lambda(r-1)-\lambda ph}{2h}(t_n^*(x))^{p-r}}{1+\frac{\lambda}{2h}(t_n^*(x))^p} \\
= {}& 1 - (r-1-ph) a_n b_n^{-1} + O(b_n^{-r-p}).
\end{aligned}
\tag{3.9}
$$

Note that

$$(t_n^*(x))^{-r} = \frac{r}{q} a_n b_n^{-1} + O((a_n b_n^{-1})^2),$$
$$\Big(1 + \frac{\lambda}{2h}(t_n^*(x))^p\Big)^{-2} = \Big((t_n^*(x))^p((t_n^*(x))^{-p} + \frac{\lambda}{2h})\Big)^{-2}.$$

Using (3.3), we have

$$\begin{aligned}
0 < s(t_n^*(x)) &< \frac{Aq^3}{r^3\sqrt{2\pi}}(1 + \frac{\lambda}{2h}(t_n^*(x))^p)^{h-2}\exp(-(t_n^*(x))^r/q)\\
&\cdot \Big(\frac{\lambda^2(r-1-ph)(2r-1-ph)}{4h^2}(t_n^*(x))^{2p-3r+1}\\
&+ \Big(\frac{2\lambda r^2 - 3\lambda r + \lambda}{h} - \frac{3\lambda pr}{2} + \lambda p + \frac{\lambda p^2}{2}\Big)(t_n^*(x))^{p-3r+1}\\
&+ (r-1)(2r-1)(t_n^*(x))^{1-3r}\Big)\\
&= \frac{Aq}{r\sqrt{2\pi}}(1 + \frac{\lambda}{2h}(t_n^*(x))^p)^h \exp\Big(-\frac{(t_n^*(x))^r}{q}\Big)(t_n^*(x))^{1-r} \times \frac{q^2}{r^2}\\
&\times (1 + \frac{\lambda}{2h}(t_n^*(x))^p)^{-2} \cdot \Big(\frac{\lambda^2(r-1-ph)(2r-1-ph)}{4h^2}(t_n^*(x))^{2p-2r}\\
&+ \Big(\frac{2\lambda r^2 - 3\lambda r + \lambda}{h} - \frac{3\lambda pr}{2} + \lambda p + \frac{\lambda p^2}{2}\Big)(t_n^*(x))^{p-2r}\\
&+ (r-1)(2r-1)(t_n^*(x))^{-2r}\Big)\\
&= \frac{Aq}{r\sqrt{2\pi}}(1 + \frac{\lambda}{2h}(t_n^*(x))^p)^h \exp\Big(-\frac{(t_n^*(x))^r}{q}\Big)(t_n^*(x))^{1-r} \times (t_n^*(x))^{-2p}\\
&\times \frac{q^2}{r^2}((t_n^*(x))^{-p} + \frac{\lambda}{2h})^{-2} \cdot \Big(\frac{\lambda^2(r-1-ph)(2r-1-ph)}{4h^2}(t_n^*(x))^{2p-2r}\\
&+ \Big(\frac{2\lambda r^2 - 3\lambda r + \lambda}{h} - \frac{3\lambda pr}{2} + \lambda p + \frac{\lambda p^2}{2}\Big)(t_n^*(x))^{p-2r}\\
&+ (r-1)(2r-1)(t_n^*(x))^{-2r}\Big).
\end{aligned}$$

Noting

$$\begin{aligned}
&\frac{q^2}{r^2}((t_n^*(x))^{-p} + \frac{\lambda}{2h})^{-2} \cdot \Big(\frac{\lambda^2(r-1-ph)(2r-1-ph)}{4h^2}(t_n^*(x))^{2p-2r}\\
&+\Big(\frac{2\lambda r^2 - 3\lambda r + \lambda}{h} - \frac{3\lambda pr}{2} + \lambda p + \frac{\lambda p^2}{2}\Big)(t_n^*(x))^{p-2r}\\
&+(r-1)(2r-1)(t_n^*(x))^{-2r}\Big)
\end{aligned}$$

is bounded, combining $(t_n^*(x))^{-2p} = \frac{2p}{q} a_n b_n^{-1} + O((a_n b_n^{-1})^2)$ and (3.8) yields

