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CHANGE OF SCALE FORMULAS FOR A GENERALIZED CONDITIONAL WIENER INTEGRAL

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Abstract. Let $C[0,t]$ denote the space of real-valued continuous functions on $[0,t]$ and define a random vector $Z_n : C[0,t] \to \mathbb{R}^n$ by $Z_n(x) = (\int_0^{t_1} h(s)dx(s), \ldots, \int_0^{t_n} h(s)dx(s))$, where $0 < t_1 < \cdots < t_n = t$ is a partition of $[0,t]$ and $h \in L_2[0,t]$ with $h \neq 0$ a.e. Using a simple formula for a conditional expectation on $C[0,t]$ with Z_n, we evaluate a generalized analytic conditional Wiener integral of the function $G_r(x) = F(x)\Psi(\int_0^t v_1(s)dx(s), \ldots, \int_0^t v_r(s)dx(s))$ for F in a Banach algebra and for $\Psi = f + \phi$ which need not be bounded or continuous, where $f \in L_p(\mathbb{R}^r)(1 \leq p \leq \infty)$, $\{v_1, \ldots, v_r\}$ is an orthonormal subset of $L_2[0,t]$ and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^r$. Finally we establish various change of scale transformations for the generalized analytic conditional Wiener integrals of G_r with the conditioning function Z_n.

1. Introduction

Let $C_0[0,t]$ denote the Wiener space, the space of continuous real-valued functions x on $[0,t]$ with $x(0) = 0$. As mentioned in [14] the Wiener measure and Wiener measurability behave badly under change of scale transformation and under translation [1, 2]. Various kinds of the change of scale formulas for Wiener integrals of bounded functions were developed on the classical and abstract Wiener spaces [3, 12, 13, 15]. Chang, Kim, Song and Yoo [14] established a change of scale formula for the Wiener integral of function on the abstract Wiener space $\mathbb{B}$ which have the form

$$F_1(x) = G(x)\Psi((e_1, x)^\sim, \ldots, (e_r, x)^\sim)$$

for $G \in \mathcal{F}(\mathbb{B})$, the Fresnel class [5] and $\Psi = \psi + \phi$, where $\psi \in L_p(\mathbb{R}^r), 1 \leq p < \infty$, $(\cdot,\cdot)^\sim$ denotes a stochastic inner product on $\mathbb{B}$ [10] and ϕ is the Fourier

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transform of a measure of bounded variation over $\mathbb{R}^r$. Furthermore the author and his coauthors [6, 8, 11] introduced various kinds of the change of scale formulas for the conditional Wiener integrals of the function of the form F_1 defined on $C_0[0,t]$, $C_0(\mathbb{B})$, the infinite dimensional Wiener space and $C[0,t]$, an analogue of Wiener space [9] which is the space of real-valued continuous paths on $[0,t]$.

Let $h \in L_2[0,t]$ with $h \neq 0$ a.e. on $[0,t]$. Define a stochastic process $Z : C[0,t] \times [0,t] \to \mathbb{R}$ by $Z(x,s) = \int_0^s h(u)dx(u)$ for $x \in C[0,t]$ and $s \in [0,t]$, where the integral denotes the Paley-Wiener-Zygmund integral, and let

$$Z_n(x) = (Z(x,t_1), \ldots, Z(x,t_n)).$$

On the space $C[0,t]$ the author [7] derived a simple formula for a generalized conditional Wiener integral given the vector-valued conditioning function Z_n.

Using the simple formula on $C[0,t]$ with the conditioning function Z_n, we evaluate a generalized analytic conditional Wiener integral of the function G_r having the form

$$G_r(x) = F(x)\Psi\bigg(\int_0^t v_1(s)dx(s), \ldots, \int_0^t v_r(s)dx(s)\bigg)$$

for F in a Banach algebra which corresponds to the Cameron-Storvick's Banach algebra $\mathcal{S}$ [4] and for $\Psi = f + \phi$ which need not be bounded or continuous, where $f \in L_p(\mathbb{R}^r)(1 \le p \le \infty)$, $\{v_1, \ldots, v_r\}$ is an orthonormal subset of $L_2[0,t]$ and ϕ is the Fourier transform of a measure of bounded variation over $\mathbb{R}^r$. Finally we establish various kinds of new change of scale transformations for the generalized analytic conditional Wiener integral of G_r with the conditioning function Z_n. We note that the results of this paper are different from those in [6, 8, 11].

2. A generalized conditional Wiener integral

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\mathbb{C}_+^\sim$ denote the sets of complex numbers, complex numbers with positive real parts and nonzero complex numbers with nonnegative real parts, respectively.

Let $(C[0,t], \mathcal{B}(C[0,t]), w_\varphi)$ be the analogue of Wiener space associated with a probability measure φ on the Borel class of $\mathbb{R}$, where $\mathcal{B}(C[0,t])$ denotes the Borel class of $C[0,t]$ [9]. For $v \in L_2[0,t]$ and $x \in C[0,t]$ let $(v,x) = \int_0^t v(s)dx(s)$ denote the Paley-Wiener-Zygmund integral of v according to x. The inner product on the real Hilbert space $L_2[0,t]$ is denoted by $\langle \cdot, \cdot \rangle$. Furthermore the dot product on the r-dimensional Euclidean space $\mathbb{R}^r$ is also denoted by $\langle \cdot, \cdot \rangle_{\mathbb{R}^r}$.

Let $F : C[0,t] \to \mathbb{C}$ be integrable and let X be a random vector on $C[0,t]$. Then we have the conditional expectation $E[F|X]$ given X from a well-known probability theory. Furthermore there exists a P_X-integrable function ψ on the value space of X such that $E[F|X](x) = (\psi \circ X)(x)$ for w_φ-a.e. $x \in C[0,t]$, where P_X is the probability distribution of X. The function ψ is called the conditional Wiener w_φ-integral of F given X and it is also denoted by $E[F|X]$.

Let $0 = t_0 < t_1 < \cdots < t_n = t$ be a partition of $[0,t]$, where n is a positive integer. Let $h \in L_2[0,t]$ be of bounded variation with $h \neq 0$ a.e. For $j = 1, \ldots, n$ let $\alpha_j = \frac{1}{\|\chi_{(t_{j-1},t_j]}h\|}\chi_{(t_{j-1},t_j]}h$ and let V be the subspace of $L_2[0,t]$ generated by $\{\alpha_1, \ldots, \alpha_n\}$. Let $V^\perp$ be the orthogonal complement of V. Let $\mathcal{P} : L_2[0,t] \to V$ be the orthogonal projection given by

$$\mathcal{P}v = \sum_{j=1}^{n} \langle v, \alpha_j \rangle \alpha_j$$

and $\mathcal{P}^\perp : L_2[0,t] \to V^\perp$ be the orthogonal projection. For $x \in C[0,t]$ define the stochastic integral by

$$Z(x,s) = \int_0^s h(u)dx(u), \quad 0 \leq s \leq t$$

and let $Z_n : C[0,t] \to \mathbb{R}^n$ be given by

$$Z_n(x) = (Z(x,t_1), \ldots, Z(x,t_n)). \tag{1}$$

Let $b(s) = \int_0^s (h(u))^2 du$ and for $x \in C[0,t]$ define the polygonal function $[Z(x,\cdot)]_b$ of $Z(x,\cdot)$ by

$$\begin{aligned}(2) \quad & [Z(x,\cdot)]_b(s) \\ &= \sum_{j=1}^{n} \chi_{(t_{j-1},t_j]}(s)\bigg(Z(x,t_{j-1}) + \frac{b(s)-b(t_{j-1})}{b(t_j)-b(t_{j-1})}(Z(x,t_j) - Z(x,t_{j-1}))\bigg)\end{aligned}$$

