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REDUCING SUBSPACES OF WEIGHTED SHIFTS WITH OPERATOR WEIGHTS

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Abstract. We characterize reducing subspaces of weighted shifts with operator weights as wandering invariant subspaces of the shifts with additional structures. We show how some earlier results on reducing subspaces of powers of weighted shifts with scalar weights on the unit disk and the polydisk can be fitted into our general framework.

1. Introduction

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Let $\Omega \subset B(H)$ be a set of operators. A closed subspace X is an invariant subspace of Ω if for every $T \in \Omega$, T maps X into X. The space X is a reducing subspace of Ω if X is invariant under both T and T^* for every $T \in \Omega$. The space X is a minimal invariant (or reducing) subspace of Ω if the only invariant (or reducing) subspaces contained in X are X and $\{0\}$. An operator T is irreducible if the only reducing subspaces of T are H and $\{0\}$.

Previous work focused on reducing subspaces of powers of weighted shifts with scalar weights [12] and related analytic Toeplitz operators [1]. It is well-known that the unweighted unilateral shift of multiplicity one is irreducible. The structure of the reducing subspace lattice for unweighted unilateral shifts of high multiplicities was described in [3] and [7]. The reducing subspaces of some analytic Toeplitz operators on the Bergman space of the unit disk were studied in [12], and more general weighted shifts were discussed in [11]. The paper by Zhu [12] was also inspirational in the recent development of reducing subspaces of analytic Toeplitz operators with Blaschke product symbols on the Bergman space [2]. The reducing subspaces of some analytic Toeplitz operators on the Bergman space of the bidisk were characterized in [6] and [9].

In this note, we characterize the reducing subspaces of weighted shifts with operator weights as wandering invariant subspaces of the shifts with additional structures. Our approach gives simple insights into why these results hold. Our

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framework can provide direct and uniform proofs of some previous results from [11], [6] and [9]. We remark that weighted shifts have been studied extensively in the past several decades. A classical reference is [10], and a recent interest development is the weighted shifts on trees [4].

Let E be a complex Hilbert space. Let $l^2(E)$ be the E-valued l^2 space such that

$$l^2(E) = \left\{ y = (y_0, y_1, \ldots),\ \|y\|^2 = \sum_{i=0}^{\infty} \|y_i\|^2 < \infty \right\}.$$

Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$, where the 1 is in the n-th place. Then we write

$$y = (y_0, y_1, \ldots) = \sum_{i=0}^{\infty} y_i e_i.$$

We identify E as a subspace of $l^2(E)$ by mapping y to ye_0 for $y \in E$. By an abuse of notation, we just write y instead of ye_0 for $y \in E$. Let $\Phi = \{\Phi_n, n \geq 0\} \subset B(E)$ be a sequence of invertible operators. The weighted shift S_Φ with weight Φ is an operator on $l^2(E)$ defined by

$$S_\Phi y e_n = [\Phi_n y]\, e_{n+1},\ n \geq 0, y \in E.$$

It follows that $S_\Phi^* y = 0$ for $y \in E$ and

$$S_\Phi^* y e_{n+1} = [\Phi_n^* y]\, e_n,\ n \geq 0, y \in E.$$

Thus $\ker(S_\Phi^*) = E$. Since

$$\begin{aligned} \left\| S_\Phi \sum_{i=0}^{\infty} y_i e_i \right\|^2 &= \left\| \sum_{i=0}^{\infty} \Phi_i y_i e_{i+1} \right\|^2 \\ &= \sum_{i=0}^{\infty} \|\Phi_i y_i\|^2 \leq \sup_{i \geq 0} \|\Phi_i\|^2 \sum_{i=0}^{\infty} \|y_i\|^2, \end{aligned}$$

S_Φ is a bounded operator if and only if $\sup_{n\geq 0} \|\Phi_n\| < \infty$ and $\|S_\Phi\| = \sup_{n \geq 0} \|\Phi_n\|$.

2. Reducing subspaces of S_Φ

Throughout the paper, we will often write S instead of S_Φ. Let

$$W_n = \Phi_n \cdots \Phi_1 \Phi_0 \ (n \geq 0),\ W_{-1} = I_E.$$

Note that $\Phi_n = W_n W_{n-1}^{-1}$. In this paper, $\mathrm{Span}\{v_i, i \in I\}$ always means the closed linear span of $\{v_i, i \in I\}$.

