

INT-SOFT FILTERS IN LATTICE IMPLICATION ALGEBRAS

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ABSTRACT. The notion of int-soft (implicative) filters in lattice implication algebras is introduced, and related properties are investigated. Characterizations of int-soft (implicative) filters are discussed. Conditions for an int-soft filter to be an int-soft implicative filter are provided. Extension property for int-soft implicative filters is established.

1. Introduction

In the field of many-valued logic, lattice-valued logic plays an important role for two aspects: One is that it extends the chain-type truth-value field of some well-known presented logic [4] to some relatively general lattices. The other is that the incompletely comparable property of truth value characterized by general lattice can more efficiently reflect the uncertainty of people's thinking, judging and decision. Hence, lattice-valued logic is becoming a research field which strongly influences the development of Algebraic Logic, Computer Science and Artificial Intelligence Technology. Therefore Goguen [10], Pavelka [25] and Novak [23] researched on this lattice-valued logic formal systems. In order to research the logical system whose propositional value is given in a lattice, Xu [26] proposed the concept of lattice implication algebras, and discussed their some properties. For the general development of lattice implication algebras, filter theory and its fuzzification play an important role. Xu and Qin [27] introduced the notion of (implicative) filters in a lattice implication algebra, and investigated their properties. Jun (together with Xu and Qin) [11, 19] discussed positive implicative and associative filters of a lattice implication algebra, and Jun [12] considered the fuzzification of positive implicative and associative filters of a lattice implication algebra. In [28], Xu and Qin considered the fuzzification of (implicative) filters.

Molodtsov [22] introduced the concept of soft set as a new mathematical tool for dealing with uncertainties that is free from the difficulties that have troubled the usual theoretical approaches. Molodtsov pointed out several directions

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for the applications of soft sets. At present, works on the soft set theory are progressing rapidly. Maji et al. [21] described the application of soft set theory to a decision making problem. Maji et al. [20] also studied several operations on the theory of soft sets. Chen et al. [7] presented a new definition of soft set parametrization reduction, and compared this definition to the related concept of attributes reduction in rough set theory. The algebraic structure of set theories dealing with uncertainties has been studied by some authors. Çağman et al. [6] introduced fuzzy parameterized (FP) soft sets and their related properties. They proposed a decision making method based on FP-soft set theory, and provided an example which shows that the method can be successfully applied to the problems that contain uncertainties. Feng [8] considered the application of soft rough approximations in multicriteria group decision making problems. Aktaş and Çağman [2] studied the basic concepts of soft set theory, and compared soft sets to fuzzy and rough sets, providing examples to clarify their differences. They also discussed the notion of soft groups. After than, many algebraic properties of soft sets are studied (see [1, 3, 9, 13, 14, 15, 16, 17, 18, 24, 30]).

In this paper, we introduce the notion of int-soft (implicative) filters in lattice implication algebras, and investigated related properties. We discuss characterizations of int-soft (implicative) filters. We provide conditions for an int-soft filter to be an int-soft implicative filter. We establish extension property for int-soft implicative filters.

2. Preliminaries

By a *lattice implication algebra* we mean a bounded lattice $L := (L, \vee, \wedge, 0, 1)$ with order-reversing involution “ \prime ” and a binary operation “ \rightarrow ” satisfying the following axioms:

- (I1) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$,
- (I2) $x \rightarrow x = 1$,
- (I3) $x \rightarrow y = y' \rightarrow x'$,
- (I4) $x \rightarrow y = y \rightarrow x = 1 \Rightarrow x = y$,
- (I5) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,
- (L1) $(x \vee y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$,
- (L2) $(x \wedge y) \rightarrow z = (x \rightarrow z) \vee (y \rightarrow z)$,

for all $x, y, z \in L$.

In a lattice implication algebra L , the following hold (see [26]):

- (a1) $0 \rightarrow x = 1$, $1 \rightarrow x = x$ and $x \rightarrow 1 = 1$,
- (a2) $x \rightarrow y \leq (y \rightarrow z) \rightarrow (x \rightarrow z)$,
- (a3) $x \leq y$ implies $y \rightarrow z \leq x \rightarrow z$ and $z \rightarrow x \leq z \rightarrow y$,
- (a4) $x' = x \rightarrow 0$,
- (a5) $x \vee y = (x \rightarrow y) \rightarrow y$,
- (a6) $((y \rightarrow x) \rightarrow y')' = x \wedge y = ((x \rightarrow y) \rightarrow x')'$,
- (a7) $x \leq (x \rightarrow y) \rightarrow y$,

where $x \leq y$ means $x \rightarrow y = 1$.

A subset F of a lattice implication algebra L is called a *filter* of L (see [27]) if it satisfies:

- (F1) $1 \in F$,
- (F2) $x \in F$ and $x \rightarrow y \in F$ imply $y \in F$

for all $x, y \in L$.

A subset F of a lattice implication algebra L is called an *implicative filter* of L (see [27]) if it satisfies (F1) and

- (F3) $x \rightarrow y \in F$ and $x \rightarrow (y \rightarrow z) \in F$ imply $x \rightarrow z \in F$

for all $x, y, z \in L$. Note that every implicative filter is a filter, but the converse is not true in general.

A soft set theory is introduced by Molodtsov [22], and Çağman et al. [5] provided new definitions and various results on soft set theory.

In what follows, let U be an initial universe set and E be a set of parameters. Let $\mathcal{P}(U)$ denotes the power set of U and $A, B, C, \dots \subseteq E$.

A *soft set* $(\tilde{\alpha}, A)$ over U is defined to be the set of ordered pairs

$$(\tilde{\alpha}, A) := \{(x, \tilde{\alpha}(x)) : x \in E, \tilde{\alpha}(x) \in \mathcal{P}(U)\},$$

where $\tilde{\alpha} : E \rightarrow \mathcal{P}(U)$ such that $\tilde{\alpha}(x) = \emptyset$ if $x \notin A$.