for $s \in [0,t]$, where $\chi_{(t_{j-1},t_j]}$ denotes the indicator function on the interval $(t_{j-1}, t_j]$. Similarly for $\vec{\xi} = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ the polygonal function $[\vec{\xi}]_b$ of $\vec{\xi}$ is given by (2) replacing $Z(x,t_j)$ by $\xi_j (j = 1, \ldots, n)$ with $\xi_0 = 0$. For a function $F : C[0,t] \to \mathbb{C}$ such that $F(Z(x,\cdot))$ is integrable over x, we have by Theorem 2.12 in [7]

$$E[F(Z(x,\cdot))|Z_n](\vec{\xi}) = E[F(Z(x,\cdot) - [Z(x,\cdot)]_b + [\vec{\xi}]_b)] \tag{3}$$

for P_{Z_n}-a.e. $\vec{\xi} \in \mathbb{R}^n$ (for a.e. $\vec{\xi} \in \mathbb{R}^n$), where P_{Z_n} is the probability distribution of Z_n on the Borel class of $\mathbb{R}^n$. For $\lambda > 0$ let $F_Z^\lambda(x) = F(\lambda^{-\frac{1}{2}} Z(x,\cdot))$ and $Z_n^\lambda(x) = Z_n(\lambda^{-\frac{1}{2}} x)$ for $x \in C[0,t]$, where Z_n is given by (1). Suppose that $E[F_Z^\lambda]$ exists. By the definition of the conditional Wiener w_φ-integral and (3)

$$E[F_Z^\lambda|Z_n^\lambda](\vec{\xi}) = E[F(\lambda^{-\frac{1}{2}}(Z(x,\cdot) - [Z(x,\cdot)]_b) + [\vec{\xi}]_b)] \tag{4}$$

for $P_{Z_n^\lambda}$-a.e. $\vec{\xi} \in \mathbb{R}^n$, where $P_{Z_n^\lambda}$ is the probability distribution of Z_n^λ on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$. Let $I_{F_Z}^\lambda(\vec{\xi})$ be the right-hand side of (4). If $I_{F_Z}^\lambda(\vec{\xi})$ has the analytic extension $J_{F_Z}^\lambda(\vec{\xi})$ on $\mathbb{C}_+$, then it is called the conditional analytic Wiener w_φ-integral of F_Z given Z_n with the parameter λ and denoted by

$$E^{anw_\lambda}[F_Z|Z_n](\vec{\xi}) = J_{F_Z}^\lambda(\vec{\xi})$$

for $\vec{\xi} \in \mathbb{R}^n$. Moreover if for nonzero real q, $E^{anw_\lambda}[F_Z|Z_n](\vec{\xi})$ has the limit as λ approaches $-iq$ through $\mathbb{C}_+$, then it is called the conditional analytic Feynman w_φ-integral of F_Z given Z_n with the parameter q and denoted by

$$E^{anf_q}[F_Z|Z_n](\vec{\xi}) = \lim_{\lambda \to -iq} E^{anw_\lambda}[F_Z|Z_n](\vec{\xi}).$$

Lemma 2.1. *Let $v \in L_2[0,t]$. Then for w_φ-a.e. $x \in C[0,t]$*

$$(v, [Z(x,\cdot)]_b) = (\mathcal{P}(vh), x).$$

Proof. By the definition of the Paley-Wiener-Zygmund integral

$$\begin{aligned}
&(v, [Z(x,\cdot)]_b) \\
&= \sum_{j=1}^n \frac{Z(x,t_j) - Z(x,t_{j-1})}{b(t_j) - b(t_{j-1})} \int_{t_{j-1}}^{t_j} v(s)db(s) \\
&= \sum_{j=1}^n \frac{\int_{t_{j-1}}^{t_j} v(s)(h(s))^2 ds}{\|\chi_{(t_{j-1},t_j]} h\|^2} \bigg(\int_0^{t_j} h(s)dx(s) - \int_0^{t_{j-1}} h(s)dx(s) \bigg) \\
&= \sum_{j=1}^n \langle vh, \alpha_j \rangle (\alpha_j, x) = (\mathcal{P}(vh), x)
\end{aligned}$$

which completes the proof. □

3. Generalized analytic conditional Feynman integrals

Throughout this paper let $h \in L_2[0,t]$ be of bounded variation with $h \neq 0$ a.e. and $\{v_1, v_2, \ldots, v_r\}$ be an orthonormal subset of $L_2[0,t]$ such that $\{\mathcal{P}^\perp(hv_1), \ldots, \mathcal{P}^\perp(hv_r)\}$ is an independent set. Let

$$\{e_1, \ldots, e_r\} \tag{5}$$

be the orthonormal set obtained from $\{\mathcal{P}^\perp(hv_1), \ldots, \mathcal{P}^\perp(hv_r)\}$ by the Gram-Schmidt orthonormalization process. Now for $l = 1, \ldots, r$ let $\mathcal{P}^\perp(hv_l) = \sum_{j=1}^r \alpha_{lj} e_j$ be the linear combinations of the e_js and let

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1r} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{r1} & \alpha_{r2} & \cdots & \alpha_{rr} \end{bmatrix} \tag{6}$$

be the coefficient matrix of the combinations. We can also regard A as the linear transformation $T_A : \mathbb{R}^r \to \mathbb{R}^r$ given by $T_A(\vec{z}) = \vec{z}A$, where $\vec{z}$ is an arbitrary row-vector in $\mathbb{R}^r$. We note that A is invertible so that T_A is an isomorphism.

Remark 3.1. An example of h and $\{v_1, \ldots, v_r\}$ satisfying the above conditions can be obtained by the following process. Let

$$h(s) = \sum_{j=1}^n \chi_{(t_{j-1},t_j]}(s) \frac{2(-1)^j}{t_j - t_{j-1}} \left(s - \frac{t_{j-1} + t_j}{2} \right) + \chi_{\{0\}}(s)$$

and for $l = 1, \dots, r$ let

$$h_l(s) = \sum_{j=1}^{n} \chi_{(t_{j-1},t_j]}(s) \frac{(-1)^j 2^{2l-1}}{(t_j - t_{j-1})^{2l-1}} \left(s - \frac{t_{j-1} + t_j}{2} \right)^{2l-1} + \chi_{\{0\}}(s)$$

for $s \in [0, t]$. For a.e $s \in [0, t]$ let $\sum_{l=1}^{r} c_l h_l(s) = 0$. Fix $k \in \{1, \dots, n\}$ and take distinct points $a_1, \dots, a_r$ in $(\frac{t_{k-1}+t_k}{2}, t_k)$ satisfying the above equality. Let $b_m = \frac{2}{t_k - t_{k-1}} a_m - \frac{t_k + t_{k-1}}{t_k - t_{k-1}}$ for $m = 1, \dots, r$. Replacing s by a_m we have the linear equation system with unknowns $c_1, \dots, c_r$; $\sum_{l=1}^{r} b_m^{2l-1} c_l = 0$ for $m = 1, \dots, r$. The determinant of the coefficient matrix is given by

$$\begin{vmatrix} b_1 & b_1^3 & \cdots & b_1^{2r-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_r & b_r^3 & \cdots & b_r^{2r-1} \end{vmatrix} = \left(\prod_{m=1}^{r} b_m \right) \left(\prod_{1 \le j < k \le r} (b_k^2 - b_j^2) \right) \neq 0$$

so that $c_1 = \cdots = c_r = 0$, which shows that $\{h_1, \dots, h_r\}$ is an independent set. Let $\{v_1, \dots, v_r\}$ be the orthonormal set obtained from $\{h_1, \dots, h_r\}$ by the Gram-Schmidt orthonormalization process. Now let $v_l = \sum_{j=1}^{r} \beta_{lj} h_j$ for $l = 1, \dots, r$. Then we have

$$\mathcal{P}^{\perp}(hv_l) = \sum_{j=1}^{r} \beta_{lj} hh_j - \sum_{j=1}^{r} \sum_{k=1}^{n} \beta_{lj} \langle hh_j, \alpha_k \rangle \alpha_k.$$