$$s(t_n^*(x)) = O(n^{-1}(a_n b_n^{-1})^2). \tag{3.10}$$

Combining (3.8), (3.9) and (3.10), it follows that

$$\begin{aligned}
&F^n(t_n^*(x)) - \Lambda(x) \\
&= \Big(1 - n^{-1}e^{-x}\Big(1-(r-1)a_nb_n^{-1}\Big(\frac{x^2}{2} + \Big(1+\frac{hp}{1-r}\Big)x + \frac{r-1-ph}{r-1}\Big) \\
&\quad - (r_n-1)x - \delta_n + O(b_n^{-r-p} + (r_n-1)^2 + \delta_n^2)\Big)\Big)^n - \Lambda(x) \\
&= \Lambda(x)e^{-x}\Big((r-1)a_nb_n^{-1}\Big(\frac{x^2}{2} + \Big(1+\frac{hp}{1-r}\Big)x + \frac{r-1-ph}{r-1}\Big) + (r_n-1)x \\
&\quad + \delta_n + O(b_n^{-r-p} + (r_n-1)^2 + \delta_n^2)\Big).
\end{aligned}$$

The proof is completed. □

Lemma 3.3. *Suppose $r-1 > 2p$ and $r > p > 1$. Then, there exists a positive integer n_6 such that, for $n > n_6$,*

$$\Big(\frac{1+\frac{\lambda}{2h}(a_nx+b_n)^p}{1+\frac{\lambda}{2h}b_n^p}\Big)^h \exp\Big(-\frac{r-1}{2}a_nb_n^{-1}x^2\Big) < 1 \text{ for } x > 0. \tag{3.11}$$

Proof. Write the left-hand side of (3.11) as

$$\exp\Big(-\frac{r-1}{2}a_nb_n^{-1}x^2 + h\log\Big(1+\frac{\lambda}{2h}(a_nx+b_n)^p\Big) - h\log\Big(1+\frac{\lambda}{2h}b_n^p\Big)\Big).$$

Noting that $a_nb_n^{-1} \sim 1/(2r\log n)$, then the inequality can be checked easily. □

4. The proofs of main results

For convenience, we first prove Theorem 2.2, which partially completes the proof of Theorem 2.1.

Proof of Theorem 2.2. Firstly, we prove the following asymptotic expansions of b_n:

$$b_n = \beta_n + o((\log n)^{1/r-1}) \tag{4.1}$$

and

$$b_n = \beta_n - \frac{(r-1)\Delta^2}{2q^{-1/r}r^2(\log n)^{2-1/r}} + o\Big(\frac{(\log\log n)^2}{(\log n)^{2-1/r}}\Big).$$

We can obtain (4.1) similarly to the arguments of Example 2 on pages 71–72 in Resnick [20] or the proof of Theorem 3.2 in Lin and Jiang [15]. Now put

$$b_n = \beta_n + \theta_n,$$

where $\theta_n = o((\log n)^{\frac{1}{r}-1})$. Note $\log(1-x) = -x + \frac{1}{2}x^2 + O(x^3)$ and $(1-x)^v = 1 - vx + \frac{v(v-1)}{2}x^2 + O(x^3)$ as $x \to 0$. Substituting $b_n = \beta_n + \theta_n$ into

$$-\log\Big(\frac{Aq}{r\sqrt{2\pi}}\Big) - h\log\Big(1+\frac{\lambda}{2h}b_n^p\Big) + \frac{b_n^r}{q} + (r-1)\log b_n = \log n,$$

this yields

$$\Big[(\log n)\Big(\Big(1-\tfrac{\Delta}{r\log n}\Big)^{r}-1\Big)+\tfrac{r-1-hp}{r}\log(q\log n)+(r-1-hp)\log\Big(1-\tfrac{\Delta}{r\log n}\Big)$$
$$-\log\Big(\tfrac{Aq}{r\sqrt{2\pi}}\Big)-h\log\Big(\tfrac{\lambda}{2h}\Big)-h\log\Big(1+\tfrac{2h}{\lambda}(q\log n)^{-\frac{p}{r}}\Big(1-\tfrac{\Delta}{r\log n}\Big)^{-p}\Big)\Big]$$
$$\times(rq^{-\frac{1}{r}}(\log n)^{1-\frac{1}{r}}\theta_n)^{-1}+1+o(1)=0$$