The function $\tilde{\alpha}$ is called approximate function of the soft set $(\tilde{\alpha}, A)$. The subscript A in the notation $\tilde{\alpha}$ indicates that $\tilde{\alpha}$ is the approximate function of $(\tilde{\alpha}, A)$.

For a soft set $(\tilde{\alpha}, A)$ over U and a subset ε of U , the ε -inclusive set of $(\tilde{\alpha}, A)$, denoted by $i_A(\tilde{\alpha}; \varepsilon)$, is defined to be the set

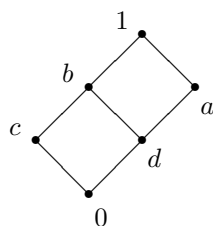
$$i_A(\tilde{\alpha}; \varepsilon) := \{x \in A \mid \varepsilon \subseteq \tilde{\alpha}(x)\}.$$

3. Int-soft filters

In what follows, let $S(U, L)$ denote the set of all soft sets of L over U where L is a lattice implication algebra unless otherwise specified.

Definition. A soft set $(\tilde{\alpha}, L) \in S(U, L)$ is an *int-soft filter* of L over U if the ε -inclusive set $i_L(\tilde{\alpha}; \varepsilon)$ of $(\tilde{\alpha}, L)$ is a filter of L for all $\varepsilon \in \mathcal{P}(U)$ with $i_L(\tilde{\alpha}; \varepsilon) \neq \emptyset$.

Example 3.1. Let $L = \{0, a, b, c, d, 1\}$ be a set with the following Hasse diagram and Cayley tables:



| x | x' |
|-----|------|
| 0 | 1 |
| a | c |
| b | d |
| c | a |
| d | b |
| 1 | 0 |

| \rightarrow | 0 | a | b | c | d | 1 |
|---------------|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| a | c | 1 | b | c | b | 1 |
| b | d | a | 1 | b | a | 1 |
| c | a | a | 1 | 1 | a | 1 |
| d | b | b | 1 | 1 | b | 1 |
| 1 | 0 | 0 | a | b | c | d |

Then L is a lattice implication algebra (see [29]). Let $(\tilde{\alpha}, L)$ be a soft set over $U = \mathbb{Z}$ in L given as follows:

$$\tilde{\alpha} : L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} 2\mathbb{Z} & \text{if } x \in \{a, 1\}, \\ 2\mathbb{N} & \text{otherwise.} \end{cases}$$

Then $(\tilde{\alpha}, L)$ is an int-soft filter of L over U .

Theorem 3.2. *A soft set $(\tilde{\alpha}, L) \in S(U, L)$ is an int-soft filter of L over U if and only if the following assertions are valid.*

$$(3.1) \quad (\forall x \in L) (\tilde{\alpha}(1) \supseteq \tilde{\alpha}(x)),$$

$$(3.2) \quad (\forall x, y \in L) (\tilde{\alpha}(y) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(x \rightarrow y)).$$

Proof. Let $(\tilde{\alpha}, L) \in S(U, L)$ be satisfy the conditions (3.1) and (3.2). Let $\varepsilon \in \mathcal{P}(U)$ be such that $i_L(\tilde{\alpha}; \varepsilon) \neq \emptyset$. Then there exists $a \in i_L(\tilde{\alpha}; \varepsilon)$, and so $\tilde{\alpha}(a) \supseteq \varepsilon$. It follows from (3.1) that $\tilde{\alpha}(1) \supseteq \tilde{\alpha}(a) \supseteq \varepsilon$ and so that $1 \in i_L(\tilde{\alpha}; \varepsilon)$. Let $x, y \in L$ be such that $x \in i_L(\tilde{\alpha}; \varepsilon)$ and $x \rightarrow y \in i_L(\tilde{\alpha}; \varepsilon)$. Then $\varepsilon \subseteq \tilde{\alpha}(x)$ and $\varepsilon \subseteq \tilde{\alpha}(x \rightarrow y)$. Using (3.2), we get $\tilde{\alpha}(y) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(x \rightarrow y) \supseteq \varepsilon$. Thus $y \in i_L(\tilde{\alpha}; \varepsilon)$, and hence $i_L(\tilde{\alpha}; \varepsilon)$ is a filter of L for all $\varepsilon \in \mathcal{P}(U)$ with $i_L(\tilde{\alpha}; \varepsilon) \neq \emptyset$. Therefore $(\tilde{\alpha}, L)$ is an int-soft filter of L over U .

Conversely, suppose that $(\tilde{\alpha}, L)$ is an int-soft filter of L over U . For any $x \in L$, let $\tilde{\alpha}(x) = \varepsilon_x$. Then $x \in i_L(\tilde{\alpha}; \varepsilon_x)$, and so $i_L(\tilde{\alpha}; \varepsilon_x) \neq \emptyset$. Hence $i_L(\tilde{\alpha}; \varepsilon_x)$ is a filter of L , and thus $1 \in i_L(\tilde{\alpha}; \varepsilon_x)$. Hence $\tilde{\alpha}(1) \supseteq \varepsilon_x = \tilde{\alpha}(x)$ for all $x \in L$. For any $x, y \in L$, let $\tilde{\alpha}(x) \cap \tilde{\alpha}(x \rightarrow y) = \delta$. Then $\tilde{\alpha}(x) \supseteq \delta$ and $\tilde{\alpha}(x \rightarrow y) \supseteq \delta$, that is, $x \in i_L(\tilde{\alpha}; \delta)$ and $x \rightarrow y \in i_L(\tilde{\alpha}; \delta)$. It follows from (F2) that $y \in i_L(\tilde{\alpha}; \delta)$ and so that $\tilde{\alpha}(y) \supseteq \delta = \tilde{\alpha}(x) \cap \tilde{\alpha}(x \rightarrow y)$ for all $x, y \in L$. \square