We note that

$$\langle hh_j, \alpha_k \rangle = \frac{1}{\|\chi_{(t_{k-1},t_k]} h\|} \int_{t_{k-1}}^{t_k} (h(s))^2 h_j(s) ds = 0$$

so that for a.e. $s \in [0, t]$

$$\begin{aligned} &\mathcal{P}^{\perp}(hv_l)(s) \\ &= \sum_{j=1}^{r} \sum_{p=1}^{n} \beta_{lj} \chi_{(t_{p-1},t_p]}(s) \frac{2^{2j}}{(t_p - t_{p-1})^{2j}} \left(s - \frac{t_{p-1} + t_p}{2} \right)^{2j} + \sum_{j=1}^{r} \beta_{lj} \chi_{\{0\}}(s). \end{aligned}$$

To prove the independence of $\{\mathcal{P}^{\perp}(hv_l) : l = 1, \dots, r\}$ let

$$\sum_{l=1}^{r} c'_l (\mathcal{P}^{\perp}(hv_l))(s) = 0 \quad \text{for a.e. } s \in [0, t].$$

Fix $p \in \{1, \dots, n\}$ and take distinct points $a'_1, \dots, a'_r$ in $(\frac{t_{p-1}+t_p}{2}, t_p)$ satisfying the above two equalities. Let $b'_m = \frac{2}{t_p - t_{p-1}} a'_m - \frac{t_p + t_{p-1}}{t_p - t_{p-1}}$ for $m = 1, \dots, r$. Replacing s by a'_m we have $\sum_{l=1}^{r} (\sum_{j=1}^{r} \beta_{lj} (b'_m)^{2j}) c'_l = 0$ for $m = 1, \dots, r$. The determinant of the coefficient matrix is given by

$$\begin{vmatrix} \sum_{j=1}^{r} (b'_1)^{2j} \beta_{1j} & \cdots & \sum_{j=1}^{r} (b'_1)^{2j} \beta_{rj} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{r} (b'_r)^{2j} \beta_{1j} & \cdots & \sum_{j=1}^{r} (b'_r)^{2j} \beta_{rj} \end{vmatrix}$$

$$= \left(\prod_{m=1}^{r}(b'_m)^2\right)\left(\prod_{1\le j<m\le r}((b'_m)^2-(b'_j)^2)\right)\begin{vmatrix} \beta_{11} & \beta_{21} & \cdots & \beta_{r1} \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{1r} & \beta_{2r} & \cdots & \beta_{rr} \end{vmatrix} \ne 0.$$

Hence $c'_1 = \cdots = c'_r = 0$, which shows the independence of $\{\mathcal{P}^\perp(hv_l) : l = 1,\ldots,r\}$.

Let $\hat{\mathrm{M}}(\mathbb{R}^r)$ be the space of all functions ϕ on $\mathbb{R}^r$ defined by

$$\phi(\vec{u}) = \int_{\mathbb{R}^r} \exp\{i\langle\vec{u},\vec{z}\rangle_{\mathbb{R}^r}\}d\rho(\vec{z}), \tag{7}$$

where ρ is a complex Borel measure of bounded variation over $\mathbb{R}^r$. Let

$$\mathcal{M}(L_2[0,t])$$

be the class of all $\mathbb{C}$-valued Borel measures of bounded variation over $L_2[0,t]$ and let $\mathcal{S}_{w_\varphi}$ be the space of all functions F which for $\sigma \in \mathcal{M}(L_2[0,t])$ have the form

$$F(x) = \int_{L_2[0,t]} \exp\{i(v,x)\}d\sigma(v) \tag{8}$$

for w_φ-a.e. $x \in C[0,t]$. We note that $\mathcal{S}_{w_\varphi}$ is a Banach algebra [4, 9].

Let $(\vec{v},x) = ((v_1,x),\ldots,(v_r,x))$ and $(h\vec{v},x) = ((hv_1,x),\ldots,(hv_r,x))$ for $x \in C[0,t]$. For a complete orthonormal basis $\{e_1,\ldots,e_r,e_{r+1},\ldots\}$ containing (5) and $v \in L_2[0,t]$ let

$$c_j(v) = \langle v, e_j\rangle \quad \text{for } j = 1,\ldots,r,r+1,\ldots. \tag{9}$$

Theorem 3.2. *Let $\Psi(x) = \phi(\vec{v},x)F(x)$, where ϕ and F are given by* (7) *and* (8)*, respectively. For $\lambda \in \mathbb{C}_+^\sim$, $v \in L_2,[0,t]$, $\vec{\xi} \in \mathbb{R}^n$ and $\vec{z} \in \mathbb{R}^r$ let*

$$A_1(\vec{\xi},v,\vec{z}) = \exp\{i[(v,[\vec{\xi}]_b) + \langle(\vec{v},[\vec{\xi}]_b),\vec{z}\rangle_{\mathbb{R}^r}]\} \tag{10}$$

and

$$\begin{aligned} &A_2(\lambda,v,\vec{z}) \\ &\quad = \exp\left\{-\frac{1}{2\lambda}[\|\mathcal{P}^\perp(hv)\|^2 - \|\vec{c}(\mathcal{P}^\perp(hv))\|^2_{\mathbb{R}^r} + \|\vec{c}(\mathcal{P}^\perp(hv)) + T_A\vec{z}\|^2_{\mathbb{R}^r}]\right\}, \end{aligned} \tag{11}$$

where $\vec{c} = (c_1,\ldots,c_r)$ and the c_js are given by (9)*. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$*

$$E^{anw_\lambda}[\Psi_Z|Z_n](\vec{\xi}) = \int_{L_2[0,t]}\int_{\mathbb{R}^r} A_1(\vec{\xi},v,\vec{z})A_2(\lambda,v,\vec{z})d\rho(\vec{z})d\sigma(v).$$

Moreover for a nonzero real q, $E^{anf_q}[\Psi_Z|Z_n](\vec{\xi})$ is given by the right hand side of the above equality replacing λ by $-iq$.

Proof. For $\lambda > 0$ and a.e. $\vec{\xi} \in \mathbb{R}^n$ we have by Lemma 2.1

$$\begin{aligned}
I^{\lambda}_{\Psi_Z}(\vec{\xi}) &= E[\Psi(\lambda^{-\frac{1}{2}}(Z(x,\cdot) - [Z(x,\cdot)]_b) + [\vec{\xi}]_b)] \\
&= \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \int_{C[0,t]} \exp\{i\lambda^{-\frac{1}{2}}[(vh - \mathcal{P}(vh), x) \\
&\quad + \langle (h\vec{v} - \mathcal{P}(h\vec{v}), x), \vec{z}\rangle_{\mathbb{R}^r}]\} dw_\varphi(x) d\rho(\vec{z}) d\sigma(v) \\
&= \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \int_{C[0,t]} \exp\{i\lambda^{-\frac{1}{2}}[(\mathcal{P}^{\perp}(vh), x) \\
&\quad + \langle (\mathcal{P}^{\perp}(h\vec{v}), x), \vec{z}\rangle_{\mathbb{R}^r}]\} dw_\varphi(x) d\rho(\vec{z}) d\sigma(v),
\end{aligned}$$

where for a functional $g : L_2[0,t] \to L_2[0,t]$

$$(g(h\vec{v}), x) = ((g(hv_1), x), \ldots, (g(hv_r), x)).$$

By the same process as used in the proof of Theorem 2.6 in [8] we can obtain

$$I^{\lambda}_{\Psi_Z}(\vec{\xi}) = \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) A_2(\lambda, v, \vec{z}) d\rho(\vec{z}) d\sigma(v).$$

By the Morera's theorem and the dominated convergence theorem we have the theorem. □

For $1 \le p \le \infty$ let $\mathcal{A}_r^{(p)}$ be the space of the cylinder functions having the following form

$$F_r(x) = f(\vec{v}, x) \tag{12}$$

for w_φ-a.e. $x \in C[0,t]$, where $f \in L_p(\mathbb{R}^r)$. Without loss of generality we can take f to be Borel measurable.