and hence

$$\begin{aligned}\theta_n\sim\Big[&-(\log n)\Big(\Big(1-\tfrac{\Delta}{r\log n}\Big)^{r}-1\Big)-\tfrac{r-1-hp}{r}\log(q\log n)\\&-\log\Big(1-\tfrac{\Delta}{r\log n}\Big)^{r-1-hp}+\log\Big(\tfrac{Aq}{r\sqrt{2\pi}}\Big)+h\log\Big(\tfrac{\lambda}{2h}\Big)\\&+h\log\Big(1+\tfrac{2h}{\lambda}(q\log n)^{-\frac{p}{r}}\Big(1-\tfrac{\Delta}{r\log n}\Big)^{-p}\Big)\Big]\\&\Big/(rq^{-\frac{1}{r}}(\log n)^{1-\frac{1}{r}}).\end{aligned}$$

Once again let

$$\theta_n=-\frac{(r-1)\Delta^2}{2q^{-1/r}r^2(\log n)^{2-1/r}}+\nu_n$$

where $\nu_n=o\Big(\frac{(\log\log n)^2}{(\log n)^{2-1/r}}\Big)$. So, we have

$$b_n=\beta_n-\frac{(r-1)\Delta^2}{2q^{-1/r}r^2(\log n)^{2-1/r}}+\nu_n. \tag{4.2}$$

Noting the definition of β_n, (4.2) deduces the following results:

$$a_nb_n^{-1}\sim\frac{1}{r\log n},\ r_n-1=\frac{\alpha_n}{a_n}-1\sim-\frac{(r-1-hp)(r-1)}{r^2}\frac{\log\log n}{\log n},$$
$$\delta_n=\frac{\beta_n-b_n}{a_n}\sim\frac{(r-1)(r-1-hp)^2}{2r^3}\frac{(\log\log n)^2}{\log n}$$

for large n. So the desired result follows by Lemma 3.2. □

Proof of Theorem 2.1. Letting $r_n=1$, $\delta_n=0$ in (3.7) and noting $a_nb_n^{-1}\sim\frac{1}{r\log n}$, by Lemma 3.2 there exists a constant $C_1(h,\lambda,r,p)$ such that

$$\sup_{-\infty<x<\infty}|F^n(t_n(x))-\Lambda(x)|=\sup_{-\infty<x<\infty}|F^n(a_nx+b_n)-\Lambda(x)|>\frac{C_1(h,\lambda,r,p)}{\log n}$$

for $n>n_0$ (n_0 is a positive integer). For the upper bound, we need to prove

$$\sup_{0\le x<+\infty}|F^n(t_n(x))-\Lambda(x)|<d_{21}a_nb_n^{-1}, \tag{4.3}$$

$$\sup_{-c_n<x<0}|F^n(t_n(x))-\Lambda(x)|<d_{22}a_nb_n^{-1}, \tag{4.4}$$

$$\sup_{-\infty<x\le-c_n}|F^n(t_n(x))-\Lambda(x)|<d_{23}a_nb_n^{-1} \tag{4.5}$$

for $n \geq n_2$, where $d_{2i} > 0$, $i = 1, 2, 3$ are positive constants and

$$c_n = \log\log b_n^r > Ç \ \text{ with } Ç \text{ satisfying Lemma 3.1.}$$

According to (4.1), we have

$$q\log n - \frac{q(r-1-hp)}{r}\log(q\log n) < b_n^r < q\log n$$

and there furthermore exists a positive integer n_3 such that

$$\text{(4.6)} \quad \sup_{n\geq n_3} \frac{\log\log b_n^r}{b_n^r} < \frac{r}{q}\sup_{n\geq n_3}\frac{\log\log(q\log n)}{r\log n - (r-1-hp)\log(q\log n)} < \frac{r}{q}.$$

So,

$$b_n - a_n c_n > 0$$

for $n > n_4$.