Lemma 1. *For a closed subspace E_0 of E, let $V(E_0)$ be defined by*

$$V(E_0) := \mathrm{Span}\left\{ S_\Phi^n x, n \geq 0, x \in E_0 \right\}. \tag{1}$$

Then $V(E_0)$ is a reducing subspace of S_Φ if and only if E_0 is an invariant subspace of the sequence of operators $\Omega = \left\{ W_{n-2}^{-1} \Phi_{n-1}^* \Phi_{n-1} W_{n-2}, n \geq 1 \right\}$.

Proof. By definition, $V(E_0)$ is invariant for S. The space $V(E_0)$ is also invariant for S^*, if and only if $S^*S^nx \in V(E_0)$ for any $x \in E_0$ and $n \geq 0$. For $n = 0$, $S^*x = 0$. Now assume $n \geq 1$. Note that

$$\begin{aligned} S^*S^nxe_0 &= S^*\left[(\Phi_{n-1}\cdots\Phi_1\Phi_0x)\,e_n\right] \\ &= \left(\Phi_{n-1}^*\Phi_{n-1}\cdots\Phi_1\Phi_0x\right)e_{n-1}. \end{aligned}$$

By (1), $S^*S^nx \in V(E_0)$ if and only if there exists $y \in E_0$ such that

$$\begin{aligned} S^*S^nxe_0 &= \left(\Phi_{n-1}^*\Phi_{n-1}\cdots\Phi_1\Phi_0x\right)e_{n-1} \\ &= S^{n-1}y = (\Phi_{n-2}\cdots\Phi_1\Phi_0y)\,e_{n-1}. \end{aligned} \tag{2}$$

Therefore

$$(\Phi_{n-2}\cdots\Phi_1\Phi_0)^{-1}\,\Phi_{n-1}^*\Phi_{n-1}\cdots\Phi_1\Phi_0x = y \in E_0$$

Equivalently, E_0 is invariant for Ω. □

Remark 2. Note that for $x \in E_0$, since

$$S_\Phi^{*n}S_\Phi^nxe_0 = W_{n-1}^*W_{n-1}xe_0,$$

then $S^{*n}S^nxe_0 \in V(E_0)$ implies that E_0 is invariant for $W_{n-1}^*W_{n-1}$. By the above lemma the invariance of E_0 for $W_{n-1}^*W_{n-1}$ is implied by the invariance of E_0 for Ω. Here is a direct proof: If E_0 is invariant for Ω, equivalently, (2) holds. By using (2) repeatedly,

$$\begin{aligned} S^{*n}S^nx &= S^{*n-1}S^*S^nx = S^{*n-1}S^{n-1}y_1 \\ &= S^{*n-2}S^{n-2}y_2 = \cdots = y_n \end{aligned}$$

for some $y_1, y_2, \ldots, y_n \in E_0$. Thus E_0 is invariant for $W_{n-1}^*W_{n-1}$. The space E_0 is also invariant for other operators involving Φ_n and W_n by considering the invariance of X for $S^{*m}S^n$ for any $m, n \geq 0$.

Theorem 3. *A closed subspace X is a reducing subspace of S_Φ if and only if*

$$X = \operatorname{Span}\left\{S_\Phi^nx, n \geq 0, x \in E_0\right\}, \tag{3}$$

where $E_0 \subseteq E$ is an invariant subspace of the sequence of operators

$$\Omega = \left\{W_{n-2}^{-1}\Phi_{n-1}^*\Phi_{n-1}W_{n-2}, n \geq 1\right\}.$$

Furthermore X is a minimal reducing subspace of S_Φ if and only if E_0 is a minimal invariant subspace of Ω.

Proof. We only need to prove that if X is a reducing subspace of S, then X is given by (3) for some $E_0 \subseteq E$. Set $E_0 = X \ominus SX$. If SX is not closed, we replace SX by the closure of SX. We first prove that $E_0 \subset E$. Let $f \in E_0$, then

$$\langle f, Sg\rangle = \langle S^*f, g\rangle = 0 \text{ for all } g \in X.$$

Since X is also invariant for S^*, $S^*f \in X$. Hence $S^*f = 0$ and $f \in E$. This proves that $E_0 \subseteq E$. We claim

$$X = V(E_0) := \operatorname{Span}\left\{S^nx, n \geq 0, x \in E_0 = X \ominus SX\right\}.$$