Theorem 3.3. *Let $1 \le p \le \infty$ and $F_r \in \mathcal{A}_r^{(p)}$ be given by* (12). *Then for $\lambda \in \mathbb{C}_+$ we have*

$$E^{anw_\lambda}[(F_r)_Z | Z_n](\vec{\xi}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) \exp\left\{-\frac{\lambda}{2}\|\vec{u}\|^2_{\mathbb{R}^r}\right\} d\vec{u}$$

for a.e. $\vec{\xi} \in \mathbb{R}^n$, where A^T is the transpose of A given by (6). *Furthermore if $p = 1$, then for a non-zero real q $E^{anf_q}[(F_r)_Z | Z_n](\vec{\xi})$ is given by the right hand side of the above equality replacing λ by $-iq$.*

Proof. By the same process as used in the proof of Theorem 3.1 in [8]

$$\begin{aligned}
I^{\lambda}_{(F_r)_Z}(\vec{\xi}) &= E[F_r(\lambda^{-\frac{1}{2}}(Z(x,\cdot) - [Z(x,\cdot)]_b) + [\vec{\xi}]_b)] \\
&= \int_{C[0,t]} f(\lambda^{-\frac{1}{2}}(\mathcal{P}^{\perp}(h\vec{v}), x) + (\vec{v}, [\vec{\xi}]_b)) dw_\varphi(x) \\
&= \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) \exp\left\{-\frac{\lambda}{2}\|\vec{u}\|^2_{\mathbb{R}^r}\right\} d\vec{u}
\end{aligned}$$

for $\lambda > 0$ and a.e. $\vec{\xi} \in \mathbb{R}^n$. By the Morera's theorem we have the first part of the theorem. If $p = 1$, then the final result follows from the dominated convergence theorem. $\square$

Theorem 3.4. *Let $G_r = FF_r$, where $F \in \mathcal{S}_{w_\varphi}$ and $F_r \in \mathcal{A}_r^{(p)}(1 \le p \le \infty)$ are given by* (8) *and* (12), *respectively. For $\lambda \in \tilde{\mathbb{C}}_+$, $v \in L_2[0,t]$ and $\vec{u} \in \mathbb{R}^r$ let*

$$A_3(\lambda, v, \vec{u}) = \exp\bigg\{-\frac{1}{2\lambda}[\|\mathcal{P}^\perp(hv)\|^2 - \|\vec{c}(\mathcal{P}^\perp(hv))\|^2_{\mathbb{R}^r}] - \frac{\lambda}{2}\|\vec{u}\|^2_{\mathbb{R}^r} + i\langle \vec{c}(\mathcal{P}^\perp(hv)), \vec{u}\rangle_{\mathbb{R}^r}\bigg\}, \tag{13}$$

where $\vec{c} = (c_1, \dots, c_r)$ and the c_js are given by (9). Then we have for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$

$$E^{anw_\lambda}[(G_r)_Z|Z_n](\vec{\xi}) = \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{L_2[0,t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{\mathbb{R}^r} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) \times A_3(\lambda, v, \vec{u}) d\vec{u} d\sigma(v),$$

where A^T is the transpose of A given by (6). *Furthermore if $p = 1$, then for a real q $E^{anf_q}[(G_r)_Z|Z_n](\vec{\xi})$ is given by the right hand side of the above equality replacing λ by $-iq$.*

Proof. By the same process as used in the proof of Theorem 3.3 in [8]

$$\begin{aligned}
&I^\lambda_{(G_r)_Z}(\vec{\xi})\\
&= E[G_r(\lambda^{-\frac{1}{2}}(Z(x,\cdot) - [Z(x,\cdot)]_b) + [\vec{\xi}]_b)]\\
&= \int_{L_2[0,t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{C[0,t]} \exp\{i\lambda^{-\frac{1}{2}}(\mathcal{P}^\perp(vh), x)\} f(\lambda^{-\frac{1}{2}}(\mathcal{P}^\perp(h\vec{v}), x)\\
&\quad + (\vec{v}, [\vec{\xi}]_b)) dw_\varphi(x) d\sigma(v)\\
&= \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{L_2[0,t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{\mathbb{R}^r} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) A_3(\lambda, v, \vec{u}) d\vec{u} d\sigma(v)
\end{aligned}$$

for $\lambda > 0$ and a.e. $\vec{\xi} \in \mathbb{R}^n$. By the Morera's theorem we have the first part of the theorem. If $p = 1$, then the final result follows from the dominated convergence theorem. $\square$

From Theorems 3.2 and 3.4 we have the following corollary by the linearities of the generalized conditional Wiener and Feynman integrals on the analogue of Wiener space.

Corollary 3.5. *Let ϕ, F and $F_r \in \mathcal{A}_r^{(p)}(1 \le p \le \infty)$ be given by* (7), (8) *and* (12), *respectively. Furthermore let q be a nonzero real number. Then*

$E^{anw_\lambda}[((\phi(\vec{v},\cdot)+F_r)F)_Z|Z_n](\vec{\xi})$ *exists for* $\lambda\in\mathbb{C}_+$ *and a.e.* $\vec{\xi}\in\mathbb{R}^n$*, and it is given by*

$$\begin{aligned}&E^{anw_\lambda}[((\phi(\vec{v},\cdot)+F_r)F)_Z|Z_n](\vec{\xi})\\&=\int_{L_2[0,t]}\bigg[\int_{\mathbb{R}^r}A_1(\vec{\xi},v,\vec{z})A_2(\lambda,v,\vec{z})d\rho(\vec{z})+\exp\{i(v,[\vec{\xi}]_b)\}\\&\quad\times\left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}}\int_{\mathbb{R}^r}f(\vec{u}A^T+(\vec{v},[\vec{\xi}]_b))A_3(\lambda,v,\vec{u})d\vec{u}\bigg]d\sigma(v),\end{aligned}$$

where A_1*,* A_2 *and* A_3 *are given by* (10)*,* (11) *and* (13)*, respectively. In particular if* $F_r\in\mathcal{A}_r^{(1)}$*, then* $E^{anf_q}[((\phi(\vec{v},\cdot)+F_r)F)_Z|Z_n](\vec{\xi})$ *exists for a.e.* $\vec{\xi}\in\mathbb{R}^n$ *and it is obtained with replacing* λ *by* $-iq$ *in the right-hand side of the above equality.*

4. A change of scale formula using the polygonal function

In this section we derive change of scale formulas for the generalized conditional Wiener integrals of unbounded functions on the analogue of Wiener space using the polygonal function.

Let $\{e_j : j=1,2,\ldots\}$ be a complete orthonormal basis for $L_2[0,t]$ containing $\{e_1,\ldots,e_r\}$ which is given by (5). For $m\in\mathbb{N}$, $\lambda\in\mathbb{C}_+^\sim$ and $x\in C[0,t]$ let

$$K_m(\lambda,x)=\exp\bigg\{\frac{1-\lambda}{2}\sum_{j=1}^m(e_j,x)^2\bigg\}. \tag{14}$$

Theorem 4.1. *Let* $1\le p\le\infty$ *and* F_r *be given by* (12)*. Then for* $\lambda\in\mathbb{C}_+$ *and a.e.* $\vec{\xi}\in\mathbb{R}^n$ *we have*

$$E^{anw_\lambda}[(F_r)_Z|Z_n](\vec{\xi})=\lambda^{\frac{r}{2}}\int_{C[0,t]}K_r(\lambda,x)F_r(Z(x,\cdot)-[Z(x,\cdot)]_b+[\vec{\xi}]_b)dw_\varphi(x),$$

where K_r *is given by* (14) *replacing* m *by* r*. Moreover if* $p=1$ *and* q *is a nonzero real number, then*

$$\begin{aligned}&E^{anf_q}[(F_r)_Z|Z_n](\vec{\xi})\\&=\lim_{m\to\infty}\lambda_m^{\frac{r}{2}}\int_{C[0,t]}K_r(\lambda_m,x)F_r(Z(x,\cdot)-[Z(x,\cdot)]_b+[\vec{\xi}]_b)dw_\varphi(x)\end{aligned}$$

for any sequence $\{\lambda_m\}_{m=1}^\infty$ *in* $\mathbb{C}_+$ *converging to* $-iq$ *as* m *approaches* ∞*.*