We firstly consider the case of $x \geq -c_n$ and define

$$R_n(x) = -(n\log F(a_n x + b_n) + n\psi_n(x)), \ B_n(x) = \exp(-R_n),$$
$$A_n(x) = \exp(-n\psi_n(x) + e^{-x}),$$

where $\psi_n(x) = 1 - F(a_n x + b_n)$. Using the following simple result

$$b_n^r - (b_n - a_n c_n)^r = \int_{b_n - a_n c_n}^{b_n} rt^{r-1}dt < ra_n c_n b_n^{r-1}$$

for $r > 1$, and according to (3.1) and (1.4), it follows that

$$\begin{aligned}
&\psi_n(x) < \psi_n(-c_n)\\
&\quad < \tfrac{Aq}{r\sqrt{2\pi}}(b_n - a_n c_n)^{1-r}\Big(1 + \tfrac{\lambda}{2h}(b_n - a_n c_n)^p\Big)^h \exp\Big(-\tfrac{(b_n - a_n c_n)^r}{q}\Big)\\
&\quad = n^{-1}(1 - a_n c_n b_n^{-1})^{1-r}\left(\tfrac{1+\frac{\lambda}{2h}(b_n - a_n c_n)^p}{1+\frac{\lambda}{2h}b_n^p}\right)^h \exp\left(\tfrac{b_n^r}{q} - \frac{(b_n - a_n c_n)^r}{q}\right)\\
&\quad < \sup_{n\geq n_3}\left\{\left(1 - \tfrac{q\log\log b_n^r}{rb_n^r}\right)^{1-r}\left(\tfrac{1+\frac{\lambda}{2h}b_n^p(1-\frac{q}{r}b_n^{-r}\log\log b_n^r)^p}{1+\frac{\lambda}{2h}b_n^p}\right)^h (n^{-1}\log b_n^r)\right\}\\
&\quad = Ç_5(p, q, r, h) < 1.
\end{aligned}$$

Thus,

$$\inf_{x\geq -c_n}(1 - \psi_n(x)) > 1 - Ç_5(p, q, r, h) > 0.$$

Noting that

$$\log(1-x) < -x, \ \log(1-x) > -x - \frac{x^2}{2(1-x)} \text{ for } 0 < x < 1,$$

we have

$$0 < R_n(x) = -(n\log(1 - \psi_n(x)) + n\psi_n(x)) < \frac{n\psi_n^2(-c_n)}{2(1 - Ç_5(p, q, r, h))}$$

$$< \frac{(1-a_nc_nb_n^{-1})^{2(1-r)}\Big(\frac{1+\frac{\lambda}{2h}(b_n-a_nc_n)^p}{1+\frac{\lambda}{2h}b_n^p}\Big)^{2h}}{2(1-Ç_5(p,q,r,h))a_n^{-1}b_n}(r(qn)^{-1}b_n^r\exp(2c_n)). \tag{4.7}$$

Noting $b_n^r < q\log n$ for $n > n_3$, it follows that

$$n^{-1}\frac{r}{q}b_n^r\exp(2c_n) < \frac{r}{q}n^{-1}(q\log n)^3 < Ç_6(q,r),\ \text{ for } n > n_5.$$

Substituting this into (4.7) yields

$$\begin{aligned} R_n(x) < \sup_{n\ge n_4}&\Big((1-a_nc_nb_n^{-1})^{2(1-r)}\Big(\frac{1+\frac{\lambda}{2h}(b_n-a_nc_n)^p}{1+\frac{\lambda}{2h}b_n^p}\Big)^{2h}\Big)\\ &\times\frac{Ç_6(q,r)}{2(1-Ç_5(p,q,r,h))}a_nb_n^{-1} < Ç_7(p,q,r,h)a_nb_n^{-1}. \end{aligned}$$

So,

$$|B_n(x)-1| < R_n(x) < Ç_7(p,q,r,h)a_nb_n^{-1} \text{ for } n > \max(n_3,n_4,n_5). \tag{4.8}$$