Since $E_0 \subseteq X$, $X \supseteq V(E_0)$. Let $y \in X \ominus V(E_0)$. We need to show that $y = 0$. Write $y = (y_0, y_1, \ldots)$. Since X is invariant for S^*, $S^{*n}y \in X$. For all $x \in E_0 = X \ominus SX$, $n \geq 0$, note that $y \in X \ominus V(E_0)$ implies that

$$0 = \langle y, S^n x\rangle = \langle S^{*n}y, x\rangle .$$

That is, $S^{*n}y \in X \ominus [X \ominus SX] = SX$. But $SX \subseteq S\left[l^2(E)\right]$ and $S\left[l^2(E)\right]$ is orthogonal to E. Now

$$S^{*n}y = (S_\Phi^{*n} y_n e_n, \ldots) = \left(\Phi_0^*\Phi_1^* \cdots \Phi_{n-1}^* y_n, \ldots\right) = (W_{n-1}^* y_n, \ldots) \in S\left[l^2(E)\right].$$

Thus $W_{n-1}^* y_n = 0$ for $n \geq 0$. By assumption W_{n-1} is invertible, so $y_n = 0$ for $n \geq 0$. In conclusion, $y = 0$. The proof is complete. □

If $\Phi = \{\Phi_n, n \geq 0\}$ is double commuting, that is, for all $i \neq j$,

$$\Phi_i\Phi_j = \Phi_j\Phi_i, \Phi_i\Phi_j^* = \Phi_j^*\Phi_i,$$

then $\Omega = \{\Phi_n^*\Phi_n, n \geq 0\}$.

By the above theorem, the lattice of reducing subspaces of S_Φ is completely determined by the lattice of invariant subspaces of Ω. This topic has been discussed extensively in literature, and many results are known, in particular when Ω is a set of finite matrices, see the book [8].

Since any power S_Φ^k for $k \geq 1$ is a weighted shift with operator weights, the above theorem also applies to S_Φ^k. This will become clear as we rephrase the results of [6], [9] and [11] in our framework.

3. Multiplication by z on weighted Hardy space

It is well-known from [10] that a weighted shift with scalar weights is unitarily equivalently to multiplication by z on the weighted Hardy spaces with positive scalar weights. But weighted shifts S_Φ with operator weights from last section are slightly more general than multiplication by z on the weighted Hardy spaces with operator weights defined below in (4). First note that for $A \in B(H)$ and $h \in H$,

$$\langle Ah, Ah\rangle = \langle A^*Ah, h\rangle = \left\langle \sqrt{A^*A}h, \sqrt{A^*A}h \right\rangle,$$

and $\sqrt{A^*A} \geq 0$. Thus in the definition of weighted Hardy space we will use positive operators. Let $\Delta = \{W_n, n \geq 0\}$ be a sequence of invertible positive operators in $B(E)$. The weighted Hardy space $H_\Delta^2(E)$ is defined by

$$H_\Delta^2(E) = \left\{ f(z) = \sum_{i=0}^{\infty} f_i z^i,\ f_i \in E, \|f(z)\|^2 = \sum_{i=0}^{\infty} \|W_{i-1}f_i\|^2 < \infty \right\}, \tag{4}$$

where $W_{-1} = I_E$. Then the multiplication by z on $H_\Delta^2(E)$, denoted by M_z, can be identified with the weighted shift S_Φ on $l^2(E)$ with $\Phi = \left\{\Phi_n = W_n W_{n-1}^{-1}, n \geq 0\right\}$. More precisely, let U be the operator from $l^2(E)$ onto $H_\Delta^2(E)$ defined by

$$U y_n e_n = \left(W_{n-1}^{-1} y_n\right) z^n,\ n \geq 0,\ y_n \in E.$$

Then

$$\|Uy_ne_n\| = \left\|W_{n-1}^{-1}y_nz^n\right\| = \left\|W_{n-1}W_{n-1}^{-1}y_n\right\| = \|y_n\|.$$

Thus U is an onto isometry. Furthermore

$$\begin{aligned} M_zUy_ne_n &= M_z\left(W_{n-1}^{-1}y_nz^n\right) = W_{n-1}^{-1}y_nz^{n+1}, \\ US_\Phi y_ne_n &= U(\Phi_ny_ne_{n+1}) = W_n^{-1}\Phi_ny_nz^{n+1} \\ &= W_n^{-1}W_nW_{n-1}^{-1}y_nz^{n+1} = W_{n-1}^{-1}y_nz^{n+1}. \end{aligned}$$

That is

$$M_zU = US_\Phi,$$

and the reducing subspaces (or minimal reducing subspaces) of M_z and S_Φ are in one to one correspondence. Now Theorem 3 can be reformulated as the following simple and elegant result.