Proof. For $\lambda\in\mathbb{C}_+$ and a.e. $\vec{\xi}\in\mathbb{R}^n$ we have by Lemma 2.1

$$\begin{aligned}\Gamma(\lambda,r,\vec{\xi})&\equiv\lambda^{\frac{r}{2}}\int_{C[0,t]}K_r(\lambda,x)F_r(Z(x,\cdot)-[Z(x,\cdot)]_b+[\vec{\xi}]_b)dw_\varphi(x)\\&=\lambda^{\frac{r}{2}}\int_{C[0,t]}K_r(\lambda,x)f((\mathcal{P}^\perp(h\vec{v}),x)+(\vec{v},[\vec{\xi}]_b))dw_\varphi(x)\end{aligned}$$

$$= \lambda^{\frac{r}{2}} \int_{C[0,t]} K_r(\lambda, x) f((\vec{e}, x)A^T + (\vec{v}, [\vec{\xi}]_b)) dw_\varphi(x),$$

where $(\vec{e}, x) = ((e_1, x), \dots, (e_r, x))$ and A^T is the transpose of A given by (6). By the generalized Wiener integration theorem [9, Theorem 3.5] and Theorem 3.3

$$\begin{aligned} &\Gamma(\lambda, r, \vec{\xi}) \\ &= \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{\mathbb{R}^r} \exp\left\{\frac{1-\lambda}{2}\|\vec{u}\|^2_{\mathbb{R}^r}\right\} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) \exp\left\{-\frac{1}{2}\|\vec{u}\|^2_{\mathbb{R}^r}\right\} d\vec{u} \\ &= E^{anw_\lambda}[(F_r)_Z | Z_n](\vec{\xi}), \end{aligned}$$

which completes the proof of the first part of the theorem. If $p = 1$, then the final result follows from the dominated convergence theorem. □

Theorem 4.2. *Let Ψ be as given in Theorem 3.2. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$ we have*

$$\begin{aligned} (15) \qquad & E^{anw_\lambda}[\Psi_Z | Z_n](\vec{\xi}) \\ &= \lim_{m\to\infty} \lambda^{\frac{m}{2}} \int_{C[0,t]} K_m(\lambda, x) \Psi(Z(x,\cdot) - [Z(x,\cdot)]_b + [\vec{\xi}]_b) dw_\varphi(x), \end{aligned}$$

where K_m is given by (14). *Moreover if q is a nonzero real number and $\{\lambda_m\}_{m=1}^\infty$ is a sequence in $\mathbb{C}_+$ converging to $-iq$ as m approaches ∞, then $E^{anf_q}[\Psi_Z | Z_n](\vec{\xi})$ is given by the right hand side of* (15) *replacing λ by λ_m.*

Proof. For $m > r$, $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$ we have by Lemma 2.1

$$\begin{aligned} \Gamma(\lambda, m, \vec{\xi}) &\equiv \int_{C[0,t]} K_m(\lambda, x) \Psi(Z(x,\cdot) - [Z(x,\cdot)]_b + [\vec{\xi}]_b) dw_\varphi(x) \\ &= \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \int_{C[0,t]} K_m(\lambda, x) \exp\{i[(v, Z(x,\cdot) \\ &\quad - [Z(x,\cdot)]_b) + \langle(\vec{v}, Z(x,\cdot) - [Z(x,\cdot)]_b), \vec{z}\rangle_{\mathbb{R}^r}]\} dw_\varphi(x) d\rho(\vec{z}) d\sigma(v) \\ &= \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \int_{C[0,t]} K_m(\lambda, x) \exp\{i[(\mathcal{P}^\perp(vh), x) \\ &\quad + \langle(\mathcal{P}^\perp(h\vec{v}), x), \vec{z}\rangle_{\mathbb{R}^r}]\} dw_\varphi(x) d\rho(\vec{z}) d\sigma(v), \end{aligned}$$

where A_1 and K_m are given by (10) and (14), respectively. By the similar method as used in the proof of Lemma 8 in [11]

$$\begin{aligned} &\Gamma(\lambda, m, \vec{\xi}) \\ &= \lambda^{-\frac{m}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \exp\left\{\frac{\lambda - 1}{2\lambda} \sum_{j=1}^m (c_j(\mathcal{P}^\perp(hv))^2 - \frac{1}{\lambda}\right. \\ &\quad \left. \times \langle \vec{c}(\mathcal{P}^\perp(hv)), T_A\vec{z}\rangle_{\mathbb{R}^r} - \frac{1}{2\lambda}\|T_A\vec{z}\|^2_{\mathbb{R}^r} - \frac{1}{2}\|\mathcal{P}^\perp(hv)\|^2\right\} d\rho(\vec{z}) d\sigma(v), \end{aligned}$$

where $\vec{c} = (c_1, \ldots, c_r)$ and the c_js are given by (9). Now we have by the dominated convergence theorem and the Parseval's identity

$$\begin{aligned}
&\lim_{m\to\infty} \lambda^{\frac{m}{2}}\Gamma(\lambda, m, \vec{\xi}) \\
&= \int_{L_2[0,t]}\int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \exp\bigg\{-\frac{1}{2\lambda}\|\mathcal{P}^\perp(hv)\|^2 - \frac{1}{\lambda}\langle \vec{c}(\mathcal{P}^\perp(hv)), T_A\vec{z}\rangle_{\mathbb{R}^r} \\
&\quad - \frac{1}{2\lambda}\|T_A\vec{z}\|^2_{\mathbb{R}^r}\bigg\} d\rho(\vec{z})d\sigma(v) \\
&= \int_{L_2[0,t]}\int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) A_2(\lambda, v, \vec{z}) d\rho(\vec{z})d\sigma(v),
\end{aligned}$$

where A_2 is given by (11). Now the proof of the first part of the theorem is completed by Theorem 3.2. The remainder part of the theorem immediately follows from the dominated convergence theorem. □

Theorem 4.3. *Let G_r be as given in Theorem 3.4. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$ $E^{anw_\lambda}[(G_r)_Z|Z_n](\vec{\xi})$ is given by the right hand side of* (15) *replacing Ψ by G_r. Moreover if $p = 1$, q is a nonzero real number and $\{\lambda_m\}_{m=1}^\infty$ is a sequence in $\mathbb{C}_+$ converging to $-iq$ as m approaches ∞, then $E^{anf_q}[(G_r)_Z|Z_n](\vec{\xi})$ is given by the right hand side of* (15)*, where λ and Ψ are replaced by λ_m and G_r, respectively.*

Proof. For $m > r$, $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$ we have by Lemma 2.1

$$\begin{aligned}
\Gamma(\lambda, m, \vec{\xi}) &\equiv \int_{C[0,t]} K_m(\lambda, x) G_r(Z(x,\cdot) - [Z(x,\cdot)]_b + [\vec{\xi}]_b) dw_\varphi(x) \\
&= \int_{L_2[0,t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{C[0,t]} K_m(\lambda, x)\exp\{i(\mathcal{P}^\perp(vh), x)\} \\
&\quad \times f((\mathcal{P}^\perp(h\vec{v}), x) + (\vec{v}, [\vec{\xi}]_b)) dw_\varphi(x) d\sigma(v).
\end{aligned}$$