Proposition 3.3. *Every int-soft filter $(\tilde{\alpha}, L)$ of L over U satisfies:*

$$(3.3) \quad (\forall x, y \in L) (x \rightarrow y = 1 \Rightarrow \tilde{\alpha}(x) \subseteq \tilde{\alpha}(y)).$$

Proof. Let $x, y \in L$ be such that $x \rightarrow y = 1$. Then

$$\tilde{\alpha}(y) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(x \rightarrow y) = \tilde{\alpha}(x) \cap \tilde{\alpha}(1) = \tilde{\alpha}(x)$$

by (3.2) and (3.1). \square

Theorem 3.4. *A soft set $(\tilde{\alpha}, L) \in S(U, L)$ is an int-soft filter of L over U if and only if it satisfies (3.1) and*

$$(3.4) \quad (\forall x, y, z \in L) (\tilde{\alpha}(x \rightarrow z) \supseteq \tilde{\alpha}(x \rightarrow y) \cap \tilde{\alpha}(y \rightarrow z)).$$

Proof. Assume that $(\tilde{\alpha}, L)$ is an int-soft filter of L over U . Since

$$(x \rightarrow y) \rightarrow ((y \rightarrow z) \rightarrow (x \rightarrow z)) = 1$$

for all $x, y, z \in L$, it follows from (3.3) that $\tilde{\alpha}(x \rightarrow y) \subseteq \tilde{\alpha}((y \rightarrow z) \rightarrow (x \rightarrow z))$ and so from (3.2) that

$$\begin{aligned} \tilde{\alpha}(x \rightarrow z) &\supseteq \tilde{\alpha}(y \rightarrow z) \cap \tilde{\alpha}((y \rightarrow z) \rightarrow (x \rightarrow z)) \\ &\supseteq \tilde{\alpha}(y \rightarrow z) \cap \tilde{\alpha}(x \rightarrow y) \end{aligned}$$

for all $x, y, z \in L$.

Conversely, let $(\tilde{\alpha}, L)$ satisfy (3.1) and (3.4). Taking $x = 1$ in (3.4) and using (a1), we have

$$\begin{aligned}\tilde{\alpha}(z) &= \tilde{\alpha}(1 \rightarrow z) \supseteq \tilde{\alpha}(1 \rightarrow y) \cap \tilde{\alpha}(y \rightarrow z) \\ &= \tilde{\alpha}(y) \cap \tilde{\alpha}(y \rightarrow z)\end{aligned}$$

for all $y, z \in L$. Therefore $(\tilde{\alpha}, L)$ is an int-soft filter of L over U . \square

Theorem 3.5. For any $(\tilde{\alpha}, L) \in S(U, L)$, the following assertions are equivalent.

- (1) $(\tilde{\alpha}, L)$ is an int-soft filter of L over U .
- (2) $(\forall x, y, z \in L) (x \rightarrow (y \rightarrow z) = 1 \Rightarrow \tilde{\alpha}(z) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y))$.

Proof. Suppose that $(\tilde{\alpha}, L)$ is an int-soft filter of L over U . Let $x, y, z \in L$ be such that $x \rightarrow (y \rightarrow z) = 1$. Using (3.2) and (3.3) implies that $\tilde{\alpha}(z) \supseteq \tilde{\alpha}(y) \cap \tilde{\alpha}(y \rightarrow z) \supseteq \tilde{\alpha}(y) \cap \tilde{\alpha}(x)$.

Assume that the second condition is valid. Since $x \rightarrow (x \rightarrow 1) = 1$ for all $x \in L$, we have $\tilde{\alpha}(1) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(x) = \tilde{\alpha}(x)$ for all $x \in L$. Note that $y \rightarrow ((y \rightarrow x) \rightarrow x) = 1$ for all $x, y \in L$. Hence $\tilde{\alpha}(x) \supseteq \tilde{\alpha}(y) \cap \tilde{\alpha}(y \rightarrow x)$ for all $x, y \in L$. Therefore $(\tilde{\alpha}, L)$ is an int-soft filter of L over U . \square

Theorem 3.6. A soft set $(\tilde{\alpha}, L) \in S(U, L)$ is an int-soft filter of L over U if and only if it satisfies (3.1), (3.3) and

$$(3.5) \quad (\forall x, y \in L) (\tilde{\alpha}((x \rightarrow y)')) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(y).$$

Proof. Assume that $(\tilde{\alpha}, L)$ is an int-soft filter of L over U . Then the conditions (3.1) and (3.3) are valid by Theorem 3.2 and Proposition 3.3. Using (3.1), (3.2) and (I2), we have

$$\begin{aligned}\tilde{\alpha}((x \rightarrow y)') &\supseteq \tilde{\alpha}(y) \cap \tilde{\alpha}(y \rightarrow (x \rightarrow y)') \\ &\supseteq \tilde{\alpha}(y) \cap \tilde{\alpha}(x) \cap \tilde{\alpha}(x \rightarrow (y \rightarrow (x \rightarrow y)')) \\ &= \tilde{\alpha}(y) \cap \tilde{\alpha}(x) \cap ((x \rightarrow y)' \rightarrow (x \rightarrow y)') \\ &= \tilde{\alpha}(x) \cap \tilde{\alpha}(y) \cap \tilde{\alpha}(1) \\ &= \tilde{\alpha}(x) \cap \tilde{\alpha}(y)\end{aligned}$$

for all $x, y \in L$. Hence (3.5) is valid.