Theorem 4. *Any reducing subspace of M_z on $H^2_\Delta(E)$ is of the form $H^2_\Delta(E_0)$ where $E_0 \subseteq E$ is an invariant subspace of $\Omega = \{W_n, n \geq 0\}$. Furthermore $H^2_\Delta(E_0)$ is a minimal reducing subspace of M_z if and only if E_0 is a minimal invariant subspace of Ω.*

Proof. We need to explain the set Ω. By Theorem 3,

$$\Omega = \left\{W_{n-2}^{-1}\Phi_{n-1}^*\Phi_{n-1}W_{n-2}, n \geq 1\right\}.$$

Since now W_n is assumed to be positive, using $\Phi_n = W_nW_{n-1}^{-1}$, we have

$$W_{n-2}^{-1}\Phi_{n-1}^*\Phi_{n-1}W_{n-2} = W_{n-2}^{-1}W_{n-2}^{-1}W_{n-1}W_{n-1}W_{n-2}^{-1}W_{n-2} = W_{n-2}^{-2}W_{n-1}^2.$$

Since $W_{-1}^{-2}W_0^2 = W_0^2, W_0^2W_0^{-2}W_1^2 = W_1^2, W_1^2W_1^{-2}W_2^2 = W_2^2$ and so on, if E_0 is invariant for $\left\{W_{n-2}^{-2}W_{n-1}^2, n \geq 1\right\}$, then it is invariant for $\left\{W_n^2, n \geq 0\right\}$. Since W_n^2 is invertible, if E_0 is invariant for $\left\{W_n^2, n \geq 0\right\}$, then it is invariant for $\left\{W_{n-2}^{-2}W_{n-1}^2, n \geq 1\right\}$. Lastly, E_0 is invariant for a positive operator W_n if and only it is invariant for W_n^2. □

If E is a finite dimensional complex Hilbert space and $E_0 \subseteq E$ is a nontrivial invariant subspace of $\Omega = \{W_n, n \geq 0\}$, then E_0 contains a minimal invariant subspace of Ω. Since W_n is positive, E_0 is in fact a reducing subspace of Ω and it is the direct sum of several minimal invariant subspaces of Ω.

Corollary 5. *Assume $N = \dim(E) < \infty$. Then any nontrivial reducing subspace of M_{z^k} on $H^2_\Delta(E)$ contains a minimal reducing subspace. Furthermore it is a direct sum of at most Nk minimal reducing subspaces of M_{z^k}.*

4. Remarks on previous results

Now we turn our attention to the results in [11]. Let $\omega = \{\omega_0, \omega_1, \omega_2, \ldots\}$ be a sequence of positive numbers. Let $\mathbb{C}$ denote the set of complex numbers viewed as a one dimensional Hilbert space. Let H^2_ω be as in [10] and [11]:

$$H^2_\omega = \left\{ f(z) = \sum_{i=0}^{\infty} f_i z^i, f_i \in \mathbb{C}, \|f(z)\|^2 = \sum_{i=0}^{\infty} \omega_i |f_i|^2 \right\}. \tag{5}$$

For $N \geq 2$, let E be the N-dimensional subspace of H^2_ω defined by

$$E = \left\{ f(z) = \sum_{i=0}^{N-1} f_i z^i, f_i \in \mathbb{C}, \|f(z)\|^2 = \sum_{i=0}^{N-1} \omega_i |f_i|^2 \right\}, \tag{6}$$

and $\left\{ z^i/\sqrt{\omega_i}, 0 \leq i \leq N-1 \right\}$ is the standard basis of E. Let

$$\Delta = \left\{ W_n = V_0^{-1} V_{n+1}, n \geq 0 \right\},$$

where V_n is the diagonal matrix (with respect the standard basis of E) defined by

$$V_n = D(\sqrt{\omega_{nN}}, \sqrt{\omega_{nN+1}}, \ldots, \sqrt{\omega_{nN+N-1}}).$$