Using the inequality (4.8), we have, for $x \ge -c_n$

$$\begin{aligned} &|F^n(a_nx+b_n)-\Lambda(x)|\\ &= |F^n(a_nx+b_n)-\exp(-e^{-x})F^n(a_nx+b_n)e^{n(1-F(a_nx+b_n))}\\ &\quad+\exp(-e^{-x})F^n(a_nx+b_n)e^{n(1-F(a_nx+b_n))}-\exp(-e^{-x})|\\ &\le |F^n(a_nx+b_n)-\exp(-e^{-x})F^n(a_nx+b_n)e^{n(1-F(a_nx+b_n))}|\\ &\quad+\exp(-e^{-x})|F^n(a_nx+b_n)e^{n(1-F(a_nx+b_n))}-1|\\ &\le \exp(-e^{-x})F^n(a_nx+b_n)e^{n(1-F(a_nx+b_n))}\\ &\quad|\exp(-n(1-F(a_nx+b_n)))\exp(e^{-x})-1|\\ &\quad+|F^n(a_nx+b_n)e^{n(1-F(a_nx+b_n))}-1|\\ &= \Lambda(x)B_n(x)|A_n(x)-1|+|B_n(x)-1|\\ &< \Lambda(x)|A_n(x)-1|+Ç_7(p,q,r,h)a_nb_n^{-1}. \end{aligned} \tag{4.9}$$

We now prove (4.3). Note that for $r > 2$,

$$(1+x)^r > 1+rx+\frac{r(r-1)}{2}x^2 \quad \text{for } x > 0.$$

Take $x = a_nb_n^{-1}$ in the inequality, and multiply by b_n^r. By using (1.4), we have

$$x-\frac{(a_nx+b_n)^r-b_n^r}{q} < -\frac{(r-1)}{2}a_nb_n^{-1}x^2 \quad \text{for } x > 0. \tag{4.10}$$

Using (1.4), (4.10) and the definition of $A_n(x)$, for $x > 0$, there exists n_6 such that, for $n > n_6$,

$$\begin{aligned} A_n'(x) &= \exp(-n\psi_n(x)+e^{-x})(-n(\psi_n(x))'-e^{-x})\\ &= -A_n(x)e^{-x}(1-ne^xa_nF'(a_nx+b_n)) \end{aligned}$$

$$= -A_n(x)e^{-x}\Big(1 - e^x\Big(\frac{1+\frac{\lambda}{2h}(a_nx+b_n)^p}{1+\frac{\lambda}{2h}b_n^p}\Big)^h \exp\Big(\frac{b_n^r - (a_nx+b_n)^r}{q}\Big)\Big)$$
$$< -A_n(x)e^{-x}\Big(1 - \Big(\frac{1+\frac{\lambda}{2h}(a_nx+b_n)^p}{1+\frac{\lambda}{2h}b_n^p}\Big)^h \exp\Big(-\frac{r-1}{2}a_nb_n^{-1}x^2\Big)\Big) < 0,$$

where the last inequality is due to Lemma 3.3.

Noting that $A_n(x) \to 1$ as $x \to \infty$, we have

$$\begin{aligned}\sup_{x\geq 0}|A_n(x)-1| &= |A_n(0)-1| = |\exp(-n(1-F(b_n))+1)-1| \\ &= |\exp(-n(n^{-1}-r(b_n))+1)-1| \\ &\leq nr(b_n)\exp(nr(b_n)) \leq Ç_8(p,q,r,h)a_nb_n^{-1}.\end{aligned}$$

The above inequalities stem from the facts: $e^x - 1 \leq xe^x$ for $0 \leq x \leq 1$, and

$$\begin{aligned}nr(b_n) &< \frac{Aq^2}{r^2\sqrt{2\pi}}nb_n^{-r}\Big((r-1)b_n^{1-r} + \frac{\lambda(r-1)-\lambda ph}{2h}b_n^{1+p-r}\Big) \\ &\quad \Big(1+\frac{\lambda}{2h}b_n^p\Big)^{h-1}\exp(-b_n^r/q) \\ &= Ç_9(p,q,r,h)a_nb_n^{-1}.\end{aligned}$$

Combining with (4.9), we have, for $n > \max(n_3, n_4, n_5, n_6)$,

$$\sup_{0\leq x<+\infty}|F^n(t_n(x)) - \Lambda(x)| < (Ç_8(p,q,r,h) + Ç_7(p,q,r,h))a_nb_n^{-1}.$$