Conversely, let $(\tilde{\alpha}, L) \in S(U, L)$ satisfy conditions (3.1), (3.3) and (3.5). Note that $(x \rightarrow (x \rightarrow y)')' \rightarrow y = 1$ for all $x, y \in L$. It follows from (3.3) and (3.5) that

$$\tilde{\alpha}(y) \supseteq \tilde{\alpha}((x \rightarrow (x \rightarrow y)')) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(x \rightarrow y)$$

for all $x, y \in L$. Therefore $(\tilde{\alpha}, L)$ is an int-soft filter of L over U by Theorem 3.2. \square

Let $(\tilde{\alpha}, L) \in S(U, L)$ where $U = \mathbb{Z}$ and

$$\tilde{\alpha} : L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} 8\mathbb{N} & \text{if } x \in \{0, c\}, \\ 4\mathbb{Z} & \text{if } x = a, \\ 4\mathbb{N} & \text{if } x = b, \\ 2\mathbb{Z} & \text{if } x = 1. \end{cases}$$

Then $[\tilde{\alpha}(b)] = \{a, b, 1\}$ is not a filter of L since $a \rightarrow c = 1 \in [\tilde{\alpha}(b)]$ and $a \in [\tilde{\alpha}(b)]$, but $c \notin [\tilde{\alpha}(b)]$.

We provide conditions for the set $[\tilde{\alpha}(a)]$ to be a filter of L for $a \in L$.

Theorem 3.9. *Let $a \in L$. If $(\tilde{\alpha}, L)$ is an int-soft filter of L over U , then $[\tilde{\alpha}(a)]$ is a filter of L .*

Proof. Obviously $1 \in [\tilde{\alpha}(a)]$ by (3.1). Let $x, y \in L$ be such that $x \rightarrow y \in [\tilde{\alpha}(a)]$ and $x \in [\tilde{\alpha}(a)]$. Then $\tilde{\alpha}(x \rightarrow y) \supseteq \tilde{\alpha}(a)$ and $\tilde{\alpha}(x) \supseteq \tilde{\alpha}(a)$. It follows from (3.2) that

$$\tilde{\alpha}(y) \supseteq \tilde{\alpha}(x) \cap \tilde{\alpha}(x \rightarrow y) \supseteq \tilde{\alpha}(a).$$

Thus $y \in [\tilde{\alpha}(a)]$ and $[\tilde{\alpha}(a)]$ is a filter of L . □

Theorem 3.10. *For any $a \in L$ and $(\tilde{\alpha}, L) \in S(U, L)$, we have the following assertions:*

(1) *If $[\tilde{\alpha}(a)]$ is a filter of L , then $(\tilde{\alpha}, L)$ satisfies the following implication.*

$$(3.7) \quad (\forall x, y \in L) (\tilde{\alpha}(a) \subseteq \tilde{\alpha}(x \rightarrow y) \cap \tilde{\alpha}(x) \Rightarrow \tilde{\alpha}(a) \subseteq \tilde{\alpha}(y)).$$

(2) *If $(\tilde{\alpha}, L)$ satisfies (3.1) and (3.7), then $[\tilde{\alpha}(a)]$ is a filter of L .*

Proof. (1) Assume that $[\tilde{\alpha}(a)]$ is a filter of L for $a \in L$. Let $x, y \in L$ be such that

$$\tilde{\alpha}(a) \subseteq \tilde{\alpha}(x \rightarrow y) \cap \tilde{\alpha}(x).$$

Then $x \rightarrow y \in [\tilde{\alpha}(a)]$ and $x \in [\tilde{\alpha}(a)]$. Since $[\tilde{\alpha}(a)]$ is a filter of L , it follows that $y \in [\tilde{\alpha}(a)]$, that is, $\tilde{\alpha}(a) \subseteq \tilde{\alpha}(y)$.

(2) Suppose that $(\tilde{\alpha}, L)$ satisfies (3.1) and (3.7). Let $x, y \in L$ be such that $x \rightarrow y \in [\tilde{\alpha}(a)]$ and $x \in [\tilde{\alpha}(a)]$. Then $\tilde{\alpha}(a) \subseteq \tilde{\alpha}(x \rightarrow y)$ and $\tilde{\alpha}(a) \subseteq \tilde{\alpha}(x)$, which implies that

$$\tilde{\alpha}(a) \subseteq \tilde{\alpha}(x \rightarrow y) \cap \tilde{\alpha}(x).$$

It follows from (3.7) that $\tilde{\alpha}(a) \subseteq \tilde{\alpha}(y)$, i.e., $y \in [\tilde{\alpha}(a)]$. Since $(\tilde{\alpha}, L)$ satisfies (3.1), we have $1 \in [\tilde{\alpha}(a)]$. Therefore $[\tilde{\alpha}(a)]$ is a filter of L . □

For a fixed element $a \in L$, let $(\tilde{\alpha}_a, L) \in S(U, L)$ be given as follows:

$$\tilde{\alpha}_a : L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} \varepsilon_1 & \text{if } a \rightarrow x = 1, \\ \varepsilon_2 & \text{otherwise} \end{cases}$$

where $\varepsilon_1, \varepsilon_2 \in \mathcal{P}(U)$ with $\varepsilon_1 \supseteq \varepsilon_2$.

Let $L = \{0, a, b, c, 1\}$ be the lattice implication algebra in Example 3.8. For $b \in L$, the soft set $(\tilde{\alpha}_b, L)$ over $U = \{1, 2, 3, 4, 5, 6\}$ which is given by

$$\tilde{\alpha}_b : L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} \{1, 2, 3, 5\} & \text{if } b \rightarrow x = 1, \\ \{3, 5\} & \text{otherwise} \end{cases}$$

is not an int-soft filter of L over U since $\tilde{\alpha}_b(a) = \{3, 5\} \not\supseteq \{1, 2, 3, 5\} = \tilde{\alpha}_b(c) \cap \tilde{\alpha}_b(c \rightarrow a)$.

Given $a \in L$, we provide conditions for the soft set $(\tilde{\alpha}_a, L) \in S(U, L)$ to be an int-soft filter of L over U .