Then M_{z^N} on H^2_ω can be identified with M_z on $H^2_\Delta(E)$. More precisely, let U be the linear operator from H^2_ω onto $H^2_\Delta(E)$ defined by

$$U \sum_{i=0}^{\infty} f_i z^i = \sum_{k=0}^{\infty} g_k z^k, \text{ where } g_k = \left(\sum_{j=0}^{N-1} f_{j+kN} z^j \right) \in E.$$

Note that formally

$$\sum_{i=0}^{\infty} f_i z^i = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{N-1} f_{j+kN} z^j \right) z^{kN} = \sum_{k=0}^{\infty} g_k z^{kN} \neq \sum_{k=0}^{\infty} g_k z^k,$$

so U maps z^N in H^2_ω to z in $H^2_\Delta(E)$. It is easy to verify that U is an onto isometry and $UM_{z^N} = M_z U$. Since Δ consists of diagonal matrices, it is relatively straightforward to determine the invariant subspaces of Δ, as we demonstrate now. Instead of recalling terminology and restating results of [11], we state a lemma which, combined with Theorem 4, will recover results in [11]. Of course, the results in this lemma are essentially also proved in [11], albeit using quite different terminology and techniques. In fact, these results also hold if Ω is a set of diagonal operators on an infinite dimensional separable Hilbert space.

Let $\mathbb{C}^N$ be the N-dimensional complex Hilbert space.

Lemma 6. (i) *Let Ω be a set of invertible diagonal matrices on $\mathbb{C}^N$ with respect to an orthonormal basis $\{e_1, \ldots, e_N\}$. Then any minimal invariant subspace of Ω is one dimensional.*

(ii) *Any invariant subspace of Ω is the orthogonal sum of several one dimensional invariant subspaces of Ω.*

(iii) *Let* $v = \sum_{i=1}^{k} v_{n_i} e_{n_i}$, *where all* v_{n_i} *are nonzero. Then* $\operatorname{Span}\{v\}$ *is invariant for* Ω *if and only if each diagonal matrix in* Ω *restricted to* $\operatorname{Span}\{e_{n_1}, \ldots, e_{n_k}\}$ *is a constant multiple of the identity matrix.*

Proof. Let $E_0 \subseteq \mathbb{C}^N$ be an invariant subspace of Ω. Let $v \in E_0$. Write $v = \sum_{i=1}^{N} v_i e_i$. The length of v is the number of nonzero coefficients v_i. Let k be the minimum length of all nonzero vectors in E_0. Pick $v \in E_0$ such that the length of v is k. Without loss of generality, write $v = \sum_{i=1}^{k} v_i e_i$ where all $v_i \neq 0$ for $1 \leq i \leq k$. For any $A \in \Omega$, either $Av = \lambda v$ for some λ or $Av \neq \lambda v$ for any λ. If for each $A \in \Omega$, Av is a multiple of v, then $\operatorname{Span}\{v\}$ is invariant for Ω. Otherwise, there exists $A \in \Omega$ such that $Av = \sum_{i=1}^{k} \lambda_i v_i e_i$ where not all the λ_i are the same. Thus the length of $\lambda_1 v - Av$ is strictly less than k, which contradicts the definition of k. If E_0 is minimal, then E_0 is equal to $\operatorname{Span}\{v\}$. This proves (i).

Assume E_0 is not $\operatorname{Span}\{v\}$. It follows from the above argument that each $A \in \Omega$ is a constant multiple of the identity on $\operatorname{Span}\{e_1, \ldots, e_k\}$. Replace the basis $\{e_1, \ldots, e_k\}$ by the orthonormal basis $\{v/\|v\|, g_2, \ldots, g_k\}$. Then, for each $A \in \Omega$, the matrix of A with respect to the new basis $\{v/\|v\|, g_2, \ldots, g_k, e_{k+1}, \ldots, e_N\}$ is the same diagonal matrix we started with. Thus if

$$F = \operatorname{Span}\{g_2, \ldots, g_k, e_{k+1}, \ldots, e_N\},$$

then F is reducing for Ω. Note that $\Omega|F$ (the restriction of each matrix in Ω to F) is still a set of invertible diagonal matrices. If $u \in E_0$ and u is not in $\operatorname{Span}\{v\}$, then $u - \lambda v \in E_0 \cap F$ for some λ. Thus

$$E_0 = \operatorname{Span}\{v\} \oplus E_0 \cap F,$$

where $E_0 \cap F$ is an invariant subspace of $\Omega|F$. Continuing this process, we get (ii). The proof of (iii) is similar. □