The following we consider the case of $-c_n \leq x < 0$. Using (1.4) and Lemma 3.1 yields the result

$$\begin{aligned}&-n\psi_n(x) + e^{-x} \\ =\ &-n\Big(\tfrac{Aq}{r\sqrt{2\pi}}(a_nx+b_n)^{1-r}\Big(1+\tfrac{\lambda}{2h}(a_nx+b_n)^p\Big)^h \\ &\qquad \exp(-(a_nx+b_n)^r/q) - r(a_nx+b_n)\Big) + e^{-x} \\ =\ &-n\Big\{\tfrac{Aq}{r\sqrt{2\pi}}(a_nx+b_n)^{1-r}\Big(1+\tfrac{\lambda}{2h}(a_nx+b_n)^p\Big)^h\exp\Big(-\tfrac{(a_nx+b_n)^r}{q}\Big) \\ &-\tfrac{Aq^2}{r^2\sqrt{2\pi}}\Big[(r-1)(a_nx+b_n)^{1-2r} + \tfrac{\lambda(r-1)-\lambda ph}{2h}(a_nx+b_n)^{1+p-2r}\Big] \\ &\times\Big(1+\tfrac{\lambda}{2h}(a_nx+b_n)^p\Big)^{h-1}\exp\Big(-\tfrac{(a_nx+b_n)^r}{q}\Big)d_n(a_nx+b_n)\Big\} + e^{-x} \\ =\ &-b_n^{r-1}(a_nx+b_n)^{1-r}\Big(\tfrac{1+\frac{\lambda}{2h}(a_nx+b_n)^p}{1+\frac{\lambda}{2h}b_n^p}\Big)^h\exp\Big(-\tfrac{(a_nx+b_n)^r-b_n^r}{q}\Big) \\ &+\tfrac{qd_n(a_nx+b_n)}{r}b_n^{r-1}\Big((r-1)(a_nx+b_n)^{1-2r} + \tfrac{\lambda(r-1)-\lambda ph}{2h}(a_nx+b_n)^{1+p-2r}\Big) \\ &\times\Big(1+\tfrac{\lambda}{2h}(a_nx+b_n)^p\Big)^{h-1}\Big(1+\tfrac{\lambda}{2h}b_n^p\Big)^{-h}\exp\Big(-\tfrac{(a_nx+b_n)^r-b_n^r}{q}\Big) + e^{-x}\end{aligned}$$

$$= (a_n b_n^{-1} x + 1)^{1-r} e^{-x} \Big\{ -\Big[\Big(\tfrac{1+\frac{\lambda}{2h}(a_n x + b_n)^p}{1+\frac{\lambda}{2h} b_n^p}\Big)^h - \tfrac{q d_n(a_n x + b_n)}{r} \Big(\tfrac{(r-1) b_n^{-r}}{(a_n b_n^{-1} x + 1)^{-r}}$$
$$+ \tfrac{\lambda(r-1) - \lambda p h}{2h} b_n^{p-r} (a_n b_n^{-1} x + 1)^{p-r} \Big) \Big(1 + \tfrac{\lambda}{2h}(a_n x + b_n)^p\Big)^{h-1} \Big(1 + \tfrac{\lambda}{2h} b_n^p\Big)^{-h}\Big]$$
$$\times \exp\Big(-\tfrac{(a_n x + b_n)^r - b_n^r - qx}{q}\Big) + (a_n b_n^{-1} x + 1)^{r-1}\Big\}$$
$$=: (a_n b_n^{-1} x + 1)^{1-r} e^{-x} D_n(x),$$

where $0 < d_n(a_n x + b_n) < 1$ and

$$D_n(x) = -\Big\{\Big(\tfrac{1+\frac{\lambda}{2h}(a_n x + b_n)^p}{1+\frac{\lambda}{2h} b_n^p}\Big)^h - \tfrac{q}{r}\Big[(r-1) b_n^{-r} (a_n b_n^{-1} x + 1)^{-r}$$
$$+ \tfrac{\lambda(r-1) - \lambda p h}{2h} b_n^{p-r} (a_n b_n^{-1} x + 1)^{p-r}\Big]$$
$$\Big(1 + \tfrac{\lambda}{2h}(a_n x + b_n)^p\Big)^{h-1} \Big(1 + \tfrac{\lambda}{2h} b_n^p\Big)^{-h}$$
$$\times d_n(a_n x + b_n)\Big\} \exp\Big(-\tfrac{(a_n x + b_n)^r - b_n^r - qx}{q}\Big) + (a_n b_n^{-1} x + 1)^{r-1}.$$