Theorem 3.11. *Given $a \in L$, the soft set $(\tilde{\alpha}_a, L) \in S(U, L)$ is an int-soft filter of L over U if and only if the following assertion is valid.*

$$(3.8) \quad (\forall x, y \in L) (a \rightarrow (y \rightarrow x) = 1, a \rightarrow y = 1 \Rightarrow a \rightarrow x = 1).$$

Proof. Suppose that $(\tilde{\alpha}_a, L)$ is an int-soft filter of L over U and let $x, y \in L$ be such that $a \rightarrow (y \rightarrow x) = 1$ and $a \rightarrow y = 1$. Then $\tilde{\alpha}_a(y \rightarrow x) = \varepsilon_1 = \tilde{\alpha}_a(y)$, and so

$$\tilde{\alpha}_a(x) \supseteq \tilde{\alpha}_a(y) \cap \tilde{\alpha}_a(y \rightarrow x) = \varepsilon_1.$$

Thus $a \rightarrow x = 1$, which satisfies the condition (3.8).

Conversely, assume that the condition (3.8) is valid. Note that

$$i_L(\tilde{\alpha}_a; \varepsilon) = \begin{cases} L & \text{if } \varepsilon \subseteq \varepsilon_2, \\ \{x \in L \mid a \rightarrow x = 1\} & \text{if } \varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1, \\ \emptyset & \text{otherwise.} \end{cases}$$

For the case of $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$, obviously $1 \in i_L(\tilde{\alpha}_a; \varepsilon)$. Let $x, y \in L$ be such that $x \in i_L(\tilde{\alpha}_a; \varepsilon)$ and $x \rightarrow y \in i_L(\tilde{\alpha}_a; \varepsilon)$. Then $a \rightarrow x = 1$ and $a \rightarrow (x \rightarrow y) = 1$, which imply from the hypothesis that $a \rightarrow y = 1$, that is, $y \in i_L(\tilde{\alpha}_a; \varepsilon)$. Hence $i_L(\tilde{\alpha}_a; \varepsilon)$ is a filter of L whenever it is nonempty. Therefore $(\tilde{\alpha}_a, L)$ is an int-soft filter of L over U . □

Theorem 3.12. *For a subset J of L , let $(\tilde{\beta}, L) \in S(U, L)$ be given as follows:*

$$\tilde{\beta} : L \rightarrow \mathcal{P}(U), x \mapsto \begin{cases} \varepsilon_1 & \text{if } x \in J, \\ \varepsilon_2 & \text{otherwise,} \end{cases}$$

where $\varepsilon_1, \varepsilon_2 \in \mathcal{P}(U)$ with $\varepsilon_1 \supseteq \varepsilon_2$. Then $(\tilde{\beta}, L)$ is an int-soft filter of L over U if and only if the following assertion is valid.

$$(3.9) \quad (\forall x, y, z \in L) (x, y \in J, y \rightarrow (x \rightarrow z) = 1 \Rightarrow z \in J).$$

Proof. Note that

$$i_L(\tilde{\beta}; \varepsilon) = \begin{cases} L & \text{if } \varepsilon \subseteq \varepsilon_2, \\ J & \text{if } \varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Assume that $(\tilde{\beta}, L)$ is an int-soft filter of L over U . Then $J = i_L(\tilde{\beta}; \varepsilon)$ for $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$, and J is a filter of L . Let $x, y, z \in L$ be such that $x, y \in J$ and $y \rightarrow (x \rightarrow z) = 1$. Since $1 \in J$, it follows that $z \in J$.

Conversely, let $(\tilde{\beta}, L) \in S(U, L)$ and suppose that (3.9) is valid. Since $y \rightarrow (x \rightarrow 1) = y \rightarrow 1 = 1$ for all $x, y \in L$, we have $1 \in J$ by (3.9), and so $1 \in i_L(\tilde{\beta}; \varepsilon)$ for $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$. Let $x, y \in L$ be such that $y \in J = i_L(\tilde{\beta}; \varepsilon)$ and $y \rightarrow x \in J = i_L(\tilde{\beta}; \varepsilon)$ for $\varepsilon_2 \subsetneq \varepsilon \subseteq \varepsilon_1$. Since $y \rightarrow ((y \rightarrow x) \rightarrow x) = 1$, it follows from (3.9) that $x \in J = i_L(\tilde{\beta}; \varepsilon)$. Hence $i_L(\tilde{\beta}; \varepsilon)$ is a filter of L for all $\varepsilon \in \mathcal{P}(U)$ with $i_L(\tilde{\beta}; \varepsilon) \neq \emptyset$. Therefore $(\tilde{\beta}, L)$ is an int-soft filter of L over U . \square

4. Int-soft implicative filters

Definition. A soft set $(\tilde{\alpha}, L) \in S(U, L)$ is an *int-soft implicative filter* of L over U if the ε -inclusive set $i_L(\tilde{\alpha}; \varepsilon)$ of $(\tilde{\alpha}, L)$ is an implicative filter of L for all $\varepsilon \in \mathcal{P}(U)$ with $i_L(\tilde{\alpha}; \varepsilon) \neq \emptyset$.

Theorem 4.1. A soft set $(\tilde{\alpha}, L) \in S(U, L)$ is an int-soft implicative filter of L over U if and only if it satisfies the condition (3.1) and

$$(4.1) \quad (\forall x, y \in L) (\tilde{\alpha}(x \rightarrow z) \supseteq \tilde{\alpha}(x \rightarrow y) \cap \tilde{\alpha}(x \rightarrow (y \rightarrow z))).$$

Proof. The proof is similar to the proof of Theorem 3.2. \square

Theorem 4.2. Every int-soft implicative filter is an int-soft filter.