Example 7. (i) Let $\Omega_1 = \{A, B\}$, where A and B are diagonal matrices on $\mathbb{C}^3$,

$$A = D(\alpha, \alpha, \beta), B = D(\gamma, \beta, \beta).$$

Here α, β and γ are three distinct complex numbers. Then $\operatorname{Span}\{e_1\}$, $\operatorname{Span}\{e_2\}$ and $\operatorname{Span}\{e_3\}$ are the three only minimal invariant subspaces of Ω.

(ii) Let $A = D(\alpha, \alpha, \beta)$ where α and β are two distinct complex numbers. Then the minimal invariant subspaces of A are $\operatorname{Span}\{e_3\}$ and $\operatorname{Span}\{c_1 e_1 + c_2 e_2\}$ for any c_1 and c_2 such that not both are zero.

Lemma 6 can be extended to the set of diagonal operators on an infinite dimensional separable Hilbert space. In fact we can relax slightly the invertibility condition of Ω. Let $\mathbb{N}$ be the set of positive integers. In the infinite dimensional case, all subspaces are assumed to be closed.

Lemma 8. (i) *Let* Ω *be a set of injective diagonal operators on* l^2 *with respect to an orthonormal basis* $\{e_n, n \in \mathbb{N}\}$. *Let* $v = \sum_{i=1}^{\infty} v_{n_i} e_{n_i}$ *where all* v_{n_i} *are nonzero. Then* $\operatorname{Span}\{v\}$ *is invariant for* Ω *if and only if the restriction of* *each diagonal operator in* Ω *to* $\text{Span}\{e_{n_1}, e_{n_2}, \ldots\}$ *is a constant multiple of the identity operator.*

(ii) *Any minimal invariant subspace of* Ω *is one dimensional.*

(iii) *Any invariant subspace of* Ω *is the orthogonal sum of finitely or infinitely many one dimensional invariant subspaces of* Ω.

Proof. Let $v = \sum_{i=1}^{\infty} v_{n_i} e_{n_i}$ where all v_{n_i} are nonzero. Assume $\text{Span}\{v\}$ is invariant for Ω. Let $A \in \Omega$. Then

$$Av = \sum_{i=1}^{\infty} \lambda_i v_{n_i} e_{n_i} = \lambda \sum_{i=1}^{\infty} v_{n_i} e_{n_i}$$

for some nonzero $\lambda,\ \lambda_i,\ i \geq 1$. Therefore $\lambda_i = \lambda$ and A restricted to $\text{Span}\{e_{n_1}, e_{n_2}, \ldots\}$ is a constant multiple of the identity operator. This proves (i).

Let $E_0 \subseteq l^2$ be an invariant subspace of Ω. Since diagonal operators are normal operators, E_0 is reducing for Ω. Let $v \in E_0$. Write $v = \sum_{i=l}^{\infty} v_i e_i$, where $v_l \neq 0$. We call l the index of vector v. Let k be the minimum index of all nonzero vectors in E_0. Let

$$E_1 = E_0 \cap \text{Span}\{e_l, l \geq k+1\},$$

then E_1 is reducing for Ω. Let $G = E_0 \ominus E_1$. Then $G \neq \{0\}$, G is reducing for Ω, and every nonzero vector in G has index k. Pick $v \in G$ and write $v = \sum_{i=1}^{\infty} v_{n_i} e_{n_i}$ where $l = n_1, \{n_i, i \geq 1\}$ is a sequence of strictly increasing positive integers and all $v_{n_i} \neq 0$. It is possible there are only finitely many n_i. But we assume there are infinitely many n_i since the argument for the finite case is similar. If $u \in G$ is another vector, not in $\text{Span}\{v\}$, then for some λ, the index of $v - \lambda u$ is strictly bigger than k, which contradicts the definition of G. Therefore $G = \text{Span}\{v\}$. This proves (ii).