Since

$$a_n x + b_n > 0 \text{ for } x > -c_n; e^{-x} \geq 1 - x \text{ for } x \in R,$$

$$1 + rx < (1+x)^r < 1 + rx + \frac{r(r-1)}{2} x^2 \text{ and}$$
$$(1+x)^{-r} < 1 - 2^{r+1} r x \text{ for } -1/2 < x < 0, \tag{4.11}$$

and

$$(1 + \frac{\lambda}{2h}(a_n x + b_n)^p)/(1 + \frac{\lambda}{2h} b_n^p) < 1 \text{ for } -c_n < x < 0,$$

we have

$$D_n(x) < -\Big(1 - \tfrac{(a_n x + b_n)^r - b_n^r - qx}{q}\Big)$$
$$\times \Big\{\Big(\tfrac{1+\frac{\lambda}{2h}(a_n x + b_n)^p}{1+\frac{\lambda}{2h} b_n^p}\Big)^h - \tfrac{q}{r}\Big[(r-1) b_n^{-r} (a_n b_n^{-1} x + 1)^{-r} + \tfrac{\lambda(r-1-ph)}{2h}$$
$$\times b_n^{p-r} (a_n b_n^{-1} x + 1)^{p-r}\Big] \cdot \tfrac{b_n^{-p}}{b_n^{-p} + \frac{\lambda}{2h}(a_n b_n^{-1} x + 1)^p} d_n(a_n x + b_n)\Big\}$$
$$+ (a_n b_n^{-1} x + 1)^{r-1}$$
$$= -\Big(1 - \tfrac{(a_n x + b_n)^r - b_n^r - qx}{q}\Big)$$
$$\times \Big\{1 + \tfrac{\frac{\lambda p}{2} a_n b_n^{-1} x}{b_n^{-p} + \frac{\lambda}{2h}} - \Big[\tfrac{q(r-1)}{r} b_n^{-r-p} (a_n b_n^{-1} x + 1)^{-r} + \tfrac{q\lambda(r-1-ph)}{2hr}$$
$$\times b_n^{-r} (a_n b_n^{-1} x + 1)^{p-r}\Big] \cdot \tfrac{1}{b_n^{-p} + \frac{\lambda}{2h}(a_n b_n^{-1} x + 1)^p} d_n(a_n x + b_n)\Big\}$$
$$+ (a_n b_n^{-1} x + 1)^{r-1}$$

$$
\begin{aligned}
&< -\Big(1-\tfrac{(a_nx+b_n)^r-b_n^r-qx}{q}\Big)\\
&\quad\times\Big(1-Ç_4\tfrac{\lambda(r-1)-\lambda ph}{2h}b_n^{-r}(a_nb_n^{-1}x+1)^{p-r}\Big)+(a_nb_n^{-1}x+1)^{r-1}\\
&< -\Big(1-\tfrac{(a_nx+b_n)^r-b_n^r-qx}{q}\Big)\\
&\quad\times\Big(1-Ç_4\tfrac{\lambda(r-1)-\lambda ph}{2h}b_n^{-r}(1-2^{r-p+1}(r-p)a_nb_n^{-1}x)\Big)\\
&\quad+(a_nb_n^{-1}x+1)^{r-1}\\
&= -\Big(1-Ç_4\tfrac{\lambda(r-1)-\lambda ph}{2h}b_n^{-r}(1-2^{r-p+1}(r-p)a_nb_n^{-1}x)\\
&\quad-\tfrac{(a_nx+b_n)^r-b_n^r-qx}{q}\\
&\quad+\Big(Ç_4\tfrac{\lambda(r-1)-\lambda ph}{2h}b_n^{-r}(1-2^{r-p+1}(r-p)a_nb_n^{-1}x)\Big)\tfrac{(a_nx+b_n)^r-b_n^r-qx}{q}\Big)\\
&\quad+(a_nb_n^{-1}x+1)^{r-1}\\
&< -1+Ç_4\tfrac{\lambda(r-1)-\lambda ph}{2h}b_n^{-r}(1-2^{r-p+1}(r-p)a_nb_n^{-1}x)\\
&\quad+\tfrac{(a_nx+b_n)^r-b_n^r-qx}{q}\\
&\quad+(a_nb_n^{-1}x+1)^{r-1}\\
&< -1+Ç_4\tfrac{\lambda(r-1)-\lambda ph}{2h}b_n^{-r}(1+2^{r-p+1}(r-p)a_nb_n^{-1}c_n)+\tfrac{r-1}{2}a_nb_n^{-1}x^2\\
&\quad+1+(r-1)a_nb_n^{-1}x\\
&< \Big(Ç_4\tfrac{\lambda(r-1)-\lambda ph}{2h}\cdot\tfrac{r(1+2^{r-p+1}(r-p))}{q}+(r-1)x+\tfrac{r-1}{2}x^2\Big)a_nb_n^{-1},
\end{aligned}
$$