Proof. Let $(\tilde{\alpha}, L)$ be an int-soft implicative filter of L over U . If we take $x = 1$ in (4.1) and use (a1), then $\tilde{\alpha}(z) \supseteq \tilde{\alpha}(y) \cap \tilde{\alpha}(y \rightarrow z)$ for all $y, z \in L$. Therefore $(\tilde{\alpha}, L)$ is an int-soft filter of L over U . \square

The converse of Theorem 4.2 is not true in general. In fact, consider the lattice implication algebra $L = \{0, a, b, c, d, 1\}$ in Example 3.1. Let $(\tilde{\alpha}, L)$ be a soft set over $U = \mathbb{N}$ in L given as follows:

$$\tilde{\alpha} : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} 3\mathbb{N} & \text{if } x = 1, \\ 6\mathbb{N} & \text{otherwise.} \end{cases}$$

Then $(\tilde{\alpha}, L)$ is an int-soft filter of L over U . But it is not an int-soft implicative filter of L over U since $\tilde{\alpha}(d \rightarrow b) \cap \tilde{\alpha}(d \rightarrow (b \rightarrow 0)) \not\subseteq \tilde{\alpha}(d \rightarrow 0)$.

Theorem 4.3. Let $a \in L$. If $(\tilde{\alpha}, L)$ is an int-soft implicative filter of L over U , then $[\tilde{\alpha}(a)]$ is an implicative filter of L .

Proof. Obviously $1 \in [\tilde{\alpha}(a)]$ by (3.1). Let $x, y, z \in L$ be such that $x \rightarrow y \in [\tilde{\alpha}(a)]$ and $x \rightarrow (y \rightarrow z) \in [\tilde{\alpha}(a)]$. Then $\tilde{\alpha}(x \rightarrow y) \supseteq \tilde{\alpha}(a)$ and $\tilde{\alpha}(x \rightarrow (y \rightarrow z)) \supseteq \tilde{\alpha}(a)$. It follows from (4.1) that

$$\tilde{\alpha}(x \rightarrow z) \supseteq \tilde{\alpha}(x \rightarrow y) \cap \tilde{\alpha}(x \rightarrow (y \rightarrow z)) \supseteq \tilde{\alpha}(a).$$

Thus $x \rightarrow z \in [\tilde{\alpha}(a)]$ and $[\tilde{\alpha}(a)]$ is an implicative filter of L . \square

We now provide conditions for an int-soft filter to be an int-soft implicative filter.

Theorem 4.4. *Let $(\tilde{\alpha}, L) \in S(U, L)$. If $(\tilde{\alpha}, L)$ is an int-soft filter of L over U satisfying the condition:*

$$(4.2) \quad (\forall x, y, z \in L) (\tilde{\alpha}(y \rightarrow z) \supseteq \tilde{\alpha}(x \rightarrow (y \rightarrow (y \rightarrow z))) \cap \tilde{\alpha}(x)),$$

then $(\tilde{\alpha}, L)$ is an int-soft implicative filter of L over U .

Proof. Note that $(x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow (x \rightarrow z))) = 1$ for all $x, y, z \in L$. It follows from (3.3) and (4.2) that

$$\begin{aligned} \tilde{\alpha}(x \rightarrow z) &\supseteq \tilde{\alpha}((x \rightarrow y) \rightarrow (x \rightarrow (x \rightarrow z))) \cap \tilde{\alpha}(x \rightarrow y) \\ &\supseteq \tilde{\alpha}(x \rightarrow (y \rightarrow z)) \cap \tilde{\alpha}(x \rightarrow y) \end{aligned}$$

for all $x, y, z \in L$. Therefore $(\tilde{\alpha}, L)$ is an int-soft implicative filter of L over U . \square

Theorem 4.5. *For a soft set $(\tilde{\alpha}, L) \in S(U, L)$, the following are equivalent.*

- (i) $(\tilde{\alpha}, L)$ is an int-soft implicative filter of L over U .
- (ii) $(\tilde{\alpha}, L)$ is an int-soft filter of L over U that satisfies the condition.

$$(4.3) \quad (\forall x, y \in L) (\tilde{\alpha}(x \rightarrow y) \supseteq \tilde{\alpha}(x \rightarrow (x \rightarrow y))).$$

- (iii) $(\tilde{\alpha}, L)$ is an int-soft filter of L over U that satisfies the condition.

$$(4.4) \quad (\forall x, y, z \in L) (\tilde{\alpha}((x \rightarrow y) \rightarrow (x \rightarrow z)) \supseteq \tilde{\alpha}(x \rightarrow (y \rightarrow z))).$$

Proof. Suppose that $(\tilde{\alpha}, L)$ is an int-soft implicative filter of L over U . Then $(\tilde{\alpha}, L)$ is an int-soft filter of L over U by Theorem 4.2. Using (3.1), (4.1) and (a1), we have

$$\begin{aligned} \tilde{\alpha}(x \rightarrow y) &\supseteq \tilde{\alpha}(x \rightarrow (x \rightarrow y)) \cap \tilde{\alpha}(x \rightarrow x) \\ &= \tilde{\alpha}(x \rightarrow (x \rightarrow y)) \cap \tilde{\alpha}(1) \\ &= \tilde{\alpha}(x \rightarrow (x \rightarrow y)) \end{aligned}$$

for all $x, y \in L$. Hence (ii) is valid.

Let $(\tilde{\alpha}, L)$ be an int-soft filter of L over U that satisfies the condition (4.3). Since $(x \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))) = 1$ for all $x, y, z \in L$, it follows from (3.3), (I1) and (4.3) that

$$\begin{aligned} \tilde{\alpha}((x \rightarrow y) \rightarrow (x \rightarrow z)) &= \tilde{\alpha}(x \rightarrow ((x \rightarrow y) \rightarrow z)) \\ &\supseteq \tilde{\alpha}(x \rightarrow (x \rightarrow ((x \rightarrow y) \rightarrow z))) \\ &= \tilde{\alpha}(x \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))) \\ &\supseteq \tilde{\alpha}(x \rightarrow (y \rightarrow z)) \end{aligned}$$

for all $x, y, z \in L$. Therefore (iii) is valid.