If E_0 is not equal to $\text{Span}\{v\}$, let $Q = \{n_i, i \geq 1\}$. It follows from (i) that $A \in \Omega$ is a constant multiple of the identity on $\text{Span}\{e_k, k \in Q\}$. Replace the basis $\{e_k, k \in Q\}$ by the orthonormal basis $\{v/\|v\|, g_k, k \in Q \setminus \{n_1\}\}$. Then, for each $A \in \Omega$, the matrix of A with respect to the new basis

$$\{v/\|v\|, g_k, k \in Q \setminus \{n_1\}\} \cup \{e_j, j \in \mathbb{N} \setminus Q\}$$

is still a diagonal operator obtained by permuting the diagonals of the operator we started with. Thus if

$$F = \text{Span}\{\{g_k, k \in Q \setminus \{n_1\}\} \cup \{e_j, j \in \mathbb{N} \setminus Q\}\},$$

then F is reducing for Ω. Note that $\Omega|F$ (the restriction of each matrix in Ω to F) is still a set of injective diagonal operators. If $u \in E_0$ and u is not in $\text{Span}\{v\}$, then $u - \lambda v \in F$ for some λ and $u - \lambda v \in E_0 \cap F$. Thus

$$E_0 = \text{Span}\{v\} \oplus E_0 \cap F,$$

where $E_0 \cap F$ is an invariant subspace of $\Omega|F$. Continuing this process, we get (iii). □

The following corollary indicates that, generically, the invariant subspaces of Ω are the obvious ones.

Corollary 9. (i) *Let* Ω *be a set of invertible diagonal matrices on* $\mathbb{C}^N$ *with respect to an orthonormal basis* $\{e_1, \ldots, e_N\}$*. The following two statements are equivalent*:

(a) *For any* $i \neq j$*, there is* $A \in \Omega$ *such that* $Ae_i = \lambda_i e_i, Ae_j = \lambda_j e_j$ *with* $\lambda_i \neq \lambda_j$*.*

(b) *There are exactly* N *minimal invariant subspaces of* Ω*. Namely,* $\text{Span}\{e_i\}$ *for* $i = 1, \ldots, N$*.*

(ii) *Let* Ω *be a set of injective diagonal operators on* l^2 *with respect to an orthonormal basis* $\{e_n, n \in \mathbb{N}\}$*. The following two statements are equivalent*:

(a) *For any* $i, j \in \mathbb{N}$ *with* $i \neq j$*, there is* $A \in \Omega$ *such that* $Ae_i = \lambda_i e_i, Ae_j = \lambda_j e_j$ *with* $\lambda_i \neq \lambda_j$*.*

(b) *The minimal invariant subspaces of* Ω *are* $\text{Span}\{e_i\}$ *for* $i \in \mathbb{N}$*.*

Statement (a) in both (i) and (ii) holds as long as Ω contains a diagonal operator with distinct entries on the diagonal.

5. Polydisk and tensor product

In this last section we turn our attention to some results on the weighted Bergman space of bidisk from [6] and [9]. For $-1 < \alpha < \infty$, the weighted Bergman space $A^2_\alpha(\mathbb{D})$ is a Hilbert space of analytic functions on the unit disk $\mathbb{D}$. The inner product of $A^2_\alpha(\mathbb{D})$ is defined by

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA_\alpha(z), \ \ f, g \in A^2_\alpha(\mathbb{D}),$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ and $dA(z)$ is the normalized Lebesgue area measure on $\mathbb{D}$. It is well-known that $A^2_\alpha(\mathbb{D})$ is the weighted Hardy space H^2_ω as in (5) with $\omega_n = \frac{n!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n)}$.

The weighted Bergman space $A^2_\alpha(\mathbb{D}^2)$ is a Hilbert space of analytic functions of two variables z_1 and z_2 on the bidisk $\mathbb{D}^2$. The inner product of $A^2_\alpha(\mathbb{D}^2)$ is defined by

$$\langle f, g\rangle = \int_{\mathbb{D}^2} f(z_1, z_2)\overline{g(z_1, z_2)}d\mu_\alpha(z_1, z_2), \ \ f, g \in A^2_\alpha(\mathbb{D}^2),$$