By the similar method as used in the proof of Lemma 7 in [11]

$$\begin{aligned}
\Gamma(\lambda, m, \vec{\xi}) &= \lambda^{-\frac{m}{2}}\left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{L_2[0,t]} \exp\bigg\{i(v, [\vec{\xi}]_b) + \frac{\lambda-1}{2\lambda}\sum_{j=1}^m (c_j(\mathcal{P}^\perp(hv)))^2 \\
&\quad - \frac{1}{2}\|\mathcal{P}^\perp(hv)\|^2 + \frac{1}{2\lambda}\|\vec{c}(\mathcal{P}^\perp(hv))\|^2_{\mathbb{R}^r}\bigg\} \int_{\mathbb{R}^r} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) \\
&\quad \times \exp\bigg\{-\frac{\lambda}{2}\|\vec{u}\|^2_{\mathbb{R}^r} + i\langle \vec{c}(\mathcal{P}^\perp(hv)), \vec{u}\rangle_{\mathbb{R}^r}\bigg\} d\vec{u}d\sigma(v),
\end{aligned}$$

where $\vec{c} = (c_1, \ldots, c_r)$, the c_js are given by (9) and A^T is the transpose of A given by (6). Now we have by the dominated convergence theorem and the Parseval's identity

$$\lim_{m\to\infty} \lambda^{\frac{m}{2}}\Gamma(\lambda, m, \vec{\xi})$$

$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{L_2[0,t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{\mathbb{R}^r} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) \exp\bigg\{ -\frac{1}{2\lambda}$$
$$\times [\|\mathcal{P}^\perp(hv)\|^2 - \vec{c}(\mathcal{P}^\perp(hv))] - \frac{\lambda}{2}\|\vec{u}\|^2_{\mathbb{R}^r} + i\langle \vec{c}(\mathcal{P}^\perp(hv)), \vec{u}\rangle_{\mathbb{R}^r} \bigg\} d\vec{u}d\sigma(v)$$
$$= \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{L_2[0,t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{\mathbb{R}^r} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) A_3(\lambda, v, \vec{u}) d\vec{u}d\sigma(v),$$

where A_3 is given by (13). Now the proof of the first part of the theorem is completed by Theorem 3.4. If $p = 1$, then the final result immediately follows from the dominated convergence theorem. □

Combining Theorems 4.2 and 4.3 we have the following corollary by the linearities of the generalized conditional Wiener and Feynman integrals on the analogue of Wiener space.

Corollary 4.4. *Let $(\phi(\vec{v}, \cdot) + F_r)F$ be as given in Corollary 3.5. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$ $E^{anw_\lambda}[((\phi(\vec{v}, \cdot) + F_r)F)_Z|Z_n](\vec{\xi})$ is given by the right hand side of* (15) *replacing Ψ by $(\phi(\vec{v}, \cdot) + F_r)F$. Moreover if $p = 1$, q is a nonzero real number and $\{\lambda_m\}_{m=1}^\infty$ is a sequence in $\mathbb{C}_+$ converging to $-iq$ as m approaches ∞, then $E^{anf_q}[((\phi(\vec{v}, \cdot) + F_r)F)_Z|Z_n](\vec{\xi})$ is given by the right hand side of* (15)*, where λ and Ψ are replaced by λ_m and $(\phi(\vec{v}, \cdot)+F_r)F$, respectively.*

Letting $\lambda = \gamma^{-2}$ in Corollary 4.4 we have the following change of scale formula for the generalized conditional Wiener integrals on the analogue of Wiener space using the polygonal function.

Corollary 4.5. *Let F, F_r and ϕ be as given in Corollary 4.4. Then for $\gamma > 0$ and a.e. $\vec{\xi} \in \mathbb{R}^n$*

$$E[F(\gamma Z(x, \cdot))(\phi(\vec{v}, \gamma Z(x, \cdot)) + F_r(\gamma Z(x, \cdot)))|\gamma Z_n(x)](\vec{\xi})$$
$$= \lim_{m\to\infty} \gamma^{-m} \int_{C[0,t]} \exp\bigg\{ \frac{2\gamma^2 - 1}{2\gamma^2} \sum_{j=1}^m (e_j, x)^2 \bigg\} F(Z(x, \cdot) - [Z(x, \cdot)]_b$$
$$+ [\vec{\xi}]_b)(\phi(\vec{v}, Z(x, \cdot) - [Z(x, \cdot)]_b + [\vec{\xi}]_b) + F_r(Z(x, \cdot) - [Z(x, \cdot)]_b + [\vec{\xi}]_b)) dw_\varphi(x).$$

5. A change of scale formula using the cylinder functions

In this section we derive a change of scale formula for the generalized conditional Wiener integrals of unbounded functions on the analogue of Wiener space using the cylinder functions.

Theorem 5.1. *Let $1 \le p \le \infty$ and A^T be the transpose of A given by* (6)*. For an orthonormal set $\{h_1, \dots, h_r\}$ in $L_2[0, t]$ let $H_r(\lambda, x) = \exp\{\frac{1-\lambda}{2} \sum_{j=1}^r (h_j, x)^2\}$. Let F_r and f be related by* (12)*. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$ we have*

$$E^{anw_\lambda}[(F_r)_Z|Z_n](\vec{\xi}) = \lambda^{\frac{r}{2}} \int_{C[0,t]} H_r(\lambda, x) f((\vec{h}, x)A^T + (\vec{v}, [\vec{\xi}]_b)) dw_\varphi(x),$$

where $(\vec{h},x)=((h_1,x),\ldots,(h_r,x))$. *Moreover if* $p=1$ *and* q *is a nonzero real number, then*

$$E^{anf_q}[(F_r)_Z|Z_n](\vec{\xi})=\lim_{m\to\infty}\lambda_m^{\frac{r}{2}}\int_{C[0,t]}H_r(\lambda_m,x)f((\vec{h},x)A^T+(\vec{v},[\vec{\xi}]_b))dw_\varphi(x)$$

for any sequence $\{\lambda_m\}_{m=1}^\infty$ *in* $\mathbb{C}_+$ *converging to* $-iq$ *as* m *approaches* ∞.

Proof. For $\lambda\in\mathbb{C}_+$ and a.e. $\vec{\xi}\in\mathbb{R}^n$ we have by Theorem 3.3

$$\begin{aligned}
&\lambda^{\frac{r}{2}}\int_{C[0,t]}H_r(\lambda,x)f((\vec{h},x)A^T+(\vec{v},[\vec{\xi}]_b))dw_\varphi(x)\\
&=\Big(\frac{\lambda}{2\pi}\Big)^{\frac{r}{2}}\int_{\mathbb{R}^r}\exp\Big\{\frac{1-\lambda}{2}\|\vec{u}\|^2_{\mathbb{R}^r}\Big\}f(\vec{u}A^T+(\vec{v},[\vec{\xi}]_b))\exp\Big\{-\frac{1}{2}\|\vec{u}\|^2_{\mathbb{R}^r}\Big\}d\vec{u}\\
&=E^{anw_\lambda}[(F_r)_Z|Z_n](\vec{\xi}),
\end{aligned}$$

which completes the proof of the first part of the theorem. If $p=1$, then the final result follows from the dominated convergence theorem. □

Theorem 5.2. *Let* A *be given by* (6) *and* Ψ *be as given in Theorem 3.2. Then for* $\lambda\in\mathbb{C}_+$ *and a.e.* $\vec{\xi}\in\mathbb{R}^n$ *we have*

$$\begin{aligned}
E^{anw_\lambda}[\Psi_Z|Z_n](\vec{\xi})=\lim_{m\to\infty}\lambda^{\frac{m}{2}}\int_{C[0,t]}K_m(\lambda,x)\int_{L_2[0,t]}\int_{\mathbb{R}^r}A_1(\vec{\xi},v,\vec{z})\exp\\
\{i[(\mathcal{P}^\perp(vh),x)+\langle(\vec{e},x),\vec{z}A\rangle_{\mathbb{R}^r}]\}d\rho(\vec{z})d\sigma(v)dw_\varphi(x),
\end{aligned}$$

where $(\vec{e},x)=((e_1,x),\ldots,(e_r,x))$, A_1 *and* K_m *are given by* (10) *and* (14), *respectively. Moreover if* q *is a nonzero real number and* $\{\lambda_m\}_{m=1}^\infty$ *is a sequence in* $\mathbb{C}_+$ *converging to* $-iq$ *as* m *approaches* ∞, *then* $E^{anf_q}[\Psi_Z|Z_n](\vec{\xi})$ *is given by the right hand side of the above equality, where* λ *is replaced by* λ_m.