Assume that $(\tilde{\alpha}, L)$ is an int-soft filter of L over U satisfying the condition (4.4). Then

$$(4.5) \quad \begin{aligned} \tilde{\alpha}(x \rightarrow z) &\supseteq \tilde{\alpha}((x \rightarrow y) \rightarrow (x \rightarrow z)) \cap \tilde{\alpha}(x \rightarrow y) \\ &\supseteq \tilde{\alpha}(x \rightarrow (y \rightarrow z)) \cap \tilde{\alpha}(x \rightarrow y) \end{aligned}$$

for all $x, y, z \in L$ by (3.2) and (iii). Therefore $(\tilde{\alpha}, L)$ is an int-soft implicative filter of L over U . \square

For a non-empty subset F of L and a fixed element $b \in L$, let $(\tilde{\alpha}_F^b, L) \in S(U, L)$ be given as follows:

$$\tilde{\alpha}_F^b : L \rightarrow \mathcal{P}(U), \quad x \mapsto \begin{cases} \varepsilon_1 & \text{if } b \rightarrow x \in F, \\ \varepsilon_2 & \text{otherwise,} \end{cases}$$

where $\varepsilon_1, \varepsilon_2 \in \mathcal{P}(U)$ with $\varepsilon_1 \supsetneq \varepsilon_2$. We know that there exists a filter F of L and an element $b \in L$ such that $\tilde{\alpha}_F^b$ may not be an int-soft filter of L over U as seen in the following example.

Example 4.6. Let $L = \{0, a, b, c, 1\}$ be the Lukasiewicz implication algebra of order 5 in Example 3.8. Consider a filter $F := \{1\}$ and an element $b \in L$. Then

$$\tilde{\alpha}_F^b(a) = \varepsilon_2 \subsetneq \varepsilon_1 = \tilde{\alpha}_F^b(b \rightarrow a) \cap \tilde{\alpha}_F^b(b).$$

Hence $\tilde{\alpha}_F^b$ is not an int-soft filter of L over U .

We provide a condition for the soft set $(\tilde{\alpha}_F^b, L)$ to be an int-soft filter of L over U .

Theorem 4.7. *Let F be a filter of L . Then $(\tilde{\alpha}_F^b, L)$ is an int-soft filter of L over U for all $b \in L$ if and only if F is an implicative filter of L .*

Proof. Assume that F is an implicative filter of L and let b be any element of L . Since $b \rightarrow 1 = 1 \in F$, we have $\tilde{\alpha}_F^b(1) = \varepsilon_1 \supseteq \tilde{\alpha}_F^b(x)$ for all $x \in L$. Let $x, y \in L$ be such that either $b \rightarrow (x \rightarrow y) \notin F$ or $b \rightarrow x \notin F$. Then either $\tilde{\alpha}_F^b(x \rightarrow y) = \varepsilon_2$ or $\tilde{\alpha}_F^b(x) = \varepsilon_2$. Hence $\tilde{\alpha}_F^b(y) \supseteq \varepsilon_2 = \tilde{\alpha}_F^b(x \rightarrow y) \cap \tilde{\alpha}_F^b(x)$. Let $x, y \in L$ be such that $b \rightarrow (x \rightarrow y) \in F$ and $b \rightarrow x \in F$. Then $b \rightarrow y \in F$ by (F3), and so $\tilde{\alpha}_F^b(y) = \varepsilon_1 = \tilde{\alpha}_F^b(x \rightarrow y) \cap \tilde{\alpha}_F^b(x)$. Therefore $(\tilde{\alpha}_F^b, L)$ is an int-soft filter of L over U for all $b \in L$.

Conversely, suppose that $(\tilde{\alpha}_F^b, L)$ is an int-soft filter of L over U for all $b \in L$. Assume that $x \rightarrow (y \rightarrow z) \in F$ and $x \rightarrow y \in F$ for all $x, y, z \in L$. Taking $b = x$ implies that $\tilde{\alpha}_F^b(y \rightarrow z) = \varepsilon_1 = \tilde{\alpha}_F^b(y)$, and thus $\tilde{\alpha}_F^b(z) \supseteq \tilde{\alpha}_F^b(y \rightarrow z) \cap \tilde{\alpha}_F^b(y) = \varepsilon_1$. Hence $\tilde{\alpha}_F^b(z) = \varepsilon_1$, and so $x \rightarrow z = b \rightarrow z \in F$. Therefore F is an implicative filter of L . \square

Theorem 4.8 (Extension property for int-soft implicative filters). *Let $(\tilde{\alpha}, L)$ and $(\tilde{\beta}, L)$ be int-soft filters of L over U such that $\tilde{\alpha}(1) = \tilde{\beta}(1)$ and $\tilde{\alpha} \subseteq \tilde{\beta}$, that is, $\tilde{\alpha}(x) \subseteq \tilde{\beta}(x)$ for all $x \in L$. If $(\tilde{\alpha}, L)$ is an int-soft implicative filter of L over U , then so is $(\tilde{\beta}, L)$.*

Proof. For any $x, y, z \in L$, we have

$$\begin{aligned}
 & \tilde{\beta}((x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))) \\
 &= \tilde{\beta}((x \rightarrow y) \rightarrow ((x \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow z))) \\
 &= \tilde{\beta}((x \rightarrow y) \rightarrow (x \rightarrow ((x \rightarrow (y \rightarrow z)) \rightarrow z))) \\
 &\supseteq \tilde{\alpha}((x \rightarrow y) \rightarrow (x \rightarrow ((x \rightarrow (y \rightarrow z)) \rightarrow z))) \\
 &\supseteq \tilde{\alpha}(x \rightarrow (y \rightarrow ((x \rightarrow (y \rightarrow z)) \rightarrow z))) \\
 &= \tilde{\alpha}((x \rightarrow (y \rightarrow z)) \rightarrow (x \rightarrow (y \rightarrow z))) \\
 &= \tilde{\alpha}(1) = \tilde{\beta}(1).
 \end{aligned}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}
 & \tilde{\beta}((x \rightarrow y) \rightarrow (x \rightarrow z)) \\
 &\supseteq \tilde{\beta}((x \rightarrow (y \rightarrow z)) \rightarrow ((x \rightarrow y) \rightarrow (x \rightarrow z))) \cap \tilde{\beta}(x \rightarrow (y \rightarrow z)) \\
 &\supseteq \tilde{\beta}(1) \cap \tilde{\beta}(x \rightarrow (y \rightarrow z)) = \tilde{\beta}(x \rightarrow (y \rightarrow z)).
 \end{aligned}$$