where $d\mu_\alpha(z_1, z_2) = dA_\alpha(z_1)dA_\alpha(z_2)$. The reducing subspaces of some multiplication operators on $A^2_\alpha(\mathbb{D})$ were studied in [12]. The reducing subspaces of multiplication operators $M_{z_1^{N_1}}, M_{z_2^{N_2}}$ and more generally $M_{z_1^{N_1}z_2^{N_2}}$ on $A^2_\alpha(\mathbb{D}^2)$ were investigated in [6] with $N_1 = N_2$ and [9] with $N_1 \neq N_2$. The space $A^2_\alpha(\mathbb{D}^2)$ can be identified with the tensor product $A^2_\alpha(\mathbb{D}) \otimes A^2_\alpha(\mathbb{D})$ where for the first $A^2_\alpha(\mathbb{D})$ we use z_1 and for the second $A^2_\alpha(\mathbb{D})$ we use z_2. Consequently, $M_{z_1^{N_1}z_2^{N_2}}$ on $A^2_\alpha(\mathbb{D}^2)$ is $M_{z_1^{N_1}} \otimes M_{z_2^{N_2}}$, and $M_{z_1^{N_1}}$ on $A^2_\alpha(\mathbb{D}^2)$ is $M_{z^{N_1}} \otimes I$ on $A^2_\alpha(\mathbb{D}) \otimes A^2_\alpha(\mathbb{D})$. As we have shown above, $M_{z_1^{N_1}}$ on $A^2_\alpha(\mathbb{D})$ is a weighted shift with matrix weights. The paper [5] is a classical reference for the connection between commuting weighted shifts with scalar weights and analytic functions in several variables. The operators $M_{z_1^{N_1}z_2^{N_2}}$, $M_{z_1^{N_1}}$ and $M_{z_2^{N_2}}$ on $A^2_\alpha(\mathbb{D}^2)$ can also be showed to be unitarily equivalent to weighted shifts with invertible diagonal operator weights.

It is relatively easy to see that $M_{z_1^{N_1}}$ or $M_{z_2^{N_2}}$ on $A^2_\alpha(\mathbb{D}^2)$ is a weighted shift with invertible diagonal operator weights. Let S_Φ be the weighted shift with invertible operator weights $\Phi = \{\Phi_n, n \geq 0\}$ defined on $l^2(E)$ by

$$S_\Phi x e_n = [\Phi_n x]\, e_{n+1},\ n \geq 0,\ x \in E.$$

Let K be another Hilbert space. Let T be any bounded operator on K. Then $S_\Phi \otimes T$ defined on $l^2(E) \otimes K$ is a weighted shift on $l^2(E \otimes K)$ with weights $\{\Phi_n \otimes T, n \geq 0\}$. But $\Phi_n \otimes T$ is not invertible unless T is invertible, for example, if $T = I_K$ and Φ_n is a diagonal operator on E, then $\Phi_n \otimes I_K$ is also a diagonal operator on $E \otimes K$. Thus $M_{z_1^{N_1}}$ on $A^2_\alpha(\mathbb{D}^2)$ can be identified with S_Φ on some $H^2_\Delta(E)$, as in (4) with weight operators being invertible diagonal operators, and Theorem 4 and Lemma 8 could be applied. We refer to Theorem 2.1, Theorem 2.2 and Theorem 2.3 in [6] for relevant concrete results.

The operator $M_{z_1^{N_1} z_2^{N_2}}$ on $A^2_\alpha(\mathbb{D}^2)$ is also unitarily equivalent to a weighted shift with invertible diagonal operator weights on $l^2(\widehat{E})$ where $\widehat{E} = \ker(M^*_{z_1^{N_1} z_2^{N_2}})$. Here is a very rough explanation. Note that

$$\widehat{E} = \ker(M^*_{z_1^{N_1} z_2^{N_2}}) = \operatorname{Span}\left\{ z_1^i z_2^j, 0 \leq i < N_1 \text{ or } 0 \leq j < N_2 \right\}.$$

If $f(z_1,z_2) \in A^2_\alpha(\mathbb{D}^2)$, then for some $g_n(z_1,z_2) \in \widehat{E}, n \geq 0$,

$$f(z_1,z_2) = \sum_{n=0}^{\infty} g_n(z_1,z_2)(z_1^{N_1} z_2^{N_2})^n, \text{ where } g_n(z_1,z_2) \in \widehat{E} \text{ for all } n \geq 0,$$

$$M_{z_1^{N_1} z_2^{N_2}} f(z_1,z_2) = \sum_{n=0}^{\infty} g_n(z_1,z_2)(z_1^{N_1} z_2^{N_2})^{n+1}.$$

We leave the details of this explanation possibly for the future.

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