Proof. Let $m>r$. For $v\in L_2[0,t]$ let $f_{m+1}=\mathcal{P}^\perp(vh)-\sum_{j=1}^m c_j(\mathcal{P}^\perp(vh))e_j$ and let $g_{m+1}=\frac{1}{\|f_{m+1}\|}f_{m+1}$ if $f_{m+1}\neq 0$, where c_j is given by (9). Let $g_{m+1}=0$ if $f_{m+1}=0$. For $\lambda\in\mathbb{C}_+$ and a.e. $\vec{\xi}\in\mathbb{R}^n$ we have by the generalized Wiener integration theorem [9, Theorem 3.5]

$$\begin{aligned}
&\Gamma(\lambda,m,\vec{\xi})\\
&\equiv\int_{C[0,t]}K_m(\lambda,x)\int_{L_2[0,t]}\int_{\mathbb{R}^r}A_1(\vec{\xi},v,\vec{z})\exp\{i[(\mathcal{P}^\perp(vh),x)\\
&\quad+\langle(\vec{e},x),\vec{z}A\rangle_{\mathbb{R}^r}]\}d\rho(\vec{z})d\sigma(v)dw_\varphi(x)\\
&=\int_{L_2[0,t]}\int_{\mathbb{R}^r}A_1(\vec{\xi},v,\vec{z})\int_{C[0,t]}K_m(\lambda,x)\exp\Big\{i\Big[\sum_{j=1}^m c_j(\mathcal{P}^\perp(vh))(e_j,x)\\
&\quad+\|f_{m+1}\|(g_{m+1},x)+\langle(\vec{e},x),\vec{z}A\rangle_{\mathbb{R}^r}\Big]\Big\}dw_\varphi(x)d\rho(\vec{z})d\sigma(v)
\end{aligned}$$

$$= \left(\frac{1}{2\pi}\right)^{\frac{m+1}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \int_{\mathbb{R}^{m+1}} \exp\bigg\{\frac{1-\lambda}{2} \sum_{j=1}^m u_j^2 + i\bigg[\sum_{j=1}^m c_j(\mathcal{P}^\perp(vh))u_j + \|f_{m+1}\|u_{m+1} + \langle \vec{u}, \vec{z}A\rangle_{\mathbb{R}^r}\bigg] - \frac{1}{2}\sum_{j=1}^{m+1} u_j^2\bigg\} d(u_1, \ldots, u_m, u_{m+1}) d\rho(\vec{z}) d\sigma(v),$$

where $\vec{u} = (u_1, \ldots, u_r)$. Using the following well-known integration formula

$$\int_{\mathbb{R}} \exp\{-au^2 + ibu\} du = \left(\frac{\pi}{a}\right)^{\frac{1}{2}} \exp\left\{-\frac{b^2}{4a}\right\} \tag{16}$$

for $a \in \mathbb{C}_+$ and any real b

$$\begin{aligned}
&\Gamma(\lambda, m, \vec{\xi}) \\
&= \left(\frac{1}{2\pi}\right)^{\frac{m}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \int_{\mathbb{R}^m} \exp\bigg\{-\frac{\lambda}{2} \sum_{j=1}^m u_j^2 + i\bigg[\sum_{j=1}^m c_j(\mathcal{P}^\perp(vh)) \\
&\quad \times u_j + \langle \vec{u}, \vec{z}A\rangle_{\mathbb{R}^r}\bigg] - \frac{1}{2}\|f_{m+1}\|^2\bigg\} d(u_1, \ldots, u_m) d\rho(\vec{z}) d\sigma(v) \\
&= \left(\frac{1}{2\pi}\right)^{\frac{m}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \int_{\mathbb{R}^m} \exp\bigg\{-\frac{\lambda}{2}\|\vec{u}\|_{\mathbb{R}^r}^2 + i[\langle \vec{c}(\mathcal{P}^\perp(vh)), \vec{u}\rangle_{\mathbb{R}^r} \\
&\quad + \langle \vec{z}A, \vec{u}\rangle_{\mathbb{R}^r}] - \frac{\lambda}{2} \sum_{j=r+1}^m u_j^2 + i \sum_{j=r+1}^m c_j(\mathcal{P}^\perp(vh))u_j - \frac{1}{2}\bigg[\|\mathcal{P}^\perp(vh)\|^2 \\
&\quad - \sum_{j=1}^m (c_j(\mathcal{P}^\perp(vh)))^2\bigg]\bigg\} d(u_1, \ldots, u_m) d\rho(\vec{z}) d\sigma(v),
\end{aligned}$$

where $\vec{c}(\mathcal{P}^\perp(vh)) = (c_1(\mathcal{P}^\perp(vh)), \ldots, c_r(\mathcal{P}^\perp(vh)))$. By (16)

$$\begin{aligned}
&\Gamma(\lambda, m, \vec{\xi}) \\
&= \lambda^{-\frac{m}{2}} \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \exp\bigg\{-\frac{1}{2\lambda}\bigg[\|\vec{c}(\mathcal{P}^\perp(vh)) + \vec{z}A\|_{\mathbb{R}^r}^2 + \sum_{j=r+1}^m \\
&\quad (c_j(\mathcal{P}^\perp(vh)))^2\bigg] - \frac{1}{2}\bigg[\|\mathcal{P}^\perp(vh)\|^2 - \sum_{j=1}^m (c_j(\mathcal{P}^\perp(vh)))^2\bigg]\bigg\} d\rho(\vec{z}) d\sigma(v).
\end{aligned}$$

By the dominated convergence theorem and the Parseval's identity

$$\lim_{m\to\infty} \lambda^{\frac{m}{2}} \Gamma(\lambda, m, \vec{\xi}) = \int_{L_2[0,t]} \int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) A_2(\lambda, v, \vec{z}) d\rho(\vec{z}) d\sigma(v),$$

where A_2 is given by (11). Now the proof of the first part of the theorem is completed by Theorem 3.2. The second part of the theorem immediately follows from the dominated convergence theorem. $\square$

Theorem 5.3. *Let A^T be the transpose of A given by* (6). *Let G_r be as given in Theorem 3.4. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$ we have*

$$E^{anw_\lambda}[(G_r)_Z|Z_n](\vec{\xi}) = \lim_{m\to\infty} \lambda^{\frac{m}{2}} \int_{C[0,t]} K_m(\lambda, x) \int_{L_2[0,t]} \exp\{i[(v, [\vec{\xi}]_b) + (\mathcal{P}^\perp(vh), x)]\} f((\vec{e}, x)A^T + (\vec{v}, [\vec{\xi}]_b)) d\sigma(v) dw_\varphi(x),$$

where $(\vec{e}, x) = ((e_1, x), \ldots, (e_r, x))$ and K_m is given by (14). *Moreover if $p = 1$, q is a nonzero real number and $\{\lambda_m\}_{m=1}^\infty$ is a sequence in $\mathbb{C}_+$ converging to $-iq$ as m approaches ∞, then $E^{anf_q}[(G_r)_Z|Z_n](\vec{\xi})$ is given by the right hand side of the above equality, where λ is replaced by λ_m.*