Using Theorem 4.5, we conclude that $(\tilde{\beta}, L)$ is an int-soft implicative filter of L over U . \square

References

- [1] U. Acar, F. Koyuncu, and B. Tanay, *Soft sets and soft rings*, Comput. Math. Appl. **59** (2010), no. 11, 3458–3463.
- [2] H. Aktaş and N. Çağman, *Soft sets and soft groups*, Inform. Sci. **177** (2007), no. 13, 2726–2735.
- [3] A. O. Atagiün and A. Sezgin, *Soft substructures of rings, fields and modules*, Comput. Math. Appl. **61** (2011), no. 3, 592–601.
- [4] L. Bolc and P. Borowik, *Many-Valued Logic*, Springer, Berlin, 1992.
- [5] N. Çağman, F. Çitak, and S. Enginoğlu, *Soft set theory and uni-int decision making*, European J. Oper. Res. **207** (2010), no. 2 848–855.
- [6] N. Çağman and S. Enginoğlu, *FP-soft set theory and its applications*, Ann. Fuzzy Math. Inform. **2** (2011), no. 2, 219–226.
- [7] D. Chen, E. C. C. Tsang, D. S. Yeung, and X. Wang, *The parametrization reduction of soft sets and its applications*, Comput. Math. Appl. **49** (2005), 757–763.
- [8] F. Feng, *Soft rough sets applied to multicriteria group decision making*, Ann. Fuzzy Math. Inform. **2** (2011), no. 1, 69–80.
- [9] F. Feng, Y. B. Jun, and X. Zhao, *Soft semirings*, Comput. Math. Appl. **56** (2008), no. 10, 2621–2628.
- [10] J. A. Goguen, *The logic of inexact concepts*, Synthese **19** (1969), 325–373.
- [11] Y. B. Jun, *Implicative filters of lattice implication algebras*, Bull. Korean Math. Soc. **34** (1997), no. 2, 193–198.
- [12] ———, *Fuzzy positive implicative and fuzzy associative filters of lattice implication algebras*, Fuzzy Sets and Systems **121** (2001), no. 2, 353–357.
- [13] ———, *Soft BCK/BCI-algebras*, Comput. Math. Appl. **56** (2008), no. 5, 1408–1413.
- [14] Y. B. Jun, H. S. Kim, and J. Neggers, *Pseudo d-algebras*, Inform. Sci. **179** (2009), no. 11, 1751–1759.
- [15] Y. B. Jun, K. J. Lee, and A. Khan, *Soft ordered semigroups*, MLQ Math. Log. Q. **56** (2010), no. 1, 42–50.

- [16] Y. B. Jun, K. J. Lee, and C. H. Park, *Soft set theory applied to ideals in d -algebras*, Comput. Math. Appl. **57** (2009), no. 3, 367–378.
- [17] Y. B. Jun, K. J. Lee, and J. Zhan, *Soft p -ideals of soft BCI-algebras*, Comput. Math. Appl. **58** (2009), no. 10, 2060–2068.
- [18] Y. B. Jun and C. H. Park, *Applications of soft sets in ideal theory of BCK/BCI-algebras*, Inform. Sci. **178** (2008), no. 11, 2466–2475.
- [19] Y. B. Jun, Y. Xu, and K. Y. Qin, *Positive implicative and associative filters of lattice implication algebras*, Bull. Korean Math. Soc. **35** (1998), no. 1, 53–61.
- [20] P. K. Maji, R. Biswas, and A. R. Roy, *Soft set theory*, Comput. Math. Appl. **45** (2003), no. 4-5, 555–562.
- [21] P. K. Maji, A. R. Roy, and R. Biswas, *An application of soft sets in a decision making problem*, Comput. Math. Appl. **44** (2002), no 8-9, 1077–1083.
- [22] D. Molodtsov, *Soft set theory – First results*, Comput. Math. Appl. **37** (1999), no. 4-5, 19–31.
- [23] V. Novak, *First order fuzzy logic*, Studia Logica **46** (1987), no. 1, 87–109.
- [24] C. H. Park, Y. B. Jun, and M. A. Öztürk, *Soft WS-algebras*, Commun. Korean Math. Soc. **23** (2008), no. 3, 313–324.
- [25] J. Pavelka, *On fuzzy logic I, II, III*, Z. Math. Logik Grundlag. Math. **25** (1979), 45–52, 119–134, 447–464.
- [26] Y. Xu, *Lattice implication algebras*, J. Southwest Jiaotong Univ. **1** (1993), 20–27.
- [27] Y. Xu and K. Y. Qin, *On filters of lattice implication algebras*, J. Fuzzy Math. **1** (1993), no. 2, 251–260.
- [28] ———, *Fuzzy lattice implication algebras*, J. Southwest Jiaotong University **30** (1995), no. 2, 121–127.
- [29] Y. Xu, D. Ruan, K. Y. Qin, and J. Liu, *Lattice-Valued Logic*, Springer-Verlag, Berlin, Heidelberg 2003.
- [30] J. Zhan and Y. B. Jun, *Soft BL-algebras based on fuzzy sets*, Comput. Math. Appl. **59** (2010), no. 6, 2037–2046.

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