where the last inequality is attributed to (1.4) and (4.6). Meanwhile (4.11) implies

$$D_n(x) > -1+(a_nb_n^{-1}x+1)^{r-1} > (r-1)a_nb_n^{-1}x.$$

Therefore,

$$|D_n(x)| < a_nb_n^{-1}\Big(Ç_4\frac{\lambda(r-1)-\lambda ph}{2h}\cdot\frac{r(1+2^{r-p+1}(r-p))}{q}+\frac{r-1}{2}x^2+(r-1)|x|\Big).$$

Furthermore, for $n > n_3$,

$$
\begin{aligned}
&|e^{-x}-n\psi_n(x)|\\
&< (a_nb_n^{-1}x+1)^{1-r}e^{-x}\\
&\quad\times a_nb_n^{-1}\Big(Ç_4\frac{\lambda(r-1)-\lambda ph}{2h}\cdot\frac{r(1+2^{r-p+1}(r-p))}{q}+\frac{r-1}{2}x^2+(r-1)|x|\Big)\\
&< (1-a_nb_n^{-1}c_n)^{1-r}e^{c_n}\\
&\quad\times a_nb_n^{-1}\Big(Ç_4\frac{\lambda(r-1)-\lambda ph}{2h}\cdot\frac{r(1+2^{r-p+1}(r-p))}{q}+\frac{r-1}{2}c_n^2+(r-1)|c_n|\Big)\\
&< Ç_{10}(p,q,r,h).
\end{aligned}
$$

So,

$$\begin{aligned}
&\Lambda(x)|A_n(x)-1|\\
&=\Lambda(x)|\exp(-n\psi_n(x)+e^{-x})-1|\\
&<\Lambda(x)\exp((-n\psi_n(x)+e^{-x})\theta)|-n\psi_n(x)+e^{-x}|\\
&<\ \exp(Ç_{10}(p,q,r,h))(1-a_nb_n^{-1}c_n)^{1-r}a_nb_n^{-1}\\
&\quad\times\sup_{-c_n\le x<0}\Big(\Big(\frac{Ç_4 r\lambda(r-1-ph)(1+2^{r-p+1}(r-p))}{2hq}+\frac{(r-1)(x^2+2|x|)}{2}\Big)\\
&\quad e^{-x}\Lambda(x)\Big)\\
&<Ç_{11}(p,q,r,h)a_nb_n^{-1}.
\end{aligned}$$

Now combining this with (4.9), for $n>\max(n_3,n_4,n_5)$, we complete the proof of (4.4).

Finally, focus on the case of $-\infty<x<-c_n$. Noting that

$$\Lambda(x)\le\Lambda(-c_n)=\frac{r}{q}a_nb_n^{-1},$$

we have

$$\begin{aligned}
&\sup_{x\le -c_n}|F^n(a_nx+b_n)-\Lambda(x)|\\
&<F^n(b_n-a_nc_n)+\Lambda(-c_n)\\
&<\ \sup_{-c_n\le x<0}|F^n(a_nx+b_n)-\Lambda(x)|+2\Lambda(-c_n)\\
&<(Ç_7(p,q,r,h)+Ç_{11}(p,q,r,h))a_nb_n^{-1}+\frac{2r}{q}a_nb_n^{-1}\\
&=Ç_{12}(p,q,r,h)a_nb_n^{-1}
\end{aligned}$$

and thus complete the proof of (4.5). Let $n_2=\max\{n_3,n_4,n_5,n_6\}$ and $n_1=\max\{n_0,n_2\}$, the proof is completed. □

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