Proof. For $m > r$, $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$

$$\begin{aligned}
&\Gamma(\lambda, m, \vec{\xi}) \\
&\equiv \int_{C[0,t]} K_m(\lambda, x) \int_{L_2[0,t]} \exp\{i[(v, [\vec{\xi}]_b) + (\mathcal{P}^\perp(vh), x)]\} f((\vec{e}, x)A^T \\
&\quad + (\vec{v}, [\vec{\xi}]_b)) d\sigma(v) dw_\varphi(x) \\
&= \left(\frac{1}{2\pi}\right)^{\frac{m+1}{2}} \int_{L_2[0,t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{\mathbb{R}^{m+1}} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) \exp\bigg\{\frac{1-\lambda}{2} \\
&\quad \times \sum_{j=1}^m u_j^2 + i\bigg[\sum_{j=1}^m c_j(\mathcal{P}^\perp(vh))u_j + \|f_{m+1}\| u_{m+1}\bigg] - \frac{1}{2}\sum_{j=1}^{m+1} u_j^2\bigg\} d(u_1, \ldots, \\
&\quad u_m, u_{m+1}) d\sigma(v) \\
&= \left(\frac{1}{2\pi}\right)^{\frac{m}{2}} \int_{L_2[0,t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{\mathbb{R}^m} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) \exp\bigg\{-\frac{\lambda}{2}\sum_{j=1}^m u_j^2 \\
&\quad + i\sum_{j=1}^m c_j(\mathcal{P}^\perp(vh))u_j - \frac{1}{2}\|f_{m+1}\|^2\bigg\} d(u_1, \ldots, u_m) d\sigma(v)
\end{aligned}$$

by the generalized Wiener integration theorem [9, Theorem 3.5] and (16), where $\vec{u} = (u_1, \ldots, u_r)$ and f_{m+1} is as given in the proof of Theorem 5.2. By (16)

$$\begin{aligned}
&\Gamma(\lambda, m, \vec{\xi}) \\
&= \lambda^{-\frac{m}{2}} \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{L_2[0,t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{\mathbb{R}^r} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b)) \exp\bigg\{-\frac{\lambda}{2}\|\vec{u}\|_{\mathbb{R}^r}^2 \\
&\quad + i\langle \vec{c}(\mathcal{P}^\perp(vh)), \vec{u}\rangle_{\mathbb{R}^r} - \frac{1}{2\lambda}\sum_{j=r+1}^m (c_j(\mathcal{P}^\perp(vh)))^2 - \frac{1}{2}\bigg[\|\mathcal{P}^\perp(vh)\|^2 - \sum_{j=1}^m \\
&\quad (c_j(\mathcal{P}^\perp(vh)))^2\bigg]\bigg\} d\vec{u}\, d\sigma(v),
\end{aligned}$$

where $\vec{c}(\mathcal{P}^{\perp}(vh)) = (c_1(\mathcal{P}^{\perp}(vh)), \ldots, c_r(\mathcal{P}^{\perp}(vh)))$. Now we have by the dominated convergence theorem and the Parseval's identity

$$\begin{aligned}&\lim_{m\to\infty} \lambda^{\frac{m}{2}}\Gamma(\lambda, m, \vec{\xi})\\ &= \left(\frac{\lambda}{2\pi}\right)^{\frac{r}{2}} \int_{L_2[0,t]} \exp\{i(v, [\vec{\xi}]_b)\} \int_{\mathbb{R}^r} f(\vec{u}A^T + (\vec{v}, [\vec{\xi}]_b))A_3(\lambda, v, \vec{u})d\vec{u}d\sigma(v),\end{aligned}$$

where A_3 is given by (13). Now the proof of the first part of the theorem is completed by Theorem 3.4. The second part of the theorem immediately follows from the dominated convergence theorem. □

Combining Theorems 5.2 and 5.3 we have the following corollary by the linearities of the generalized conditional Wiener and Feynman integrals on the analogue of Wiener space.

Corollary 5.4. *Let $(\phi(\vec{v}, \cdot) + F_r)F$ be as given in Corollary 3.5. Then for $\lambda \in \mathbb{C}_+$ and a.e. $\vec{\xi} \in \mathbb{R}^n$*

$$\begin{aligned}&E^{anw_\lambda}[((\phi(\vec{v}, \cdot) + F_r)F)_Z|Z_n](\vec{\xi})\\ &= \lim_{m\to\infty} \lambda^{\frac{m}{2}} \int_{C[0,t]} K_m(\lambda, x) \int_{L_2[0,t]} \exp\{i(\mathcal{P}^{\perp}(vh), x)\} \bigg[\int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z})\\ &\quad \times \exp\{i\langle(\vec{e}, x), \vec{z}A\rangle_{\mathbb{R}^r}\}d\rho(\vec{z}) + \exp\{i(v, [\vec{\xi}]_b)\}f((\vec{e}, x)A^T + (\vec{v}, [\vec{\xi}]_b))\bigg]\\ &\quad d\sigma(v)dw_\varphi(x),\end{aligned}$$

where $(\vec{e}, x) = ((e_1, x), \ldots, (e_r, x))$, A, A_1 and K_m are given by (6), (10) *and* (14), *respectively. Moreover if $p = 1$, q is a nonzero real number and $\{\lambda_m\}_{m=1}^{\infty}$ is a sequence in $\mathbb{C}_+$ converging to $-iq$ as m approaches ∞, then $E^{anf_q}[((\phi(\vec{v}, \cdot) + F_r)F)_Z|Z_n](\vec{\xi})$ is given by the right hand side of the above equality, where λ is replaced by λ_m.*

Letting $\lambda = \gamma^{-2}$ in Corollary 5.4 we have the following change of scale formula for the generalized conditional Wiener integrals on the analogue of Wiener space using the cylinder functions.

Corollary 5.5. *Let F, F_r and ϕ be as given in Corollary 4.4. Then for $\rho > 0$ and a.e. $\vec{\xi} \in \mathbb{R}^n$*

$$\begin{aligned}&E[F(\gamma Z(x, \cdot))(\phi(\vec{v}, \gamma Z(x, \cdot)) + F_r(\gamma Z(x, \cdot)))|\gamma Z_n(x)](\vec{\xi})\\ &= \lim_{m\to\infty} \gamma^{-m} \int_{C[0,t]} \exp\bigg\{\frac{2\gamma^2 - 1}{2\gamma^2}\sum_{j=1}^{m}(e_j, x)^2\bigg\} \int_{L_2[0,t]} \exp\{i(\mathcal{P}^{\perp}(vh), x)\}\\ &\quad \times \bigg[\int_{\mathbb{R}^r} A_1(\vec{\xi}, v, \vec{z}) \exp\{i\langle(\vec{e}, x), \vec{z}A\rangle_{\mathbb{R}^r}\}d\rho(\vec{z}) + \exp\{i(v, [\vec{\xi}]_b)\}f((\vec{e}, x)A^T\\ &\qquad + (\vec{v}, [\vec{\xi}]_b))\bigg]d\sigma(v)dw_\varphi(x).\end{aligned}$$

Remark 5.6. (1) The choice of the orthonormal set $\{h_1, \ldots, h_r\}$ in Theorem 5.1 is independent of $\{e_1, \ldots, e_r\}$.

(2) The results of this paper are different from those in [6, 8, 11]. If $h = 1$ a.e. on $[0, t]$, then $F(Z(x, \cdot)) = F(x - x(0))$ and $Z_n(x) = (x(t_1) - x(0), \ldots, x(t_n) - x(0))$. In this case we can take an orthonormal subset $\{v_1, v_2, \ldots, v_r\}$ of $L_2[0, t]$ such that $\mathcal{P}^\perp v_1, \ldots, \mathcal{P}^\perp v_r$ are independent [11, Remark 1]. Furthermore if $\varphi = \delta_0$, the Dirac measure concentrated at 0, then Theorems 4.2 and 4.3 generalize the equations (28) and (29) in [11].

(3) The results of this paper are independent of a particular choice of the probability measure φ.

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